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A NOTE ON SOME SYSTEMS OF GENERALIZED SYLVESTER EQUATIONS *

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Abstract. In this paper, we study two systems of generalized Sylvester operator equations. We derive necessary and sufficient conditions for the existence of a solution and provide the general form of a solution. We extend some recent resuts to more general settings.
Key words: Sylvester equations, generalized inverses, Matrix equations and identities

1. Introduction

Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ be complex Hilbert spaces and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For a given $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of operator A, respectively. The identity operator is always denoted by I. If $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ has a closed range, then there exists unique operator $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ satisfying the following equations

$$(1)\ AXA = A \quad (2)\ XAX = X \quad (3)\ (AX)^* = AX \quad (4)\ (XA)^* = XA.$$

Such operator is called the Moore-Penrose inverse of an operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ which is denoted by $A^\dagger$. If $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ satisfies the equation (1), i.e. $AXA = A$, then X is an inner generalized inverse of A, and is usually denoted by A^-. For $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ there exists a Moore-Penrose inverse, if and only if there exists its

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inner generalized inverse if and only if $\mathcal{R}(A)$ is closed. In this case, we say that A is regular. Furthermore, L_A and R_A stand for two projections $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$. induced by A, respectively.

In this paper, we study two systems of generalized Sylvester operator equations

$$A_1X_1 - X_2B_1 = C_1, \quad A_2X_3 - X_2B_2 = C_2, \tag{1.1}$$

where $A_1 \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B_1 \in \mathcal{B}(\mathcal{F},\mathcal{G})$, $C_1 \in \mathcal{B}(\mathcal{F},\mathcal{K})$, $A_2 \in \mathcal{B}(\mathcal{M},\mathcal{K})$, $B_2 \in \mathcal{B}(\mathcal{L},\mathcal{G})$, $C_2 \in \mathcal{B}(\mathcal{L},\mathcal{K})$, and

$$A_1X_1 - X_2B_1 = C_1, \quad A_2X_2 - X_3B_2 = C_2, \tag{1.2}$$

where $A_1 \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B_1 \in \mathcal{B}(\mathcal{F},\mathcal{G})$, $C_1 \in \mathcal{B}(\mathcal{F},\mathcal{K})$, $A_2 \in \mathcal{B}(\mathcal{K},\mathcal{M})$, $B_2 \in \mathcal{B}(\mathcal{G},\mathcal{N})$, $C_2 \in \mathcal{B}(\mathcal{G},\mathcal{M})$.

Systems of such type of matrix equations have been considered by many authors [3, 4, 5, 6, 7]. In this pape,r we extended recent results [7] on systems of quaternion matrix equations to infinite dimensional settings and provide much simpler proofs to existing conditions.

2. Main results

The following two lemmas play a key role in this paper:

Lemma 2.1. [1] *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B \in \mathcal{B}(\mathcal{F},\mathcal{G})$ and $C \in \mathcal{B}(\mathcal{F},\mathcal{K})$ be such that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed. Then the operator equation*

$$AXB = C$$

is consistent if and only if

$$AA^-CB^-B = C,$$

for some A^- and B^-, in which case the general solution is given by

$$X = A^-CB^- + Y - A^-AYBB^-,$$

for arbitrary $Y \in \mathcal{B}(\mathcal{G},\mathcal{H})$.

Lemma 2.2. [2] *Let E, F, G, D, N, M be Banach spaces. Let $A_1 \in \mathcal{B}(F,E)$, $A_2 \in \mathcal{B}(F,N)$, $B_1 \in \mathcal{B}(D,G)$, $B_2 \in \mathcal{B}(M,G)$ and*

$$T := (I_G - B_1B_1^-)B_2 \quad \text{and} \quad S := A_2(I_F - A_1^-A_1)$$

be all regular. Moreover, let $A_1A_1^-C_1B_1^-B_1 = C_1$ and $A_2A_2^-C_2B_2^-B_2 = C_2$. Then the equations

