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NOTES ON LEFT IDEALS OF SEMIPRIME RINGS WITH MULTIPLICATIVE GENERALIZED $(\alpha, \alpha)-$ DERIVATIONS

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Abstract. Let R be a $2-$torsion free semiprime ring, I a nonzero left ideal of R, α an automorphism on R and $F : R \to R$ a multiplicative (generalized) $(\alpha, \alpha)-$derivation of R associated with a multiplicative $(\alpha, \alpha)-$derivation d. In this note, we will give the description of commutativity of semiprime rings with help of some identities involving a multiplicative generalized $(\alpha, \alpha)-$derivation and a nonzero left ideal of R.
Keywords: Derivations, ideals, semiprime rings.

1. Introduction

Let R will be an associative ring with center Z. For any $x, y \in R$ the symbol $[x, y]$ represents commutator $xy - yx$ and the Jordan product $xoy = xy + yx$. Recall that a ring R is prime if for $x, y \in R$, $xRy = (0)$ implies either $x = 0$ or $y = 0$ and R is semiprime if for $x \in R$, $xRx = (0)$ implies $x = 0$.

An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An immediate example of a derivation is the inner derivation (i.e., a mapping $x \to [a, x]$, where a is a fixed element). By the generalized inner derivation we mean an additive mapping $F : R \to R$ such that for fixed elements $a, b \in R, F(x) = ax + xb$ for all $x \in R$. It observed that F satisfies the relation $F(xy) = F(x)y + xI_{-b}(y)$ for all $x, y \in R$, where $I_{-b}(y) = [-b, y]$ is the inner derivation of R associated with the element $(-b)$. Motivated by these observations,

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M. Brešar [3] introduced the notion of generalized derivation. Accordingly, a generalized derivation $F : R \to R$ is an additive mapping which is uniquely determined by a derivation d such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Obviously, every derivation is a generalized derivation. Thus, generalized derivations cover both the concept of derivations and left multipliers (i.e., an additive mapping such that $F(xy) = F(x)y$, for all $x, y \in R$). Generalized derivations have been primarily studied on operator algebras.

In [4], the notion of multiplicative derivation was introduced by Daif motivated by Martindale in [13]. $d : R \to R$ is called a multiplicative derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. These maps are not additive. In [10], Goldman and Šemrl gave the complete description of these maps. We have $R = C[0, 1]$, the ring of all continuous (real or complex valued) functions and define a mapping $d : R \to R$ such as

$$d(f)(x) = \left\{ \begin{array}{ll} f(x) \log |f(x)|, & f(x) \neq 0 \\ 0, & \text{otherwise} \end{array} \right\}.$$

It is clear that d is a multiplicative derivation, but d is not additive.

On the other hand, the notion of multiplicative generalized derivation was extended by Daif and Tamman El-Sayiad in [6]. $F : R \to R$ is called a multiplicative generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Dhara and Ali gave a slight generalization of this definition taking d is any mapping (not necessarily an additive mapping or a derivation) in [7]. Hence, one may observe that the concept of multiplicative generalized derivations includes the concept of derivations, generalized derivations and the left multipliers.

Over the last several years, a number of authors studied commutativity theorems for prime rings or semiprime rings admitting automorphisms or derivations on appropriate subsets of R. Herstein proved that if R is a $2-$torsion free prime ring with a nonzero derivation d of R such that $[d(x), d(y)] = 0$, for all $x, y \in R$, then R is commutative ring. In [5], Daif and Bell proved that R is semiprime ring, I is a nonzero ideal of R and d is a derivation of R such that $d([x, y]) = \pm[x, y]$, for all $x, y \in I$, then R contains a nonzero central ideal. Many authors extended these classical theorems to the class of derivations. (see [1], [2], [8], [9], [11], [12] for a partial bibliography).

