ROUGH CONTINUOUS CONVERGENCE OF SEQUENCES OF SETS

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Abstract. In this paper, we define a new type of convergence of sequences of sets by using the continuous convergence (or α-convergence) of the sequence of distance functions. Then we proved in which case it is equivalent to rough Wijsman convergence by considering the different values of the roughness degrees.

Keywords: Wijsman convergence, rough convergence, α-convergence, equicontinuity.

1. Introduction

Wijsman [11] has introduced a new type of convergence, which is considered as one of the most important contributions to the theory of convergence of sequences of sets and it is called by his name. He used pointwise convergence of distance functions to define this type of convergence. He [12] also proved a necessary and sufficient condition related to the pointwise limit and limit inferior of the sequences of distance functions under various constraints in order for a sequence of sets to be Wijsman convergent.

In the 2000s, after Phu [8] put forward the idea of rough convergence in the normed spaces, Phu's work was extended to statistical convergence by Aytar [1], and to ideal convergence by Dündar and Çakan [4]. Phu's [8] idea showed that a sequence

which is not convergent in the usual sense might be convergent to a point, with a certain degree of roughness. In 2016, by combining the two concepts (Wijsman convergence and rough convergence), the idea of rough Wijsman convergence of a sequence of sets was defined by Ölmez and Aytar [7]. Then, Subramanian and Esi [10] defined the concept of rough Wijsman convergence for a triple sequences of sets. Recently, Babaarslan and Tuncer [2] applied the theory of rough convergence to the fuzzy set theory using the double sequences.

Continuous convergence, which is a stronger type of convergence than pointwise convergence (see [6], [9]), has been referred to as α-convergence in recent years (see [3], [5]). Pointwise convergence is equivalent to α-convergence on sequences or nets of functions that are equicontinuous. Das and Papanastassiou [3] defined the concepts of α-equal convergence, α-uniform equal convergence and α-strong uniform equal convergence on the sequences of real-valued functions. Gregoriades and Papanastassiou [5] defined the concept of exhaustive, which is a property weaker than equicontinuity for sequences and nets of functions on metric spaces, and using this property, they investigated the relationships between α-convergence, pointwise convergence and uniform convergence. They also gave a generalization of Ascoli's theorem using the concept of exhaustive.

The main purpose of this article is to observe the results using α-convergence instead of pointwise convergence of distance functions. In this context, first we define the concept of rough continuous convergence. Then we examined the relations between the new definitions obtained with different roughness degrees r_1 and r_2 (see Propositions 3.1 and 3.2). As the main results of this paper, we show that in which cases the new definition coincides with the rough Wijsman convergence (see Theorem 3.1). By giving illustrative examples, the similarity (see Example 3.1) and difference (see Example 3.2) between definitions are obtained.

2. Preliminaries

Throughout this paper, we assume that X is a nonempty set and ρ_X is a metric on X and that A, A_n are nonempty closed subsets of X for each $n \in \mathbb{N}$.

Let (x_n) be a sequence in the metric space X, and r be a nonnegative real number, the sequence (x_n) is said to be rough convergent to x with the roughness degree r, denoted by $x_n \xrightarrow{r} x$, if for each $\varepsilon > 0$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that $\rho_X(x_n, x) < r + \varepsilon$ for each $n \geq n(\varepsilon)$ [8].

The *distance function* $d(\cdot, A) : X \to [0, \infty)$ is defined by the formula

$$d(x, A) = \inf\{\rho_X(x, y) : y \in A\}$$

[6, 11].

We say that the sequence (A_n) is *Wijsman convergent* to the set A if

$$\lim_{n \to \infty} d(x, A_n) = d(x, A) \text{ for all } x \in X.$$

In this case, we write $A_n \overset{W}{\to} A$, as $n \to \infty$ [11].