$$A_1XB_1 = C_1 \quad \text{and} \quad A_2XB_2 = C_2$$

have a common solution if and only if

$$(I_N - SS^-)C_2(I_M - T^-T) = (I_N - SS^-)A_2A_1^-C_1B_1^-B_2(I_M - T^-T).$$

In this case, the general common solution is given by

$$\begin{aligned} X &= (A_1^- C_1 - (I_F - A_1^- A_1)S^-(A_2 A_1^- C_1 - W))B_1^-(I_G - B_2 T^-(I_G - B_1 B_1^-)) \\ &\quad +((I_F - (I_F - A_1^- A_1)S^- A_2)A_1^- V + (I_F - A_1^- A_1)S^- C_2)T^-(I_G - B_1 B_1^-) \\ &\quad +Z - (A_1^- A_1 + (I_F - A_1^- A_1)S^- S)Z(B_1 B_1^- + TT^-(I_G - B_1 B_1^-)), \end{aligned}$$

where

$$\begin{aligned} V &= C_1 B_1^- B_2(I_M - T^- T) + A_1 A_2^-(I_N - SS^-)C_2 T^- T + A_1 A_1^- Q T^- T \\ &\quad -A_1 A_2^-(I_N - SS^-)A_2 A_1^- Q T^- T, \end{aligned}$$

$$\begin{aligned} W &= (I_N - SS^-)A_2 A_1^- C_1 + SS^- C_2(I_M - T^- T)B_2^- B_1 + SS^- P B_1^- B_1 \\ &\quad -SS^- P B_1^- B_2(I_M - T^- T)B_2^- B_1, \end{aligned}$$

in which P,Q, Z *are arbitrary elements of* $\mathcal{B}(D,N)$, $\mathcal{B}(M,E)$ *and* $\mathcal{B}(G,F)$, *respectively.*

Note that in the preceding lemmas, in the solvability conditions and formulas for general solutions, arbitrary inner generalized inverses can be replaced by the Moore-Penrose inverse. For example, in Lemma 2.1, if

$$AA^- CB^- B = C$$

holds for some A^- and B^-, then

$$AA^\dagger CB^\dagger B = AA^\dagger(AA^- CB^- B)B^\dagger B = AA^- CB^- B = C.$$

Conversely, if

$$AA^\dagger CB^\dagger B = C$$

holds, then for arbitrary A^- and B^- it follows

$$AA^- CB^- B = AA^-(AA^\dagger CB^\dagger B)B^- B = AA^\dagger CB^\dagger B = C.$$

So, for A^- and B^- in the solvability conditions and formulas for general solutions, we can choose exactly $A^\dagger$ and $B^\dagger$, respectively.

Theorem 2.1. *Let* $A_1 \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B_1 \in \mathcal{B}(\mathcal{F},\mathcal{G})$, $C_1 \in \mathcal{B}(\mathcal{F},\mathcal{K})$, $A_2 \in \mathcal{B}(\mathcal{M},\mathcal{K})$, $B_2 \in \mathcal{B}(\mathcal{L},\mathcal{G})$, $C_2 \in \mathcal{B}(\mathcal{L},\mathcal{K})$ *be such that* A_1, A_2, B_1, B_2, S *and* T *are all regular. Put*

$$\begin{aligned} &T = (I - B_1 B_1^\dagger)B_2, \quad S = (I - A_2 A_2^\dagger)A_1 A_1^\dagger, \\ &C = (I - A_2 A_2^\dagger)(C_2 - (I - A_1 A_1^\dagger)C_1 B_1^\dagger B_2)(I - T^\dagger T). \end{aligned}$$

The following statements are equivalent:

(i) *The system* (1.1) *is consistent;*

(ii) $R_{A_1}C_1L_{B_1}=0$, $R_{A_2}C_2L_{B_2}=0$, $R_SC=0$;

(iii) $R_{A_1}C_1L_{B_1}=0$, $C(I-(B_2L_T)^\dagger(B_2L_T))=0$, $R_SC=0$.

In this case, the general solution to the system (1.1) *is given by*

$$X_1=A_1^\dagger S^\dagger(R_{A_1}C_1+W)B_1^\dagger B_1+A_1^\dagger ZB_1-A_1^\dagger S^\dagger SZB_1+A_1^\dagger C_1+L_{A_1}R,$$

$$\begin{aligned}X_2=&\left(-R_{A_1}C_1+S^\dagger(R_{A_1}C_1+W)\right)B_1^\dagger(I-B_2T^\dagger)\\&+\left((I-S^\dagger)R_{A_1}V-S^\dagger C_2\right)T^\dagger+Z-(I-A_1A_1^\dagger+S^\dagger S)Z(B_1B_1^\dagger+TT^\dagger),\end{aligned}$$