In the present paper, we generalize the concept of multiplicative generalized derivations to multiplicative generalized $(\alpha, \alpha)-$derivations. A mapping $d : R \to R$ (not necessarily additive) is called a multiplicative $(\alpha, \alpha)-$derivation if there exists a map $\alpha : R \to R$ such that $d(xy) = d(x)\alpha(y) + \alpha(x)d(y)$, for all $x, y \in R$. A mapping $F : R \to R$ (not necessarily additive) is called a multiplicative generalized $(\alpha, \alpha)-$derivation if $F(xy) = F(x)\alpha(y) + \alpha(x)d(y)$, for all $x, y \in R$, where d is a multiplicative $(\alpha, \alpha)-$derivation of R. Of course a multiplicative generalized $(1, 1)-$derivation where 1 is the identity map on R is a multiplicative generalized derivation. So, it would be interesting to extend some results concerning these notions to multiplicative generalized $(\alpha, \alpha)-$derivations. Our aim is to investigate some identities with multiplicative generalized (α,α) −derivations on a nonzero left ideal of semiprime ring R.

2. Results

Throughout the paper, R be a 2−torsion free semiprime ring, I a nonzero left ideal of R, α an automorphism on R and F a multiplicative (generalized) (α,α) −derivation of R associated with a multiplicative (α,α)−derivation d. Also, we will make some extensive use of the basic commutator identities:

i) $[x,yz]=y[x,z]+[x,y]z$

ii) $[xy,z]=[x,z]y+x[y,z]$

iii) $xyoz=(xoz)y+x[y,z]=x(yoz)-[x,z]y$

iv) $xoyz=y(xoz)+[x,y]z=(xoy)z-y[z,x]$

v) $[xy,z]_{\alpha,\alpha}=x\,[y,z]_{\alpha,\alpha}+[x,\alpha(z)]\,y=x[y,\alpha(z)]+[x,z]_{\alpha,\alpha}\,y$

vi) $[x,yz]_{\alpha,\alpha}=\alpha(y)\,[x,z]_{\alpha,\alpha}+[x,y]_{\alpha,\alpha}\,\alpha(z)$

vii) $(xz\circ y)_{\alpha,\alpha}=x(z\circ y)_{\alpha,\alpha}-[x,\alpha(y)]z.$

We remind some well known results which will be useful in our proofs:

Fact : Let R be a semiprime ring, then

i) The center of R contains no nonzero nilpotent elements.

ii) If P is a nonzero prime ideal of R and $a,b\in R$ such that $aRb\subseteq P$, then either $a\in P$ or $b\in P$.

iii) The center of a nonzero one sided ideal is contained in the center of R. In particular, any commutative one sided ideal is contained in the center of R.

Lemma 2.1. [12, Lemma 5] *Let R be a 2−torsion-free semiprime ring and I a nonzero ideal of R. If $[I,I]\subseteq Z$, then R is a commutative ring.*

Theorem 2.1. *Let R be a 2−torsion free semiprime ring, I a nonzero left ideal of R, α an automorphism on R and $F:R\rightarrow R$ a multiplicative generalized (α,α) −derivation of R associated with a multiplicative (α,α)−derivation d.*

If $[d(x),F(y)]=\pm\alpha([x,y])$ for all $x,y\in I$, then $\alpha(I)\,[d(x),\alpha(x)]=(0)$ for all $x\in I$.

Proof. By the hypothesis, we have

$$[d(x),F(y)]=\pm\alpha([x,y]),\ \text{for all } x,y\in I. \tag{2.1}$$

Replacing xz by x in (2.1) and using this, we get

$$d(x)[\alpha(z), F(y)] + [\alpha(x), F(y)]d(z) = 0, \text{ for all } x, y, z \in I. \tag{2.2}$$

Replacing zx by x in (2.2), we have

$$\begin{aligned} &d(z)\alpha(x)[\alpha(z), F(y)] + \alpha(z)d(x)[\alpha(z), F(y)] \\ &+\alpha(z)[\alpha(x), F(y)]d(z) + [\alpha(z), F(y)]\alpha(x)d(z) = 0. \end{aligned} \tag{2.3}$$

Left multiplying (2.2) by $\alpha(z)$, we arrive at

$$\alpha(z)d(x)[\alpha(z), F(y)] + \alpha(z)[\alpha(x), F(y)]d(z) = 0, \text{ for all } x, y, z \in I. \tag{2.4}$$