Given $r \geq 0$, we say that a sequence (A_n) is *rough Wijsman convergent* to the set A if for every $\varepsilon > 0$ and each $x \in X$ there exists an $N(x, \varepsilon) \in \mathbb{N}$ such that

$$|d(x, A_n) - d(x, A)| < r + \varepsilon \text{ for all } n \geq N(x, \varepsilon)$$

and we write $d(x, A_n) \overset{r}{\to} d(x, A)$ or $A_n \overset{r-W}{\to} A$ as $n \to \infty$ [7].

Let (Y, ρ_Y) be another metric space and D be a subset of X. Assume the f, f_n functions from X to Y for each $n \in \mathbb{N}$. The sequence (f_n) *α-converges* to f iff for every $x \in X$ and for every sequence (x_n) of points of X converging to x, the sequence $(f_n(x_n))$ converges to $f(x)$. We shall write $f_n \overset{\alpha}{\to} f$ to denote that the sequence (f_n) α-converges to f (see [5, 6, 9]).

The open ball with centre $x \in X$ and radius $\delta > 0$ is the set

$$S(x, \delta) = \{y \in X : \rho_X(x, y) < \delta\}.$$

The sequence (f_n) is called *equicontinuous* at x if for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\rho_Y(f_n(y), f_n(x)) < \varepsilon$ whenever $y \in S(x, \delta)$, $n \in \mathbb{N}$ [6].

3. Main Results

Definition 3.1. Let $r_1 \geq 0$ and $r_2 \geq 0$. The sequence (A_n) is said to be rough α-convergent (or continuous convergent) to the set A with the roughness degree $r_1 \wedge r_2$ if for every sequence (x_n) which is $x_n \overset{r_1}{\longrightarrow} x$, the condition $d(x_n, A_n) \overset{r_2}{\longrightarrow} d(x, A)$ holds at each $x \in X$. In this case, we use the notation $A_n \overset{r_1 \wedge r_2 - \alpha}{\longrightarrow} A$.
If take $r_1 = 0$ and use the notation r instead of r_2, the sequence (A_n) is said to be rough α-convergent to the set A, and we write $A_n \overset{r-\alpha}{\longrightarrow} A$.

Let us give an illustrative example to explain the Definition 3.1 to the readers.

Example 3.1. Define

$$A_n := \begin{cases} [-3, -1] \times [-1, 1] & \text{, if } n \text{ is an odd integer} \\ [1, 3] \times [-1, 1] & \text{, if } n \text{ is an even integer} \end{cases}$$

and $A = \{0\} \times [-1, 1]$ in the space $\mathbb{R}^2$ equipped with the Euclid metric.

First we show that the sequence (A_n) is rough Wijsman convergent to the set A. Let $\varepsilon > 0$ and $(x^*, y^*) \in \mathbb{R}^2$. Then we calculate

$$d((x^*, y^*), A) = \begin{cases} \sqrt{(x^* - 0)^2 + (y^* - 1)^2} & \text{, if } x^* \in \mathbb{R} \text{ and } y^* > 1 \\ \sqrt{(x^* - 0)^2 + (y^* + 1)^2} & \text{, if } x^* \in \mathbb{R} \text{ and } y^* < -1 \\ |x^*| & \text{, if } x^* \in \mathbb{R} \text{ and } -1 \leq y^* \leq 1 \end{cases}.$$

Similarly, $d((x^*, y^*), A_n)$ can be easily calculated. Then there exists an $n_1 = n_1((x^*, y^*), \varepsilon)$ such that it can be easily obtained

$$|d((x^*, y^*), A_n) - d((x^*, y^*), A)| \leq 3 + \varepsilon$$

for each $n \geq n_1$ using the inequality $\sqrt{(x^* - x)^2 + (y^* - y)^2} \leq |x^* - x| + |y^* - y|$. Hence, it is proved that $A_n \overset{r-W}{\longrightarrow} A$, for every $r \geq 3$.