$$\begin{aligned}X_3=&A_2^\dagger\left(-R_{A_1}C_1-S^\dagger(R_{A_1}C_1+W)\right)B_1^\dagger B_2L_T\\&+A_2^\dagger\left((I-S^\dagger)R_{A_1}V+S^\dagger C_2\right)T^\dagger B_2\\&+A_2^\dagger ZB_2-A_2^\dagger(I-A_1A_1^\dagger+S^\dagger S)Z(B_1B_1^\dagger B_2+T)+A_2^\dagger C_2+L_{A_2}Y,\end{aligned}$$

where

$$\begin{aligned}V=&-R_{A_1}C_1B_1^\dagger B_2L_T-R_{A_1}R_{A_2}R_SR_{A_2}C_2T^\dagger T\\&+R_{A_1}QT^\dagger T-R_{A_1}R_{A_2}R_SR_{A_2}R_{A_1}QT^\dagger T\end{aligned}$$

and

$$\begin{aligned}W=&-R_SR_{A_2}R_{A_1}C_1-SS^\dagger C_2L_TB_2^\dagger B_1\\&+SS^\dagger PB_1^\dagger B_1-SS^\dagger PB_1^\dagger B_2L_TB_2^\dagger B_1,\end{aligned}$$

where P, Q, R and Y are arbitrary elements of $\mathcal{B}(\mathcal{F},\mathcal{K})$, $\mathcal{B}(\mathcal{G},\mathcal{K})$, $\mathcal{B}(\mathcal{F},\mathcal{H})$ and $\mathcal{B}(\mathcal{L},\mathcal{K})$, respectively.

Proof. *(i)* $\Rightarrow$ *(ii)*: Since the system (1.1) is consistent, there exists $X_2\in\mathcal{B}(\mathcal{G},\mathcal{K})$ such that equations

$$\begin{aligned}A_1X_1-X_2B_1=C_1\\A_2X_3-X_2B_2=C_2\end{aligned}$$

are solvable for X_1 and X_3, respectively. According to Lemma 2.1 equation

$$A_1X_1-X_2B_1=C_1$$

is solvable for X_1 if and only if

$$(I-A_1A_1^\dagger)(C_1+X_2B_2)=0, \tag{2.1}$$

and equation

$$A_2X_3-X_2B_2=C_2$$

is solvable for X_2 if and only if

$$(I - A_2A_2^\dagger)(C_2 + X_2B_2) = 0. \tag{2.2}$$

So, from (2.1) and (2.2) it follows that equations

$$\begin{aligned} (I - A_1A_1^\dagger)X_2B_1 &= -(I - A_1A_1^\dagger)C_1, \\ (I - A_2A_2^\dagger)X_2B_2 &= -(I - A_2A_2^\dagger)C_2 \end{aligned} \tag{2.3}$$

have a common solution. From Lemma 2.1 and Lemma 2.2 system (2.3) is consistent if and only if

$$\begin{aligned} &(I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0, \\ &(I - A_2A_2^\dagger)C_2(I - B_2^\dagger B_2) = 0, \\ &(I - SS^\dagger)C = 0. \end{aligned}$$

$(ii) \Rightarrow (i)$: If (ii) holds, then by Lemma 2.2 it follows that system (2.3) is consistent. Let $X_2 \in \mathcal{B}(G, K)$ be the solution to the system (2.3) and let $X_1 = A_1^\dagger(X_2B_1 + C_1)$ and $X_3 = A_2^\dagger(X_2B_2 + C_2)$. Then it is easy to see that such X_1, X_2 and X_3 satisfy (1.1).

$(ii) \Rightarrow (iii)$: Suppose that

$$(I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0, \tag{2.4}$$

$$(I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0 \tag{2.5}$$

and

$$(I - SS^\dagger)C = 0 \tag{2.6}$$

hold. From (2.6) we get

$$\begin{aligned} & C(I - (B_2L_T)^\dagger(B_2L_T)) \\ = \; & C(I - (B_2(I - T^\dagger T))^\dagger(B_2(I - T^\dagger T))) \\ = \; & (I - A_2A_2^\dagger)C_2(I - T^\dagger T)(I - (B_2(I - T^\dagger T))^\dagger(B_2(I - T^\dagger T))) \\ & -(I - A_2A_2^\dagger)(I - A_1A_1^\dagger)C_1B_1^\dagger B_2(I - T^\dagger T)(I - (B_2(I - T^\dagger T))^\dagger(B_2(I - T^\dagger T))) \\ = \; & (I - A_2A_2^\dagger)C_2(I - T^\dagger T)(I - (B_2(I - T^\dagger T))^\dagger(B_2(I - T^\dagger T))) \\ = \; & (I - A_2A_2^\dagger)C_2B_2^\dagger B_2(I - T^\dagger T)(I - (B_2(I - T^\dagger T))^\dagger(B_2(I - T^\dagger T))) \\ = \; & 0. \end{aligned}$$

