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FRENET CURVES IN 3-DIMENSIONAL CONTACT LORENTZIAN MANIFOLDS

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Abstract. In this paper, we give some characterizations of Frenet curves in 3-dimensional contact Lorentzian Manifolds. We define Frenet equations and the Frenet elements of these curves. We also obtain the curvatures of non-geodesic Frenet curves on 3-dimensional contact Lorentzian Manifolds. Finally we give some corollaries and examples for these curves.
Keywords: Lorentzian Manifolds, Frenet equations, Frenet curves.

1. Introduction

The differential geometry of curves in manifolds has been investigated by several authors. Especially the curves in contact and para-contact manifolds drew attention have been and studied by many authors. Olszak B. [13], derived certain necessary and sufficient conditions for an a.c.m structure on M to be normal and pointed out some of their consequences. Olszak completely characterized the local nature of normal a.c.m. structures on M by giving suitable examples. Olszak proved that any contact metric manifold of constant sectional curvature and of dimension ≥ 5 has the sectional curvature equal to 1 and is a Sasakian manifold. Moreover Olszak gave some restrictions on the scalar curvature in contact metric manifolds which are conformally flat or of constant ϕ-sectional curvature in [12].

Welyczko [15], generalized some of the results for Legendre curves to the case of 3-dimensional normal almost contact metric manifolds, especially, quasi-Sasakian

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manifolds. Welyczko [14], studied the curvature and torsion of slant Frenet curves in 3-dimensional normal almost paracontact metric manifolds.

Acet and Perktas [1] obtained curvature and torsion of Legendre curves in 3-dimensional (ε, δ) trans-Sasakian manifolds. Ji-Eun Lee defined Lorentzian cross product in a three-dimensional almost contact Lorentzian manifold. By using a Lorentzian cross product Ji-Eun Lee proved that the ratio of κ and $\tau - 1$ is constant along a Frenet slant curve in a Sasakian Lorentzian three-manifold. Moreover, Ji-Eun Lee proved that γ is a slant curve if and only if M is Sasakian for a contact magnetic curve γ in contact Lorentzian three-manifold M in [9]. Ji-Eun Lee also gave the properties of the generalized Tanaka-Webster connection in a contact Lorentzian manifold in [10].

Yildirim A. [16] obtained the curvatures of non-geodesic Frenet curves on three dimensional normal almost contact manifolds without neglecting α and β, and provided the results of their characterization.

U. C. De and K. De [18] studied the Trans-Sasakian structure on a manifold with Lorentzian metric and conformally flat Lorentzian Trans-Sasakian manifolds have been studied.

In this framework, the paper is organized in the following way. Section 2 with three subsections, we give basic definitions and propositions of a contact Lorentzian manifold. In the second subsection we give the properties of Lorentzian cross product. We give the Frenet-Serret equations of a curve in Lorentzian 3-manifold in the last subsection of this section.

We give finally the Frenet elements of a Frenet curve in 3-dimensional contact Lorentzian manifold and give theorems, corollaries and examples for these curves in the third section.

2. Preliminaries

2.1. Contact Lorentzian Manifolds

An almost contact structure (φ, ξ, η) on a (2n+1)-dimensional differentiable manifold $\bar{N}$ consists of a tensor field φ of (1,1), a global vector field ξ and a 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\varphi(\xi) = 0, \quad \eta \circ \varphi = 0. \tag{2.2}$$

If a (2n+1)-dimensional manifold $\bar{N}$ with almost contact structure (φ, ξ, η) admits a compatible Lorentzian metric such that

$$\bar{g}(\varphi X, \varphi Y) = \bar{g}(X, Y) + \eta(X)\eta(Y) \tag{2.3}$$

then we say that $\bar{N}$ has an almost contact Lorentzian structure $(\varphi, \xi, \eta, \bar{g})$. Setting $Y = \xi$, we have

