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CHARACTERIZATIONS OF NORMAL AND BINORMAL SURFACES IN G_3

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Abstract. In this paper, our aim is to give surfaces in the Galilean 3-space $\mathbf{G}_3$ with the property that there exist four geodesics through each point such that every surface built with the normal lines and the binormal lines along these geodesics is a surface with a minimal surface and a constant negative Gaussian curvature. We show that ψ should be an isoparametric surface in $\mathbf{G}_3$: A plane or a circular hyperboloid.
Key words: surfaces, Galilean 3-space, geodesics.

1. Introduction

A helical curve (or a helix) is a geometric curve which has non-vanishing constant curvature κ and non-vanishing constant torsion τ [1]. It is known that if the curve is a straight line or plane curve, then $\kappa = 0$ or $\tau = 0$, respectively [4]. On the other hand, a family of curves with constant torsion but the non-constant curvature is called anti-Salkowski curves [9].

From the view of the differential geometry, there are different characterizations of surfaces. Generally, these type characterizations of surfaces are done in terms of the Gaussian curvature and mean curvature of the surface, [2, 3]. On the contrary, if there are characterizations of the surface with constant principal curvatures only, then these surfaces are called the isoparametric surfaces.

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In connection to this matter, which focuses on characterizations of surfaces with constant principal curvatures, Tamura [13] showed that complete surfaces of constant mean curvature in E^3 on which there exist two helical geodesics through each point are planes, spheres or circular cylinders. Recently, Lopez et al. improved surfaces in E^3 with the property that there exist four geodesics through each point such that every ruled surface built with the normal lines along these geodesics is a surface with constant mean curvature.

In this paper, we improve this characterization to the Galilean 3-space $\mathbf{G}_3$ of negative curvature. To be more precise, we investigate the following results for surfaces in the Galilean 3-space: First, we define surfaces in the Galilean 3-space $\mathbf{G}_3$ with the property that there exist four geodesics through each point such that every surface built with the normal lines and the binormal lines along these geodesics is a surface, which a minimal surface, with a constant negative curvature or zero curvature. Second, we show that the surface should be an isoparametric surface in $\mathbf{G}_3$: A plane or a circular hyperboloid. For this reason, we shall show the following theorems:

Theorem 1.1. *Let ψ be a connected surface in Galilean 3-space $\mathbf{G}_3$. If there exist four geodesics through each point of ψ with the property that the normal surface constructed along these geodesics is a minimal surface with a constant negative curvature or zero curvature, then ψ is a plane or a circular hyperboloid.*

Theorem 1.2. *Let ψ be a connected surface in Galilean 3-space $\mathbf{G}_3$. If there exist four geodesics through each point of ψ with the property that the binormal surface constructed along these geodesics is a minimal surface with a constant negative curvature or zero curvature, then ψ is a plane or a circular hyperboloid.*

2. Preliminaries

The Galilean 3-space $\mathbf{G}_3$ is a Cayley-Klein space equipped with the projective metric of signature $(0, 0, +, +)$, given in [6]. The absolute figure of the Galilean space consists of an ordered triple $\{\omega, f, I\}$ in which ω is the ideal (absolute) plane, f is the line (absolute line) in ω and I is the fixed elliptic involution of f. We introduce homogeneous coordinates in $\mathbf{G}_3$ in such a way that the absolute plane ω is given by $x_0 = 0$, the absolute line f by $x_0 = x_1 = 0$ and the elliptic involution by

$$(0 : 0 : x_2 : x_3) \to (0 : 0 : x_3 : -x_2). \tag{2.1}$$

A plane is called Euclidean if it contains f, otherwise it is called isotropic or i.e., planes $x = const.$ are Euclidean, and so is the plane ω. Other planes are isotropic. In other words, an isotropic plane does not involve any isotropic direction.

Definition 2.1. ([10]) Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be any two vectors in $\mathbf{G}_3$. A vector a is called isotropic if $x_1 = 0$, otherwise it is called non-isotropic. Then the Galilean scalar product of these vectors is given by

$$\langle x, y\rangle = \begin{cases} x_1 y_1, & \text{if} x_1 \neq 0 \text{or} y_1 \neq 0 \\ x_2y_2 + x_3y_3, & \text{if } x_1 = 0 \text{and} y_1 = 0 \end{cases}. \tag{2.2}$$

Definition 2.2. ([15]) Let $x = (x_1, x_2, x_3)$ and $y= (y_1, y_2, y_3)$ be any two vectors in $\mathbf{G}_3$, the Galilean cross product is given as

$$x \wedge y = \left\{ \begin{vmatrix} 0 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \right. . \tag{2.3}$$