Subtracting (2.4) from (2.3), we find that

$$d(z)\alpha(x)[\alpha(z), F(y)] + [\alpha(z), F(y)]\alpha(x)d(z) = 0, \ \forall x, y, z \in I. \tag{2.5}$$

That is

$$d(z)\alpha(x)[\alpha(z), F(y)] = -[\alpha(z), F(y)]\alpha(x)d(z), \ \forall x, y, z \in I. \tag{2.6}$$

Replacing x with $x\alpha^{-1}(d(z))t$ in this equation, we have

$$\begin{aligned} &d(z)\alpha(x)d(z)\alpha(t)[\alpha(z), F(y)] = -[\alpha(z), F(y)]\alpha(x)d(z)\alpha(t)d(z), \\ &\forall x, y, z, t \in I. \end{aligned} \tag{2.7}$$

Right multiplying (2.6) by $\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)]$, we get

$$\begin{aligned} &d(z)\alpha(x)[\alpha(z), F(y)]\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)] \\ &= -[\alpha(z), F(y)]\alpha(x)d(z)\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)]. \end{aligned} \tag{2.8}$$

Using (2.7), it yields that

$$\begin{aligned} &d(z)\alpha(x)[\alpha(z), F(y)]\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)] \\ &= d(z)\alpha(x)d(z)\alpha(t)[\alpha(z), F(y)]\alpha(x)[\alpha(z), F(y)]. \end{aligned} \tag{2.9}$$

Using (2.5), (2.9) reduces to

$$\begin{aligned} &d(z)\alpha(x)[\alpha(z), F(y)]\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)] \\ &= -d(z)\alpha(x)[\alpha(z), F(y)]\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)]. \end{aligned}$$

That is

$$2d(z)\alpha(x)[\alpha(z), F(y)]\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)] = 0, \text{ for all } x, y, z, t \in I.$$

Since R is $2-$torsion free semiprime ring, we get

$$d(z)\alpha(x)[\alpha(z), F(y)]\alpha(t)d(z)\alpha(x)[\alpha(z), F(y)] = 0, \text{ for all } x, y, z, t \in I.$$

Replacing t with $rt, r \in R$ in this equation and left multiplying with $\alpha(t)$ gives that

$$\alpha(t)\, d(z)\alpha(x)\, [\alpha(z)\, , F(y)] R\alpha(t)\, d(z)\alpha(x)\, [\alpha(z)\, , F(y)] = (0), \text{ for all } x, y, z, t \in I, r \in R.$$

Since R is semiprime ring, we have

$$\alpha(t)\, d(z)\alpha(x)\, [\alpha(z)\, , F(y)] = 0$$

and so

$$Vd(z)V[\alpha(z)\, , F(y)] = (0), \text{ for all } y, z \in I.$$

where $\alpha(I) = V$ is a nonzero left ideal of R.

Let $\{P_\alpha | \alpha \in I\}$ be a family of prime ideals of R such that $\cap P_\alpha = (0)$. We can say

$$Vd(z) \subseteq P_\alpha \text{ or } V[\alpha(z)\, , F(y)] \subseteq P_\alpha$$

and so

$$[\alpha(z)\, , F(y)]Vd(z) \subseteq P_\alpha \text{ or } d(z)V[\alpha(z)\, , F(y)] \subseteq P_\alpha.$$

By (2.6), $[\alpha(z)\, , F(y)]Vd(z) \subseteq P_\alpha$ implies that $d(z)V[\alpha(z)\, , F(y)] \subseteq P_\alpha$ and so,

$$d(z)V[\alpha(z)\, , F(y)] \subseteq \cap P_\alpha.$$

That is

$$d(z)V[\alpha(z)\, , F(y)] = (0), \text{ for all } y, z \in I.$$

Hence we have $d(z)\alpha(x)\, [\alpha(z)\, , F(y)] = 0$ for all $x, y, z \in I$. Replacing y by yz in this equation and using this, we get

$$d(z)\alpha(x)\, [\alpha(z)\, , \alpha(y)\, d(z)] = 0, \text{ for all } x, y, z \in I. \tag{2.10}$$