$$\eta(X) = -\bar{g}(X, \xi). \tag{2.4}$$

Next, if the compatible Lorentzian metric $\bar{g}$ satisfies

$$d\eta(X,Y) = \bar{g}(X, \varphi(Y)), \tag{2.5}$$

then η is a contact form on $\bar{N}$, ξ is the associated Reeb vector field, $\bar{g}$ is an associated metric and $(\bar{N}, \varphi, \xi, \eta, \bar{g})$ is called a contact Lorentzian manifold. [9] For a contact Lorentzian manifold $\bar{N}$, one may naturally define an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}), \tag{2.6}$$

where X is a vector field tangent to $\bar{N}$, t is the coordinate of $\mathbb{R}$ and f is a function on $\bar{N} \times \mathbb{R}$. If the almost complex structure J is integrable, then the contact Lorentzian manifold $\bar{N}$ is called normal or Sasakian. It is known that a contact Lorentzian manifold $\bar{N}$ is normal if and only if $\bar{N}$ satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0, \tag{2.7}$$

where $[\varphi, \varphi]$ is the Nijenhius torsion of φ [5].

Proposition 2.1. [3, 4] *An almost contact Lorentzian manifold* $(\bar{N}^{2n+1}, \eta, \xi, \varphi, \bar{g})$ *is Sasakian if and only if*

$$(\nabla_X \varphi)Y = \bar{g}(X,Y)\xi + \eta(Y)X. \tag{2.8}$$

Using similar arguments and computations to those of [17] we obtain

Proposition 2.2. [3, 4] *Let* $(\bar{N}^{2n+1}, \eta, \xi, \varphi, \bar{g})$ *be a contact Lorentzian manifold. Then*

$$\nabla_X \xi = \varphi X - \varphi h X, \tag{2.9}$$

where $h = \frac{1}{2} L_\xi \varphi$.

If ξ is Killing vector field with respect to the Lorentzian metric $\bar{g}$, that is, $\bar{N}^{2n+1}$ is a K-contact Lorentzian manifold, then

$$\nabla_X \xi = \varphi X. \tag{2.10}$$

Proposition 2.3. [3] *Let* $\{t, n, b\}$ *be an orthonormal frame field in a Lorentzian three-manifold. Then*

$$t \wedge_L n = \varepsilon_3 b, \quad n \wedge_L b = \varepsilon_1 t, \quad b \wedge_L t = \varepsilon_2 n. \tag{2.11}$$

2.2. Lorentzian Cross Product

In [17] Camci defined a cross product in three-dimensional almost contact Riemannian manifolds $(\bar{N}, \eta, \xi, \varphi, \bar{g})$ as following

$$U \wedge V = -\bar{g}(U, \varphi V)\xi - \eta(V)\varphi U + \eta(U)\varphi V. \tag{2.12}$$

If we define the cross product $\wedge$ as in equation (2.12) in three-dimensional almost contact Lorentzian manifold $(\bar{N}, \eta, \xi, \varphi, \bar{g})$, then

$$\bar{g}(U \wedge V, U) = 2\eta(U)\bar{g}(U, \varphi V) \neq 0. \tag{2.13}$$

Proposition 2.4. [9] *Let $\{w_1, w_2, w_3\}$ be an orthonormal frame field in a Lorentzian three-manifold. Then*

$$w_1 \wedge_L w_2 = \varepsilon_3 w_3, \quad w_2 \wedge_L w_3 = \varepsilon_1 w_1, \quad w_3 \wedge_L w_1 = \varepsilon_2 w_2. \tag{2.14}$$

Now, in three-dimensional almost contact Lorentzian manifold $\bar{N}^3$, Lorentzian cross product is defined as follows:

Definition 2.1. Let $(\bar{N}^3, \varphi, \xi, \eta, \bar{g})$ be a three-dimensional almost contact Lorentzian manifold. We define a Lorentzian cross product $\wedge_L$ by

$$U \wedge_L V = \bar{g}(U, \varphi V)\xi - \eta(V)\varphi U + \eta(U)\varphi V, \tag{2.15}$$

where $U, V \in T\bar{N}$ [9].