$(iii) \Rightarrow (ii)$: Suppose that

$$(I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0, \tag{2.7}$$

$$C(I-(B_2(I-T^\dagger T))^\dagger(B_2(I-T^\dagger T)))=0 \tag{2.8}$$

and

$$(I-SS^\dagger)C=0 \tag{2.9}$$

hold. From (2.8) we get

$$\begin{aligned}
& R_{A_2}C_2(I-T^\dagger T)(I-(B_2(I-T^\dagger T))^\dagger(B_2(I-T^\dagger T))) \\
= \; & R_{A_2}R_{A_1}C_1B_1^\dagger B_2(I-T^\dagger T)L_{B_2(I-T^\dagger T)} \\
= \; & 0.
\end{aligned} \tag{2.10}$$

Note that

$$\begin{aligned}
& (I-T^\dagger T)L_{B_2} \\
= \; & (I-((I-B_1B_1^\dagger)B_2)^\dagger(I-B_1B_1^\dagger)B_2)(I-B_2^\dagger B_2) \\
= \; & I-B_2^\dagger B_2 \\
= \; & L_{B_2},
\end{aligned} \tag{2.11}$$

so from (2.11) and (2.10) we get

$$\begin{aligned}
& R_{A_2}C_2L_{B_2} \\
= \; & R_{A_2}C_2(I-T^\dagger T)L_{B_2} \\
= \; & R_{A_2}C_2(I-T^\dagger T)(B_2(I-T^\dagger T))^\dagger B_2(I-T^\dagger T)L_{B_2} \\
= \; & R_{A_2}C_2(I-T^\dagger T)(B_2(I-T^\dagger T))^\dagger(I-T^\dagger R_{B_1})B_2L_{B_2} \\
= \; & 0.
\end{aligned}$$

Suppose that system (1.1) is consistent.

Since $X_2 \in \mathcal{B}(G,K)$ is a solution to (1.1) if and only if it satisfies (2.3), its general form, according to Lemma 2.2, is given by

$$\begin{aligned}
X_2 = & \left(-R_{A_1}C_1+S^\dagger(R_{A_1}C_1+W)\right)B_1^\dagger(I-B_2T^\dagger) \\
& + \left((I-S^\dagger)R_{A_1}V-S^\dagger C_2\right)T^\dagger \\
& + Z-(I-A_1A_1^\dagger+S^\dagger S)Z(B_1B_1^\dagger+TT^\dagger),
\end{aligned}$$

where Z is an arbitrary element of $\mathcal{B}(\mathcal{G},\mathcal{K})$, and

$$\begin{aligned}
V = & -R_{A_1}C_1B_1^\dagger B_2L_T-R_{A_1}R_{A_2}R_SR_{A_2}C_2T^\dagger T \\
& + R_{A_1}QT^\dagger T-R_{A_1}R_{A_2}R_SR_{A_2}R_{A_1}QT^\dagger T
\end{aligned}$$

and

$$\begin{aligned}
W = & -R_SR_{A_2}R_{A_1}C_1-SS^\dagger C_2L_TB_2^\dagger B_1 \\
& + SS^\dagger PB_1^\dagger B_1-SS^\dagger PB_1^\dagger B_2L_TB_2^\dagger B_1,
\end{aligned}$$

where P and Q are arbitrary elements of $\mathcal{B}(\mathcal{F},\mathcal{K})$ and $\mathcal{B}(\mathcal{G},\mathcal{K})$.

From the first equation in (1.1) we have

$$A_1X_1 = X_2B_1 + C_1,$$

so, by Lemma 2.1 we get

$$\begin{aligned} X_1 &= A_1^\dagger(X_2B_1 + C_1) + L_{A_1}R \\ &= A_1^\dagger S^\dagger(R_{A_1}C_1 + W)B_1^\dagger B_1 + A_1^\dagger ZB_1 - A_1^\dagger S^\dagger SZB_1 + A_1^\dagger C_1 + L_{A_1}R, \end{aligned}$$

where R is an arbitrary element of $\mathcal{B}(\mathcal{F},\mathcal{H})$.