Left multiplying with $\alpha(zy)$ this equation, we have

$$\alpha(z)\alpha(y)\, d(z)\alpha(x)\, [\alpha(z)\, , \alpha(y)\, d(z)] = 0, \text{ for all } x, y, z \in I. \tag{2.11}$$

Replacing x by zx in (2.10) and left multiplying with $\alpha(y)$, we obtain that

$$\alpha(y)d(z)\alpha(z)\alpha(x)\, [\alpha(z)\, , \alpha(y)\, d(z)] = 0, \text{ for all } x, y, z \in I. \tag{2.12}$$

Subtracting (2.11) from (2.12), we find that

$$[\alpha(z)\, , \alpha(y)\, d(z)]\alpha(x)\, [\alpha(z)\, , \alpha(y)\, d(z)] = 0, \text{ for all } x, y, z \in I$$

and so

$$\alpha(x)\, [\alpha(z)\, , \alpha(y)\, d(z)]\alpha(r)\alpha(x)\, [\alpha(z)\, , \alpha(y)\, d(z)] = 0, \text{ for all } x, y, z \in I, r \in R.$$

Since R is a semiprime ring, it follows that $\alpha(x)\, [\alpha(z)\, , \alpha(y)\, d(z)] = 0$, for all $x, y, z \in I$. Replacing y with $\alpha^{-1}(d(z))\, y$, we have

$$\alpha(x)\, [\alpha(z)\, , d(z)\alpha(y)\, d(z)] = 0. \tag{2.13}$$

Replacing y by $y\alpha^{-1}(d(z))\,u$ in (2.13) and using this, we obtain that

$$\alpha(x)\,d(z)\alpha(y)\,[d(z),\alpha(z)]\alpha(u)\,d(z)=0,\ \text{for all } x,y,z,u\in I.$$

This implies that

$$\alpha(x)\,[d(z),\alpha(z)]\alpha(y)\,[d(z),\alpha(z)]\alpha(u)\,[d(z),\alpha(z)]=0,\ \text{for all } x,y,z,u\in I.$$

That is $(V[d(z),\alpha(z)])^3=(0)$, for all $z\in I$ where $\alpha(I)=V$ is a nonzero left ideal of R. Since a semiprime ring contains no nonzero nilpotent left ideals, it follows that

$$V[d(z),\alpha(z)]=(0)$$

and so

$$\alpha(I)[d(z),\alpha(z)]=(0),\ \text{for all } z\in I.$$

The proof is completed. □

Theorem 2.2. *Let R be a $2-$torsion free semiprime ring, I a nonzero left ideal of R, α an automorphism on R and $F:R\to R$ a multiplicative generalized $(\alpha,\alpha)-$derivation of R associated with a multiplicative $(\alpha,\alpha)-$derivation d.*

If $[d(x),F(y)]=\pm\alpha(xoy)$ for all $x,y\in I$, then $\alpha(I)\,[d(x),\alpha(x)]=(0)$ for all $x\in I$.

Proof. We assume that

$$[d(x),F(y)]=\pm\alpha(xoy)\,,\ \text{for all } x,y\in I. \tag{2.14}$$

Replacing x by xz in (2.14) and using this equation, we get

$$d(x)[\alpha(z)\,,F(y)]+\alpha(x)\,[d(z),F(y)]+[\alpha(x)\,,F(y)]d(z)=\pm\alpha(x[z,y])\,. \tag{2.15}$$

Writing zx by x in (2.15), we find that

$$\begin{aligned}&d(z)\alpha(x)[\alpha(z)\,,F(y)]+\alpha(z)\,d(x)[\alpha(z)\,,F(y)]+\alpha(zx)\,[d(z),F(y)]\\&+\alpha(z)\,[\alpha(x)\,,F(y)]d(z)+[\alpha(z)\,,F(y)]\alpha(x)\,d(z)=\pm\alpha(zx[z,y])\,.\end{aligned} \tag{2.16}$$