The Lorentzian cross product $\wedge_L$ has the following properties:

Proposition 2.5. [9] *Let $(\bar{N}^3, \varphi, \xi, \eta, \bar{g})$ be a three-dimensional almost contact Lorentzian manifold. Then, for all $U, V, W \in T\bar{N}$ the Lorentzian cross product has the following properties:*

(i) The Lorentzian cross product is bilinear and skew-symmetric

(ii) $U \wedge_L V$ is perpendicular both to U and V

(iii) $U \wedge_L \varphi V = -\bar{g}(U, V)\xi - \eta(U)V$

(iv) $\varphi U = \xi \wedge_L U$

(v) Define a mixed product by $det(U, V, W) = \bar{g}(U \wedge_L V, W)$. Then

$$det(U, V, W) = -\bar{g}(U, \varphi V)\eta(W) - \bar{g}(V, \varphi W)\eta(U) - \bar{g}(W, \varphi U)\eta(V) \tag{2.16}$$

and

$$det(U, V, W) = det(V, W, U) = det(W, U, V) \tag{2.17}$$

(vi) $\bar{g}(U, \varphi V)W + \bar{g}(V, \varphi W)U + \bar{g}(W, \varphi U)V = -(U, V, W)\xi$.

Proposition 2.6. *Let $(\bar{N}^3, \varphi, \xi, \eta, \bar{g})$ be a three-dimensional Sasakian Lorentzian manifold. Then we have*

$$\nabla_W(U \wedge_L V) = (\nabla_W U) \wedge_L V + U \wedge_L (\nabla_W V), \tag{2.18}$$

for all $U, V, W \in T\bar{N}$ [9].

2.3. Frenet Curves

Let $\zeta : I \to \bar{N}$ be a unit speed curve in Lorentzain 3-manifold $\bar{N}$ such that $\zeta^{'}$ satisfies $\bar{g}(\zeta^{'}, \zeta^{'}) = \varepsilon_1 = \mp 1$. The constant ε_1 is called the casual character of ζ. The constants ε_2 and ε_3 defined by $\bar{g}(n, n) = \varepsilon_2$ and $\bar{g}(b, b) = \varepsilon_3$ are called the second casual character and third casual character of ζ, respectively. Thus we $\varepsilon_1\varepsilon_2 = -\varepsilon_3$.
A unit speed curve ζ is said to be a spacelike or timelike if its casual character is 1 or -1, respectively. A unit speed curve ζ is said to be a Frenet curve if $\bar{g}(\zeta^{'}, \zeta^{'}) \neq 0$. A Frenet curve ζ admits an orthonormal frame field $\{t = \zeta^{'}, n, b\}$ along ζ. Then the Frenet-Serret equations given as follows:

$$\begin{aligned} \nabla_{\zeta'} t &= \varepsilon_2 \kappa n \\ \nabla_{\zeta'} n &= -\varepsilon_1 \kappa t - \varepsilon_3 \tau b \\ \nabla_{\zeta'} b &= \varepsilon_2 \tau n \end{aligned} \tag{2.19}$$

where $\kappa = |\nabla_{\zeta'} \zeta^{'}|$ is the geodesic curvature of ζ and τ is geodesic torsion [9]. The vector fields t, n and b are called the tangent vector field, the principal normal vector field and the binormal vector field of ζ, respectively.

A Frenet curve ζ is a geodesic if and only if $\kappa = 0$. A Frenet curve ζ with constant geodesic curvature and zero geodesic torsion is called a pseudo-circle. A pseudo-helix is a Frenet curve ζ whose geodesic curvature and torsion are constant.

A curve in a contact Lorentzian three-manifold is said to be slant if its tangent vector field has constant angle with the Reeb vector field, i.e. $\eta(\zeta') = -\bar{g}(\zeta', \xi) = cos\theta = constant$. If $\eta(\zeta') = -\bar{g}(\zeta', \xi) = 0$, then the curve ζ is called a Legendre curve [9].