Now we show that the sequence (A_n) is rough α-convergent to the set A. Assume that the sequence (x_n, y_n) converges to the point (x^*, y^*). Hence there exists an $n_2 = n_2((x^*, y^*), \varepsilon)$ such that it can be easily calculated

$$|d((x_n, y_n), A_n) - d((x^*, y^*), A)| \leq 3 + \varepsilon$$

for each $n \geq n_2$. This proves that $A_n \overset{r-\alpha}{\longrightarrow} A$ for each $r \geq 3$.

Lastly we show that $A_n \overset{r_1 \wedge r_2 - \alpha}{\longrightarrow} A$. Let $(x_n, y_n) \overset{r_1}{\longrightarrow} (x^*, y^*)$. Then there exists an $n_3 = n_3((x^*, y^*), \varepsilon)$ such that $|x_n - x^*| < r_1 + \varepsilon$ and $|y_n - y^*| < r_1 + \varepsilon$ for every $n \geq n_3$. Hence the inequality

$$|d((x_n, y_n), A_n) - d((x^*, y^*), A)| \leq 3 + r_1 + \varepsilon$$

is obvious for every $n \geq n_3$. If we take $r_2 = r_1 + 3$, then we get $A_n \overset{r_1 \wedge r_2 - \alpha}{\longrightarrow} A$.

Proposition 3.1. *If the sequence (A_n) is rough α-convergent to the set A with the roughness degree $r_1 \wedge r_2$ then it rough α-converges to the set A with the roughness degree r_2.*

Proof. Assume $A_n \overset{r_1 \wedge r_2 - \alpha}{\longrightarrow} A$. Take $x \in X$. Let (x_n) be a sequence such that $x_n \longrightarrow x$. We also have $x_n \overset{r_1}{\longrightarrow} x$. Since $A_n \overset{r_1 \wedge r_2 - \alpha}{\longrightarrow} A$, we get

(3.1) $$d(x_n, A_n) \overset{r_2}{\longrightarrow} d(x, A).$$

Then (3.1) holds for each sequence (x_n) such that $x_n \longrightarrow x$. Hence we have $A_n \overset{r_2 - \alpha}{\longrightarrow} A$, which completes the proof. $\square$

As can be seen following example, the converse implication of Proposition 3.1 doesn't hold in general.

Example 3.2. Define

$$A_n := \begin{cases} \left\{-2 + \dfrac{1}{n}\right\} & \text{, if } n \text{ is an odd integer} \\ \left\{2 - \dfrac{1}{n}\right\} & \text{, if } n \text{ is an even integer} \end{cases}$$

and $A = [-2, 2]$.

First we show that the sequence (A_n) is rough Wijsman convergent to the set A. We have

$$d(x, A_n) = \begin{cases} \left|x + 2 - \dfrac{1}{n}\right| & \text{, if } n \text{ is an odd integer} \\ \left|x - 2 + \dfrac{1}{n}\right| & \text{, if } n \text{ is an even integer} \end{cases}$$

and

$$d(x, A) = \begin{cases} |x + 2| & \text{, if } x < -2 \\ 0 & \text{, if } -2 \leq x \leq 2 \\ |x - 2| & \text{, if } x > 2 \end{cases}$$

for each $x \in \mathbb{R}$. Hence, for each $\varepsilon > 0$ and each x, there exists an $n_1 = n_1(x, \varepsilon)$ such that $n \geq n_1$ we have

$$|d(x, A_n) - d(x, A)| \leq 4 + \varepsilon.$$

Therefore, we get $A_n \overset{r-W}{\longrightarrow} A$ for each $r \geq 4$.