Left multiplication of (2.15) by $\alpha(z)$ yields that

$$\begin{aligned}&\alpha(z)\,d(x)[\alpha(z)\,,F(y)]+\alpha(z)\,\alpha(x)\,[d(z),F(y)]\\&+\alpha(z)\,[\alpha(x)\,,F(y)]d(z)=\pm\alpha(z)\,\alpha(x[z,y])\,.\end{aligned} \tag{2.17}$$

Subtracting (2.17) from (2.16), we have

$$d(z)\alpha(x)[\alpha(z)\,,F(y)]+[\alpha(z)\,,F(y)]\alpha(x)\,d(z)=0,\ \text{for all } x,y,z\in I. \tag{2.18}$$

The last expression is the same as the relation (2.5). Using the similar arguments as used in the Theorem 2.1, we get the required result. □

Similarly, following theorem is straightforward.

Theorem 2.3. *Let R be a $2-$torsion free semiprime ring, I a nonzero left ideal of R, α an automorphism on R and $F : R \to R$ a multiplicative generalized $(\alpha,\alpha)-$derivation of R associated with a multiplicative $(\alpha,\alpha)-$derivation d.*

If $[d(x), F(y)] = 0$ for all $x, y \in I$, then $\alpha(I)[d(x), \alpha(x)] = (0)$ for all $x \in I$.

Theorem 2.4. *Let R be a $2-$torsion free semiprime ring, I a nonzero left ideal of R, α an automorphism on R and $F : R \to R$ a multiplicative generalized $(\alpha,\alpha)-$derivation of R associated with a multiplicative $(\alpha,\alpha)-$derivation d.*

If $g : R \to R$ is a multiplicative derivation of R such that $F([x, y]) \pm [g(x), g(y)] \pm \alpha([x, y]) = 0$ for all $x, y \in I$, then $\alpha(I)[g(x), \alpha(x)] = (0)$ and $\alpha(I)[d(x), \alpha(x)] = (0)$ for all $x \in I$.

Proof. By the hypothesis, we have

$$F([x, y]) \pm [g(x), g(y)] \pm \alpha([x, y]) = 0, \text{ for all } x, y \in I. \tag{2.19}$$

Replacing yx instead of y in (2.19), we get

$$\begin{aligned} &F([x, y])\alpha(x) + \alpha([x, y])\, d(x) \\ &+[g(x), g(y)\alpha(x)] + [g(x), \alpha(y)\, g(x)] + \alpha([x, y]x) = 0. \end{aligned} \tag{2.20}$$

Right multiplying (2.19) by $\alpha(x)$, we obtain

$$F([x, y])\alpha(x) \pm [g(x), g(y)]\alpha(x) \pm \alpha([x, y])\,\alpha(x) = 0, \text{ for all } x, y \in I. \tag{2.21}$$

Now subtracting (2.21) from (2.20), for all $x, y \in I$, we arrive at

$$\alpha([x, y])\, d(x) + g(y)[g(x), \alpha(x)] + [g(x), \alpha(y)\, g(x)] = 0. \tag{2.22}$$

Substituting xy instead of y in (2.22), we obtain

$$\begin{aligned} &\alpha(x)\,\alpha([x, y])\, d(x) + g(x)\alpha(y)\,[g(x), \alpha(x)] \\ &+\alpha(x)\, g(y)[g(x), \alpha(x)] + \alpha(x)\,[g(x), \alpha(y)\, g(x)] \\ &+[g(x), \alpha(x)]\alpha(y)\, g(x) = 0. \end{aligned} \tag{2.23}$$

Left multiplying (2.22) by $\alpha(x)$ and then subtracting from (2.23), we find that

$$g(x)\alpha(y)\,[g(x), \alpha(x)] + [g(x), \alpha(x)]\alpha(y)\, g(x) = 0$$

and so

$$g(x)\alpha(y)\,[g(x), \alpha(x)] = -[g(x), \alpha(x)]\alpha(y)\, g(x), \text{ for all } x, y \in I. \tag{2.24}$$