From the second equation in (1.1) we have

$$A_2X_3 = X_2B_2 + C_2,$$

so, by Lemma 2.1 we get

$$\begin{aligned} X_3 &= A_2^\dagger(X_2B_2 + C_2) + L_{A_2}Y \\ &= A_2^\dagger\left(-R_{A_1}C_1 - S^\dagger(R_{A_1}C_1 + W)\right)B_1^\dagger B_2L_T \\ &\quad + A_2^\dagger\left((I - S^\dagger)R_{A_1}V + S^\dagger C_2\right)T^\dagger B_2 \\ &\quad + A_2^\dagger ZB_2 - A_2^\dagger(I - A_1A_1^\dagger + S^\dagger S)Z(B_1B_1^\dagger B_2 + T) + A_2^\dagger C_2 + L_{A_2}Y, \end{aligned}$$

where Y is an arbitrary element of $\mathcal{B}(\mathcal{L},\mathcal{K})$. □

Theorem 2.2. *Let $A_1 \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $B_1 \in \mathcal{B}(\mathcal{M},\mathcal{L})$, $C_1 \in \mathcal{B}(\mathcal{M},\mathcal{K})$, $A_2 \in \mathcal{B}(\mathcal{K},\mathcal{N})$, $B_2 \in \mathcal{B}(\mathcal{L},\mathcal{G})$, $C_2 \in \mathcal{B}(\mathcal{L},\mathcal{N})$ be such that A_1, A_2, B_1, B_2, S and T are all regular. Put*

$$\begin{aligned} &T = (I - B_1B_1^\dagger)(I - B_2^\dagger B_2), \quad S = A_2A_1A_1^\dagger, \\ &C = (I - (A_2A_1)(A_2A_1)^\dagger)(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)(I - B_2^\dagger B_2). \end{aligned}$$

The following statements are equivalent:

(i) *The system* (1.2) *is consistent;*

(ii) $R_{A_1}C_1L_{B_1} = 0$, $R_{A_2}C_2L_{B_2} = 0$, $CL_T = 0$;

(iii) $R_{A_1}C_1L_{B_1} = 0$, $(I - R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger)C = 0$, $CL_T = 0$.

In this case, the general solution to the system (1.2) *is given by*

$$\begin{aligned} X_1 = {} & A_1^\dagger S^\dagger A_2R_{A_1}C_1 + A_1^\dagger S^\dagger WB_1^\dagger B_1 + A_1^\dagger(I - S^\dagger)VB_1 \\ & + A_1^\dagger ZB_1 - A_1^\dagger S^\dagger SZB_1 + A_1^\dagger C_1 + R_{A_1}R, \end{aligned}$$

$$\begin{aligned} X_2 = {} & \left(-R_{A_1}C_1 + S^\dagger(A_2R_{A_1}C_1 + W)\right)B_1^\dagger(I - T^\dagger) \\ & + \left((I - S^\dagger A_2)R_{A_1}V + S^\dagger C_2L_{B_2}\right)T^\dagger \\ & + Z - (R_{A_1} + S^\dagger S)Z(B_1B_1^\dagger + TT^\dagger), \end{aligned}$$

$$X_3 = A_2\left(-R_{A_1}C_1 + S^\dagger(A_2R_{A_1}C_1 + W)\right)B_1^\dagger(I - T^\dagger)B_2^\dagger$$
$$+ A_2\left((I - S^\dagger A_2)R_{A_1}V + S^\dagger C_2L_{B_2}\right)T^\dagger B_2^\dagger$$
$$+ A_2ZB_2^\dagger - A_2(R_{A_1} + S^\dagger S)Z(B_1B_1^\dagger + TT^\dagger)B_2^\dagger - C_2B_2^\dagger + YR_{B_2},$$

where

$$V = -R_{A_1}C_1B_1^\dagger L_{B_2}L_T + R_{A_1}QT^\dagger T - R_{A_1}A_2^\dagger R_SA_2R_{A_1}QT^\dagger T$$

and

$$W = -R_SA_2R_{A_1}C_1 + SS^\dagger C_2L_{B_2}B_1 + SS^\dagger PB_1^\dagger B_1 - SS^\dagger PB_1^\dagger L_{B_2}B_1$$

with P,Q, Z *and* Y *arbitrary elements of* $\mathcal{B}(\mathcal{F},\mathcal{K})$, $\mathcal{B}(\mathcal{N},\mathcal{K})$, $\mathcal{B}(\mathcal{G},\mathcal{K})$, *and* $\mathcal{B}(\mathcal{N},\mathcal{M})$, *respectively.*