Now we show that the sequence (A_n) is rough α-convergent to the set A. Assume $x_n \longrightarrow x$. Since

$$d(x_n, A_n) = \begin{cases} \left| x_n + 2 - \frac{1}{n} \right| & , \text{ if } n \text{ is an odd integer} \\ \left| x_n - 2 + \frac{1}{n} \right| & , \text{ if } n \text{ is an even integer} \end{cases},$$

for each $\varepsilon > 0$ there exists an $n_2 = n_2(x, \varepsilon)$ such that $n \geq n_2$ we have

$$|d(x_n, A_n) - d(x, A)| \leq 4 + \varepsilon.$$

This is desired result, i.e., $A_n \overset{r-\alpha}{\longrightarrow} A$ for every $r \geq 4$.

Lastly we show that $A_n \overset{r_1 \wedge r_2 - \alpha}{\nrightarrow} A$. Take $r_1 = r_2 = 4$. Define $x_n = 6$ for each n and $x = 2$. Then the sequence (x_n) is rough α-convergent to the point x with the roughness degree $r_1 = 4$. On the other hand, we have

$$d(x_n, A_n) = \begin{cases} \left| 8 - \frac{1}{n} \right| & , \text{ if } n \text{ is an odd integer} \\ \left| 4 + \frac{1}{n} \right| & , \text{ if } n \text{ is an even integer} \end{cases}.$$

If we take $\varepsilon = 1$, then we have

$$|d(x_n, A_n) - d(x, A)| = 8 \nleq 5 = r_2 + \varepsilon$$

for every odd terms. Hence we get $A_n \overset{r_1 \wedge r_2 - \alpha}{\nrightarrow} A$.

The question may come to mind: Could the converse implication of Proposition 3.1 be obtained based on a particular selection of r_1 and r_2? Before answering this question as Proposition 3.2, we will give a simple inequality:

Lemma 3.1. *If the set A is a nonempty closed subset of X, then we have*

$$|d(x, A) - d(y, A)| \leq \rho_X(x, y)$$

for each $x, y \in X$.

The proof of Lemma 3.1 is obvious from the Lipschitz continuity of distance functions.

Proposition 3.2. *If the sequence (A_n) is α-convergent to the set A with the rough-ness degree r, then it is α-convergent to the set A with the roughness degree $r_1 \wedge r_2$ for each r_1 and r_2 such that $r_2 \geq r_1 + r$.*

Proof. Let $\varepsilon > 0$ and $x \in X$. If we assume that $x_n \overset{r_1}{\longrightarrow} x$, then it is clear that there exists a sequence $(y_n) \subset X$ such that $y_n \to x$ and $\rho_X(x_n, y_n) \leq r_1$. Since the

sequence (A_n) is α-convergent to the set A with the roughness degree r, there exists an $n_1(x, \varepsilon) \in \mathbb{N}$ such that $n \geq n_1$ we have

$$|d(y_n, A_n) - d(x, A)| < r + \varepsilon.$$

By Lemma 3.1, we get

$$|d(x_n, A_n) - d(y_n, A_n)| \leq \rho_X(x_n, y_n) \leq r_1$$

for each $n \in \mathbb{N}$. Then we have

$$
\begin{aligned}
|d(x_n, A_n) - d(x, A)| &\leq |d(x_n, A_n) - d(y_n, A_n)| + |d(y_n, A_n) - d(x, A)| \\
&< r_1 + r + \varepsilon
\end{aligned}
$$

for each $n \geq n_1$. If we take $r_2 = r_1 + r$, then we say that the sequence (A_n) is α-convergent to the set A with the roughness degree $r_1 \wedge r_2$, which completes the proof. $\square$

Before giving the main result of the paper, let's give a lemma. It will be used in the proof of Theorem 3.1.