Replacing y with $y\alpha^{-1}(g(x))t$ in this equation, we have

$$g(x)\alpha(y)\, g(x)\,\alpha(t)\,[g(x), \alpha(x)] = -[g(x), \alpha(x)]\alpha(y)\, g(x)\,\alpha(t)\, g(x). \tag{2.25}$$

Now right multiplying (2.24) by $\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)]$, for all $x,y,t\in I$, we get

$$g(x)\alpha(y)\,[g(x),\alpha(x)]\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)] \\ = -[g(x),\alpha(x)]\alpha(y)\,g(x)\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)]. \tag{2.26}$$

Using (2.25), this equation gives that

$$g(x)\alpha(y)\,[g(x),\alpha(x)]\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)] \\ = g(x)\alpha(y)\,g(x)\,\alpha(t)\,[g(x),\alpha(x)]\alpha(y)\,[g(x),\alpha(x)]. \tag{2.27}$$

Again using (2.24), it reduces to

$$g(x)\alpha(y)\,[g(x),\alpha(x)]\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)] \\ = -g(x)\alpha(y)\,[g(x),\alpha(x)]\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)]. \tag{2.28}$$

That is

$$2g(x)\alpha(y)\,[g(x),\alpha(x)]\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)] = 0, \text{ for all } x,y,t\in I.$$

Since R is $2-$torsion free semiprime ring, we have

$$g(x)\alpha(y)\,[g(x),\alpha(x)]\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)] = 0, \text{ for all } x,y,t\in I.$$

Writing $tr, r\in R$ by t in this equation, we get

$$g(x)\alpha(y)\,[g(x),\alpha(x)]\alpha(t)\,\alpha(r)g(x)\alpha(y)\,[g(x),\alpha(x)] = 0.$$

This implies that

$$\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)]R\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)] = (0).$$

By the semiprimeness of R, we get

$$\alpha(t)\,g(x)\alpha(y)\,[g(x),\alpha(x)] = 0$$

and so

$$\alpha(y)\,[g(x),\alpha(x)]R\alpha(y)\,[g(x),\alpha(x)] = (0)\,.$$

Since R is semiprime ring, we arrive at

$$\alpha(I)\,[g(x),\alpha(x)] = (0), \text{ for all } x\in I. \tag{2.29}$$

Now, replacing y with $ry, r\in R$ in (2.22) and using (2.29), we obtain

$$\alpha(r)\,\alpha([x,y])\,d(x) + \alpha([x,r]y)\,d(x) \\ +\alpha(r)\,g(y)[g(x),\alpha(x)] + \alpha(r)\,[g(x),\alpha(y)\,g(x)] \\ +[g(x),\alpha(r)]\alpha(y)\,g(x) = 0. \tag{2.30}$$

Left multiplying (2.22) by $\alpha(r)$, we get

$$\alpha(r)\,\alpha([x,y])\,d(x) + \alpha(r)\,g(y)[g(x),\alpha(x)] + \alpha(r)\,[g(x),\alpha(y)\,g(x)] = 0.$$

Subtracting this equation from (2.30), we arrive at

$$\alpha\left([x,r]y\right)d(x)+[g(x),\alpha\left(r\right)]\alpha\left(y\right)g(x)=0,\ \text{for all } x,y,\in I,r\in R. \tag{2.31}$$

Replacing yx by y in (2.31), we get

$$\alpha\left([x,r]yx\right)d(x)+[g(x),\alpha\left(r\right)]\alpha\left(yx\right)g(x)=0,\ \text{for all } x,y,\in I,r\in R. \tag{2.32}$$

Right multiplying (2.31) by $\alpha\left(x\right)$ and subtracting from (2.32), we obtain

$$\alpha\left([x,r]y\right)[d(x),\alpha(x)]+[g(x),\alpha\left(r\right)]\alpha(y)[g(x),\alpha(x)]=0.$$