Definition 2.3. ([12]) Let T be the unit tangent vector of a curve α on a surface ψ in $\mathbf{G}_3$, and N be the unit normal vector to the surface at the point $\alpha(s)$ of α, respectively. Let $B = T \wedge N$ be the tangential-normal. Then $\{T, N, B\}$ is an orthonormal frame at $\alpha(s)$ in $\mathbf{G}_3$. The frame is called a Galilean Darboux frame or a tangent-normal frame and it is expressed as

$$\begin{aligned} T'(s) &= k_g(s)B(s) + k_n(s)N(s), \\ N'(s) &= \tau_g(s)B(s), \\ B'(s) &= -\tau_g(s)N(s), \end{aligned} \tag{2.4}$$

where k_g, k_n and τ_g are the geodesic curvature, normal curvature, and geodesic torsion, respectively.

For the curvature κ of α, $\kappa^2 = k_g^2 + k_n^2$ holds. Also, a curve α is a geodesic (an asymptotic curve or a line of curvature) if and only if k_g (k_n or τ_g) vanishes, respectively.

Suppose now that α is a geodesic. Then $k_g = 0$ and $T' = k_n(s)N(s)$, which implies that, up change of orientation on ψ if necessary, $N(s)$ is the normal vector to α. So, from the Galilean Darboux frame, we can write $k_n = \kappa$ and $\tau_g = \tau$. If replacing these in Equation (2.4), then the formula becomes

$$\begin{aligned} T'(s) &= \kappa(s)N(s), \\ N'(s) &= \tau(s)B(s), \\ B'(s) &= -\tau(s)N(s). \end{aligned} \tag{2.5}$$

Let the equation of a surface $\psi = \psi(s, t)$ in $\mathbf{G}_3$ is given by

$$\psi(s, t) = (x(s, t), y(s, t), z(s, t)).$$

Then the unit isotropic normal vector field η on $\psi(s, t)$ is given by

$$\eta = \frac{\psi_{,s} \wedge \psi_{,t}}{\|\psi_{,s} \wedge \psi_{,t}\|}, \tag{2.6}$$

where the partial differentiations with respect to s and t, that is, $\psi_{,s} = \dfrac{\partial \psi(s, t)}{\partial s}$ and $\psi_{,t} = \dfrac{\partial \psi(s, t)}{\partial t}$.

From (2.1) and $w = \|\psi_{,s} \wedge \psi_{,t}\|$, we get the isotropic unit vector δ in the tangent plane of the surface as

$$\delta = \frac{x_{,2}\psi_{,s} - x_{,1}\psi_{,t}}{w}, \tag{2.7}$$

where $x_{,1} = \dfrac{\partial x(s,t)}{\partial s}$, $x_{,2} = \dfrac{\partial x(s,t)}{\partial t}$ and $\langle \eta, \delta\rangle = 0$, $\delta^2 = 1$.

Let us define

$$\begin{aligned}
&g_1 = x_{,1},\ g_2 = x_{,2},\ g_{ij} = g_i g_j,\\
&g^1 = \frac{x_{,2}}{w},\ g^2 = -\frac{x_{,1}}{w},\ g^{ij} = g^i g^j \quad (i,j = 1,2),\\
&h_{11} = \langle \widetilde{\psi}_{,1},\ \widetilde{\psi}_{,1}\rangle,\ h_{12} = \langle \widetilde{\psi}_{,1}, \widetilde{\psi}_{,2}\rangle,\ h_{22} = \langle \widetilde{\psi}_{,2}, \widetilde{\psi}_{,2}\rangle,
\end{aligned}$$

where $\widetilde{\psi}_{,1}$ and $\widetilde{\psi}_{,2}$ are the projections of vectors $\psi_{,1}$ and $\psi_{,2}$ onto the yz-plane, respectively. Then, the corresponding matrix of the first fundamental form ds^2 of the surface $\psi(s,t)$ is given by (cf. [11])

$$ds^2 = \begin{pmatrix} ds_1^2 & 0 \\ 0 & ds_2^2 \end{pmatrix}, \tag{2.8}$$

where $ds_1^2 = (g_1 ds + g_2 dt)^2$ and $ds_2^2 = h_{11}ds^2 + 2h_{12}dsdt + h_{22}dt^2$. In such case, we denote the coefficients of ds^2 by g^*_{ij}.