Lemma 3.2. *The sequence $(d(\cdot, A_n))$ of distance functions is equicontinuous.*

Proof. Let $x \in X$, $\varepsilon > 0$ and $z \in S(x, \varepsilon)$. We have

$$
\begin{aligned}
\rho_X(y, z) &\leq \rho_X(y, x) + \rho_X(x, z) \\
\rho_X(x, y) &\leq \rho_X(x, z) + \rho_X(z, y)
\end{aligned}
$$

for $y \in A_n$, where n fixed. Since

$$
\begin{aligned}
d(z, A_n) &= \inf_{y \in A_n} \rho_X(y, z) \leq \inf_{y \in A_n} (\rho_X(y, x) + \rho_X(x, z)) \\
&= \inf_{y \in A_n} \rho_X(y, x) + \rho_X(x, z) < d(x, A_n) + \varepsilon \\
d(x, A_n) &= \inf_{y \in A_n} \rho_X(x, y) \leq \inf_{y \in A_n} (\rho_X(x, z) + \rho_X(z, y)) \\
&= \inf_{y \in A_n} (\rho_X(z, y)) + \rho_X(x, z) < d(z, A_n) + \varepsilon,
\end{aligned}
$$

we get

$$-\varepsilon < d(z, A_n) - d(x, A_n) < \varepsilon.$$

Therefore, if we take $\delta = \varepsilon > 0$, then we get

$$|d(z, A_n) - d(x, A_n)| < \varepsilon$$

for each $n \in \mathbb{N}$ and each $z \in S(x, \varepsilon)$. Since the point x is arbitrary, the sequence $(d(\cdot, A_n))$ of functions is equicontinuous. $\square$

Theorem 3.1. *The concepts of rough Wijsman convergence and rough α-convergence are equivalent to each other with the same roughness degree.*

Proof. First we assume that the sequence (A_n) is rough α-convergent to the set A. Let $\varepsilon > 0$ and $x \in X$. Define $x_n = x$ for each $n \in \mathbb{N}$. Since $A_n \xrightarrow{r-\alpha} A$, there exists an $n_1(x, \varepsilon) \in \mathbb{N}$ such that $n \geq n_1$, we have

$$|d(x_n, A_n) - d(x, A)| < r + \varepsilon.$$

Then we get

$$\begin{aligned} |d(x, A_n) - d(x, A)| &= |d(x_n, A_n) - d(x, A)| \\ &< r + \varepsilon \end{aligned}$$

for each $n \geq n_1$. Therefore the sequence (A_n) is rough Wijsman convergent to the set A.

On the other hand, now we assume that the sequence (A_n) is rough Wijsman convergent to the set A with the roughness degree r. Then the sequence $(d(\cdot, A_n))$ of functions is rough convergent to the function $d(\cdot, A)$ on X with the same roughness degree r. Let $x \in X$ and $\varepsilon > 0$. Hence there exists an $n_1(x, \varepsilon) \in \mathbb{N}$ such that $n \geq n_1$ we have
$$|d(x, A_n) - d(x, A)| < r + \frac{\varepsilon}{2}.$$

By Lemma 3.2, there exists $\delta(x, \varepsilon) > 0$ such that

(3.2) $$|d(y, A_n) - d(x, A_n)| < \frac{\varepsilon}{2}$$

for each $n \in \mathbb{N}$ and each $y \in S(x, \delta)$. Take a sequence (x_n) such that $x_n \longrightarrow x$. In this case, there exists an $n_2(x, \delta) \in \mathbb{N}$ such that $\rho_X(x_n, x) < \delta$ for each $n \geq n_2$. Hence by the inequality (3.2), we get

$$|d(x_n, A_n) - d(x, A_n)| < \frac{\varepsilon}{2}$$

for each $n \geq n_2$. Define $n_0 = \max\{n_1, n_2\}$. Therefore we have

$$\begin{aligned} |d(x_n, A_n) - d(x, A)| &\leq |d(x_n, A_n) - d(x, A_n)| + |d(x, A_n) - d(x, A)| \\ &< r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = r + \varepsilon \end{aligned}$$

for each $n \geq n_0$. Since x is an arbitrary point, we say that the sequence (A_n) is rough α-convergent to the set A. $\square$

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