Using $\alpha\left(I\right)[g(x),\alpha\left(x\right)]=(0)$ in this equation, we find that

$$\alpha\left([x,r]y\right)[d(x),\alpha\left(x\right)]=0.$$

By (2.31), we get

$$[\alpha\left(x\right),r]\alpha(y)\left[d(x),\alpha\left(x\right)\right]=0,\ \text{for all } x,y\in I,r\in R.$$

In particular, $[d(x),\alpha\left(x\right)]\alpha\left(y\right)[d(x),\alpha\left(x\right)]=0$,
and so

$$\alpha\left(y\right)[d(x),\alpha\left(x\right)]R\alpha\left(y\right)[d(x),\alpha\left(x\right)]=(0),\ \text{for all } x,y\in I.$$

By the semiprimeness of R yields that $\alpha\left(I\right)[d(x),\alpha\left(x\right)]=(0)$ for all $x\in I$. This completes the proof. □

Theorem 2.5. *Let R be a $2-$torsion free semiprime ring, I a nonzero left ideal of R, α an automorphism on R and $F:R\to R$ a multiplicative generalized $(\alpha,\alpha)-$derivation of R associated with a multiplicative $(\alpha,\alpha)-$derivation d.*

If $g:R\to R$ is a multiplicative derivation of R such that $F(xoy)\pm g(x)og(y)\pm\alpha\left(xoy\right)=0$ for all $x,y\in I$, then $\alpha\left(I\right)[g(x),\alpha\left(x\right)]=(0)$ and $\alpha\left(I\right)[d(x),\alpha\left(x\right)]=(0)$ for all $x\in I$.

Proof. By our hypothesis, we have

$$F(xoy)\pm g(x)og(y)\pm\alpha\left(xoy\right)=0,\ \text{for all } x,y\in I. \tag{2.33}$$

Replacing yx by y in (2.33), we find that

$$F(xoy)\alpha\left(x\right)+\alpha\left(xoy\right)d\left(y\right)+g(x)o\left(g(y)\alpha\left(x\right)+\alpha\left(y\right)g\left(x\right)\right)\pm\alpha\left(xoy\right)\alpha\left(x\right)=0$$

and so

$$\begin{aligned}&F(xoy)\alpha\left(x\right)+\alpha\left(xoy\right)d\left(x\right)+\left(g(x)og(y)\right)\alpha\left(x\right)\\&-g\left(y\right)\left[g\left(x\right),\alpha\left(x\right)\right]+\left(g(x)o\alpha(y)\right)g\left(x\right)+\alpha\left(xoy\right)\alpha\left(x\right)=0.\end{aligned} \tag{2.34}$$

Right multiplying (2.33) by $\alpha\left(x\right)$ and subtracting from (2.34), for all $x,y\in I$, we get

$$\alpha\left(xoy\right)d\left(x\right)-g\left(y\right)\left[g\left(x\right),\alpha\left(x\right)\right]+\left(g(x)o\alpha(y)\right)g\left(x\right)=0. \tag{2.35}$$

Substituting xy instead of y in (2.35), we obtain

$$\begin{aligned}\alpha(x)\alpha\,(xoy)\,d\,(x)-\alpha\,(x)\,g\,(y)\,[g\,(x)\,,\alpha\,(x)]-g\,(x)\,\alpha(y)[g\,(x)\,,\alpha\,(x)]\\+\alpha\,(x)\,(g(x)o\alpha(y))\,g\,(x)-[g\,(x)\,,\alpha\,(x)]\alpha\,(y)\,g\,(x)=0.\end{aligned} \tag{2.36}$$

Left multiplying (2.35) by y and subtracting from (2.36), we have

$$g(x)\alpha\,(y)\,[g(x),\alpha\,(x)]+[g(x),\alpha\,(x)]\alpha\,(y)\,g(x)=0$$

and so

$$g(x)\alpha\,(y)\,[g(x),\alpha\,(x)]=-[g(x),\alpha\,(x)]\alpha\,(y)\,g(x),\ \text{for all } x,y\in I.$$

This equation is same as the relation (2.24). Using the similar arguments, we get the required result. □

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