3. Main Results

In this section we consider a 3-dimensional contact Lorentzian manifold $\bar{N}$. Let $\zeta : I \to \bar{N}$ be a non-geodesic ($\kappa \neq 0$) Frenet curve given with the arc-parameter s and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{N}$. From the basis $(\zeta^{'}, \varphi\zeta^{'}, \xi)$ we obtain an orthonormal basis $\{e_1, e_2, e_3\}$ defined by

$$\begin{aligned} e_1 &= \zeta^{'}, \\ e_2 &= \frac{\varepsilon_2 \varphi \zeta^{'}}{\sqrt{\varepsilon_1 + \rho^2}}, \\ e_3 &= \frac{-\varepsilon_3 \xi - \varepsilon_2 \rho \zeta^{'}}{\sqrt{\varepsilon_1 + \rho^2}} \end{aligned} \tag{3.1}$$

where

$$\eta(\zeta^{'}) = -\bar{g}(\zeta^{'}, \xi) = -\rho. \tag{3.2}$$

Then if we write the covariant differentiation of $\zeta^{'}$ as

$$\bar{\nabla}_{\zeta'} e_1 = \nu e_2 + \mu e_3 \tag{3.3}$$

such that

$$\nu = \bar{g}(\bar{\nabla}_{\zeta'} e_1, e_2) \tag{3.4}$$

is a function. Moreover we obtain ν by

$$\mu = \bar{g}(\bar{\nabla}_{\zeta'} e_1, e_3) = -\frac{\varepsilon_3 \rho'}{\sqrt{\varepsilon_1 + \rho^2}} + \varepsilon_3 \sqrt{\varepsilon_1 + \rho^2}, \tag{3.5}$$

where $\rho'(s) = \frac{d\rho(\zeta(s))}{ds}$. Then we find

$$\bar{\nabla}_{\zeta'} e_2 = -\nu e_1 + \left(\frac{\varepsilon_2 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} \right) e_3 \tag{3.6}$$

and

$$\bar{\nabla}_{\zeta'} e_3 = -\mu e_1 - \left(\frac{\varepsilon_2 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} \right) e_2. \tag{3.7}$$

The fundamental forms of the tangent vector ζ' on the basis of the equation (3.1) is

$$[\omega_{ij}(\zeta')] = \begin{pmatrix} 0 & \nu & \mu \\ -\nu & 0 & \frac{\varepsilon_2 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} \\ -\mu & -\frac{\varepsilon_2 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} & 0 \end{pmatrix} \tag{3.8}$$

and the Darboux vector connected to the vector ζ' is

$$\omega(\zeta') = \left(-\frac{\rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} \right) e_1 - \mu e_2 + \nu e_3. \tag{3.9}$$

So we can write

$$\bar{\nabla}_{\zeta'} e_i = \omega(\zeta') \wedge \varepsilon_i e_i \quad (1 \leq i \leq 3). \tag{3.10}$$

Furthermore, for any vector field $Z = \sum_{i=1}^{3} \theta^i e_i \in T\bar{N}$ strictly dependent on the curve ζ on $\bar{N}$, there exists the following equation

$$\bar{\nabla}_{\zeta'} Z = \omega(\zeta') \wedge Z + \sum_{i=1}^{3} \varepsilon_i e_i [\theta^i] e_i. \tag{3.11}$$

3.1. Frenet Elements of ζ

Let $\zeta : I \to \bar{N}$ be a non-geodesic ($\kappa \neq 0$) Frenet curve given with the arc parameter s and the elements $\{t, n, b, \kappa, \tau\}$. The Frenet elements of this curve are calculated as follows.

If we consider the equation (3.3), then we get

$$\varepsilon_2 \kappa n = \bar{\nabla}_{\zeta'} e_1 = \nu e_2 + \mu e_3. \tag{3.12}$$

If we consider (3.5) and (3.12) we find

$$\kappa = \sqrt{\nu^2 + \left(\frac{-\varepsilon_3\rho'}{\sqrt{\varepsilon_1+\rho^2}} + \varepsilon_3\sqrt{\varepsilon_1+\rho^2}\right)^2}. \tag{3.13}$$