Proof. $(i) \Rightarrow (ii)$: Since the system (1.1) is consistent, there exists $X_2 \in \mathcal{B}(\mathcal{G},\mathcal{K})$ such that equations

$$A_1X_1 - X_2B_1 = C_1$$
$$A_2X_2 - X_3B_2 = C_2$$

are solvable for X_1 and X_3, respectively. According to Lemma 2.1 equation

$$A_1X_1 - X_2B_1 = C_1 \tag{2.12}$$

is solvable for X_1 if and only if

$$(I - A_1A_1^\dagger)(C_1 + X_2B_2) = 0 \tag{2.13}$$

and equation

$$A_2X_2 - X_3B_2 = C_2 \tag{2.14}$$

is solvable for X_3 if and only if

$$(A_2X_2 - C_2)(I - B_2^\dagger B_2) = 0. \tag{2.15}$$

So, from (2.13) and (2.15) it follows that equations

$$\begin{aligned}(I - A_1A_1^\dagger)X_2B_1 &= -(I - A_1A_1^\dagger)C_1,\\ A_2X_2(I - B_2^\dagger B_2) &= C_2(I - B_2^\dagger B_2)\end{aligned} \tag{2.16}$$

have a common solution. From Lemma 2.1 and Lemma 2.2 system (2.16) is consistent if and only if

$$(I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0,$$
$$(I - A_2A_2^\dagger)C_2(I - B_2^\dagger B_2) = 0,$$
$$C'(I - T^\dagger T) = 0,$$

where

$$C' = (I - SS^\dagger)(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)(I - B_2^\dagger B_2).$$

Note that condition

$$C'(I - T^\dagger T) = 0 \tag{2.17}$$

is equivalent to

$$C(I - T^\dagger T) = 0, \tag{2.18}$$

since (2.17) implies

$$\begin{aligned}
& C(I - T^\dagger T) \\
= \; & R_{A_2A_1}(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)L_{B_2}L_T \\
= \; & R_{A_2A_1}SS^\dagger(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)L_{B_2}L_T \\
= \; & R_{A_2A_1}A_2A_1A_1^\dagger S^\dagger(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)L_{B_2}L_T \\
= \; & 0,
\end{aligned}$$

and (2.18) implies

$$\begin{aligned}
& C'(I - T^\dagger T) \\
= \; & R_S(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)L_{B_2}L_T \\
= \; & R_S(A_2A_1)(A_2A_1)^\dagger(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)L_{B_2}L_T \\
= \; & (I - (A_2A_1A_1^\dagger)(A_2A_1A_1^\dagger)^\dagger)(A_2A_1)(A_2A_1)^\dagger(C_2 + A_2(I - A_1A_1^-)C_1B_1^-)L_{B_2}L_T \\
= \; & 0.
\end{aligned}$$

I follows that

$$\begin{aligned}
& (I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0, \\
& (I - A_2A_2^\dagger)C_2(I - B_2^\dagger B_2) = 0, \\
& C(I - T^\dagger T) = 0.
\end{aligned}$$

$(ii) \Rightarrow (i)$: If (ii) holds, then by Lemma 2.2 it follows that system (2.16) is consistent. Let $X_2 \in \mathcal{B}(\mathcal{G}, \mathcal{K})$ be the solution to the system (2.16) and let $X_1 = A_1^\dagger(X_2B_1 + C_1)$ and $X_3 = (A_2X_2 - C_2)B_2^\dagger$. Then it is easy to see that such X_1, X_2 and X_3 satisfy (1.2).