On the other hand

$$\begin{aligned} \bar{\nabla}_{\zeta'} n &= \left(\frac{\nu}{\varepsilon_2\kappa}\right)' e_2 + \frac{\nu}{\varepsilon_2\kappa}\nabla_{\zeta'} e_2 + \left(\frac{\mu}{\varepsilon_2\kappa}\right)' e_3 + \frac{\mu}{\varepsilon_2\kappa}\nabla_{\zeta'} e_3 \\ &= -\varepsilon_1\kappa t - \varepsilon_3\tau B \end{aligned} \tag{3.14}$$

By means of the equation (3.6) and (3.7) we find

$$\begin{aligned} -\varepsilon_3\tau B &= \left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)' - \frac{\mu}{\varepsilon_2\kappa}\left(\frac{\rho\nu}{\sqrt{\varepsilon_1+\rho^2}}\right)\right] e_2 \\ &+ \left[\left(\frac{\mu}{\varepsilon_2\kappa}\right)' + \frac{\nu}{\varepsilon_2\kappa}\left(\frac{\rho\nu}{\sqrt{\varepsilon_1+\rho^2}}\right)\right] e_3. \end{aligned} \tag{3.15}$$

By a direct computation we find following

$$\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)'\right]^2 + \left[\left(\frac{\mu}{\varepsilon_2\kappa}\right)'\right]^2 = \left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)'\frac{\mu}{\varepsilon_2\kappa} - \frac{\nu}{\varepsilon_2\kappa}\left(\frac{\mu}{\varepsilon_2\kappa}\right)'\right]^2 \tag{3.16}$$

Taking the norm of the last equation by using (3.15) and if we consider the equations (3.5) and (3.16) in (3.15) we obtain

$$\tau = \left| -\varepsilon_2\frac{\rho\nu}{\sqrt{\varepsilon_1+\rho^2}} - \sqrt{\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)'\right]^2 + \left[\left(\frac{-\frac{\varepsilon_3\rho'}{\sqrt{\varepsilon_1+\rho^2}} + \varepsilon_3\sqrt{\varepsilon_1+\rho^2}}{\kappa}\right)'\right]^2}\right|. \tag{3.17}$$

Moreover we can write the Frenet vector fields of ζ as in the following theorem

Theorem 3.1. *Let $\bar{N}$ be a 3-dimensional contact Lorentzian manifold and ζ be a Frenet curve on $\bar{N}$. The Frenet vector fields t, n and b are of the form of*

$$\begin{aligned} t &= \zeta' = e_1, \\ n &= \frac{\nu}{\varepsilon_2\kappa}e_2 + \frac{\mu}{\varepsilon_2\kappa}e_3, \\ b &= -\frac{1}{\varepsilon_3\tau}\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)' - \frac{\mu}{\varepsilon_2\kappa}\left(\frac{\rho\nu}{\sqrt{\varepsilon_1+\rho^2}}\right)\right] e_2 \\ &- \frac{1}{\varepsilon_3\tau}\left[\left(\frac{\mu}{\varepsilon_2\kappa}\right)' + \frac{\nu}{\varepsilon_2\kappa}\left(\frac{\rho\nu}{\sqrt{\varepsilon_1+\rho^2}}\right)\right] e_3. \end{aligned} \tag{3.18}$$

Note that

$$(3.19)\ \xi = \varepsilon_1 \rho t + \frac{\mu\sqrt{\varepsilon_1+\rho^2}}{\varepsilon_2\kappa} n - \frac{\sqrt{\varepsilon_1+\rho^2}}{\varepsilon_3\tau}\left[\left(\frac{\mu}{\varepsilon_2\kappa}\right)' + \frac{\nu}{\varepsilon_2\kappa}\left(\frac{\rho\nu}{\sqrt{\varepsilon_1+\rho^2}}\right)\right] b$$

Let ζ be a non-geodesic Frenet curve given with the arc-parameter s in a 3-dimensional contact Lorentzian manifold $\bar{N}$. So we can give the following theorems.

Theorem 3.2. *Let $\bar{N}$ be a 3-dimensional contact Lorentzian manifold and ζ be a Frenet curve on $\bar{N}$. ζ is a slant curve ($\rho = \eta(\zeta') = cos\theta = constant$) on $\bar{N}$ if and only if the Frenet elements $\{t, n, b, \kappa, \tau\}$ of this curve ζ are as follows*

$$(3.20)\quad \begin{aligned} t &= e_1 = \zeta', \\ n &= e_2 = \frac{\varepsilon_2 \varphi \zeta'}{\sqrt{\varepsilon_1 + cos^2\theta}}, \\ b &= e_3 = \frac{-\varepsilon_3\xi - \varepsilon_2\rho\zeta'}{\sqrt{\varepsilon_1 + cos^2\theta}}, \\ \kappa &= \sqrt{\varepsilon_1 + \nu^2 + cos^2\theta}, \\ \tau &= \left| \frac{\varepsilon_2 cos\theta\nu}{\sqrt{\varepsilon_1 + cos^2\theta}} - \sqrt{\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)'\right]^2 + \left[\left(\frac{\varepsilon_3\sqrt{\varepsilon_1+\cos^2\theta}}{\varepsilon_2\kappa}\right)'\right]^2} \right|. \end{aligned}$$

Proof. Let the curve ζ be a slant curve in the 3-dimensional contact Lorentzian manifold $\bar{N}$. If we take account the condition $\rho = \eta(\zeta') = cos\theta = constant$ in the equations (3.1), (3.13) and (3.17) we find (3.20). If the equations in (3.20) hold, from the definition of slant curves it is obvious that the curve ζ is a slant curve. □

Corollary 3.1. *Let $\bar{N}$ be a 3-dimensional contact Lorentzian manifold and ζ be a slant curve on $\bar{N}$. If the geodesic curvature κ of the curve ζ is non-zero constant, then the geodesic torsion of ζ is $\tau = \left|\left(\varepsilon_2 \frac{cos\theta\nu}{\sqrt{\varepsilon_1+\delta\cos^2\theta}}\right)\right|$ and ζ is a pseudo-helix on $\bar{N}$.*

Corollary 3.2. *Let $\bar{N}$ be a 3-dimensional contact Lorentzian manifold and ζ be a slant curve on $\bar{N}$. If the geodesic curvature κ of the curve ζ is not constant and the geodesic torsion of ζ is $\tau = 0$ then ζ is a plane curve on $\bar{N}$ and function ν satisfies the equation*

$$(3.21)\qquad \nu = \frac{cos\theta}{\varepsilon_1 + \delta\cos^2\theta}\int \nu(\varepsilon_1 + \nu^2 + cos^2\theta)d\nu.$$

Theorem 3.3. *Let $\bar{N}$ be a 3-dimensional contact Lorentzian manifold and ζ is a Frenet curve on $\bar{N}$. ζ is a Legendre curve($\rho = \eta(\zeta') = 0$) in this manifold if and* *only if the Frenet elements* $\{t, n, b, \kappa, \tau\}$ *of this curve* ζ *are as follows*

$$
\begin{aligned}
t &= e_1 = \zeta', \\
n &= e_2 = \varepsilon_2 \varphi \zeta', \\
b &= e_3 = -\varepsilon_3 \xi, \\
\kappa &= \sqrt{\nu^2 + \varepsilon_1}, \\
\tau &= \left| \sqrt{\left[\left(\frac{\nu}{\varepsilon_2 \kappa} \right)' \right]^2 + \varepsilon_1 \left[\frac{\kappa'}{\kappa^2} \right]^2} \right| .
\end{aligned} \tag{3.22}
$$

Proof. Let the curve ζ be a Legendre curve in 3-dimensional contact Lorentzian manifold $\bar{N}$. If we take account the condition $\rho = \eta(\zeta') = 0$ in the equations (3.1), (3.13) and (3.17) we find (3.22). If the equations in (3.22) hold, from the definition of Legendre curves it is obvious that the curve ζ is a Legendre curve on $\bar{N}$. □

Corollary 3.3. *Let the curve* ζ *is a Legendre curve in* 3*-dimensional contact Lorentzian manifold* $\bar{N}$*. If the geodesic curvature* κ *of the curve* ζ *is non-zero constant, then the geodesic torsion of* ζ *is* $\tau = 0$ *and* ζ *is a plane curve on* $\bar{N}$*.*

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