$(ii) \Rightarrow (iii)$: Suppose that

$$(I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0, \tag{2.19}$$

$$(I - A_2A_2^\dagger)C_2(I - B_2^\dagger B_2) = 0 \tag{2.20}$$

and

$$C(I - T^\dagger T) = 0. \tag{2.21}$$

From (2.20) we obtain

$$\begin{aligned}
& (I - R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger)C \\
= \; & (I - R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger)R_{A_2A_1}(C_2 + A_2(I - A_1A_1^\dagger)C_1B_1^\dagger)L_{B_2} \\
= \; & (I - R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger)R_{A_2A_1}C_2L_{B_2} \\
& +(I - R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger)R_{A_2A_1}A_2(I - A_1A_1^\dagger)C_1B_1^\dagger L_{B_2} \\
= \; & (I - R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger)R_{A_2A_1}A_2A_2^\dagger C_2L_{B_2} \\
= \; & 0.
\end{aligned}$$

$(ii) \Rightarrow (iii)$: Suppose that

$$(I - A_1A_1^\dagger)C_1(I - B_1^\dagger B_1) = 0, \tag{2.22}$$

$$(I - R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger)C = 0 \tag{2.23}$$

and

$$C(I - T^\dagger T) = 0. \tag{2.24}$$

From (2.23) we get

$$\begin{aligned}
& (I - A_2A_2^\dagger)C_2(I - B_2^\dagger B_2) \\
= \; & (I - A_2A_2^\dagger)C \\
= \; & (I - A_2A_2^\dagger)R_{A_2A_1}A_2(R_{A_2A_1}A_2)^\dagger C \\
= \; & 0.
\end{aligned}$$

Suppose that system (1.2) is consistent. Since $X_2 \in \mathcal{B}(G, K)$ is a solution to (1.2) if and only if it is solution to (2.16), its general form, according to Lemma 2.2, is given by

$$\begin{aligned}
X_2 = & \left(-R_{A_1}C_1 + S^\dagger(A_2R_{A_1}C_1 + W)\right) B_1^\dagger(I - T^\dagger) \\
& + \left((I - S^\dagger A_2)R_{A_1}V + S^\dagger C_2L_{B_2}\right) T^\dagger \\
& + Z - (R_{A_1} + S^\dagger S)Z(B_1B_1^\dagger + TT^\dagger),
\end{aligned}$$

where

$$V = -R_{A_1}C_1B_1^\dagger L_{B_2}L_T + R_{A_1}QT^\dagger T - R_{A_1}A_2^\dagger R_SA_2R_{A_1}QT^\dagger T$$

and

$$W = -R_SA_2R_{A_1}C_1 + SS^\dagger C_2L_{B_2}B_1 + SS^\dagger PB_1^\dagger B_1 - SS^\dagger PB_1^\dagger L_{B_2}B_1$$

with P, Q, Z arbitrary elements of $\mathcal{B}(\mathcal{F},\mathcal{M})$, $\mathcal{B}(\mathcal{G},\mathcal{K})$ and $\mathcal{B}(\mathcal{G},\mathcal{K})$, respectively.

From the first equation in (1.2) we have

$$A_1X_1 = X_2B_1 + C_1,$$

so, by Lemma 2.1 we get

$$\begin{aligned} X_1 &= A_1^\dagger(X_2B_1 + C_1) + L_{A_1}R \\ &= A_1^\dagger S^\dagger(A_2R_{A_1}C_1 + W)B_1^\dagger B_1 + A_1^\dagger ZB_1 - A_1^\dagger S^\dagger SZB_1 + A_1^\dagger C_1 + L_{A_1}R, \end{aligned}$$

where R is an arbitrary element of $\mathcal{B}(\mathcal{F},\mathcal{H})$.

From the second equation in (1.2) we have

$$X_3B_2 = A_2X_2 - C_2,$$

so, by Lemma 2.1 we get

$$\begin{aligned} X_3 &= (A_2X_2 - C_2)B_2^\dagger + YR_{B_2} \\ &= A_2\left(-R_{A_1}C_1 + S^\dagger(A_2R_{A_1}C_1 + W)\right)B_1^\dagger(I - T^\dagger)B_2^\dagger \\ &\quad + A_2\left((I - S^\dagger A_2)R_{A_1}V + S^\dagger C_2L_{B_2}\right)T^\dagger B_2^\dagger \\ &\quad + A_2ZB_2^\dagger - A_2(R_{A_1} + S^\dagger S)Z(B_1B_1^\dagger + TT^\dagger)B_2^\dagger - C_2B_2^\dagger + YR_{B_2}, \end{aligned}$$

where Y is an arbitrary element of $\mathcal{B}(\mathcal{N},\mathcal{M})$. □

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