On the other hand, the function w can be represented in terms of g_i and h_{ij} as follows:

$$w^2 = g_1^2 h_{22} - 2g_1 g_2 h_{12} + g_2^2 h_{11}.$$

The Gaussian curvature and the mean curvature of a surface is defined by means of the coefficients of the second fundamental form L_{ij}, which are the normal components of $\psi_{,i,j}$ $(i,j = 1,2)$. That is,

$$\psi_{,i,j} = \sum_{k=1}^{2} \Gamma_{ij}^k \psi_{,k} + L_{ij}\eta,$$

where Γ_{ij}^k is the Christoffel symbols of the surface and L_{ij} are given by

$$L_{ij} = \frac{1}{g_1}\langle g_1\psi_{,i,j} - g_{i,j}\psi_{,1}, \eta\rangle = \frac{1}{g_2}\langle g_2\psi_{,i,j} - g_{i,j}\psi_{,2}, \eta\rangle. \tag{2.9}$$

From this, the Gaussian curvature K and the mean curvature H of the surface are expressed as [8]

$$\begin{aligned}
K &= \frac{L_{11}L_{22} - L_{12}^2}{w^2},\\
H &= \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{2w^2}.
\end{aligned} \tag{2.10}$$

3. Proof of Theorem 1.1

In this section, the below definition, proposition, and lemma to prove Theorem 1.1 are given in different steps.

Now we will start to give the definition of the normal surface as follows:

Definition 3.1. Let ψ be a surface in $\mathbf{G}_3$. The normal surface φ through α is the surface whose the base curve is α and the ruling are the straight-lines orthogonal to ψ through α.

The normal surface φ along α is a regular surface at least around α. Then φ is parametrized by

$$\varphi(s,t) = \alpha(s) + tN(s), \tag{3.1}$$

where $s \in I$ and $t \in R$. Then we obtain

$$\varphi_{,s} = \alpha'(s) + t(dN)_{\alpha(s)}(\alpha'(s)) \tag{3.2}$$

and

$$\varphi_{,t} = N(s). \tag{3.3}$$

Considering $s_0 \in I$, we get $\|\varphi_{,s}\wedge\varphi_{,t}\|(s_0,0) = \|\alpha'(s_0)\wedge N(\alpha'(s_0))\| \neq 0$. Thus, from the inverse function theorem, $\varphi(s,t)$ is an immersion.

Our first step towards proving Theorem 1.1 is to show the Gaussian curvature of the normal surface build up along the geodesic is a negative constant curvature or zero curvature and should be minimal.

Proposition 3.1. *Let ψ be a connected surface in $\mathbf{G}_3$. If the normal surface φ constructed along a geodesic of ψ is a surface with constant Gaussian curvature, then φ is either a constant negative curvature surface or flat surface and φ should be a minimal surface.*

Proof. Firstly, suppose that α is not a straightline. Then its curvature is defined as well as $T'(s) \neq 0$. Considering equations (3.2) and (3.3),

$$\varphi_{,s}\wedge\varphi_{,t} = (0,0,1), \tag{3.4}$$

then we have

$$\|\varphi_{,s}\wedge\varphi_{,t}\| = 1. \tag{3.5}$$

Using (2.6), the unit isotropic normal vector η_φ of $\varphi(s,t)$ is found as

$$\eta_\varphi = (0,0,1). \tag{3.6}$$

On the other hand, from equations (3.6) and (2.1), it is easy to show that

$$\delta_\varphi = (0,-1,0).$$

Since η_φ is the isotropic vector, using the Galilean frame, we can obtain $g_{\varphi 1} = 1$, $g_{\varphi 2} = 0$.

Considering the projection of $\varphi_{,s}$ and $\varphi_{,t}$ onto the Euclidean $yz-$plane, we obtain

$$h_{\varphi 22} = 1. \tag{3.7}$$

Using (3.3), the coefficients of the first fundamental form of the surface in Galilean space are obtained as

$$g^*_{\varphi 11} = 1, \quad g^*_{\varphi 12} = 0, \quad g^*_{\varphi 22} = 1.$$

To calculate the second fundamental form of $\varphi(s,t)$, we have to calculate the following:

$$\begin{aligned} \varphi_{,s,s} &= \left(\kappa - t\tau^2\right) N + t\tau' B \\ \varphi_{,t,s} &= \tau B, \\ \varphi_{,t,t} &= 0. \end{aligned} \tag{3.8}$$

From equations (3.8) and (2.9), the coefficients of the second fundamental form are found as

$$L_{\varphi 11} = t\tau', \; L_{\varphi 12} = \tau, \; L_{\varphi 22} = 0. \tag{3.9}$$

Thus, K_φ and H_φ are calculated as

$$K_\varphi = -\tau^2 < 0, \tag{3.10}$$

$$H_\varphi = 0. \tag{3.11}$$

Secondly, assume that α is a straight line. Then the similar calculations as the first case can be done as follows:

$$\begin{aligned} \varphi_{,s} &= T(s) + tN'(s), \; \varphi_{,t} = N(s), \\ \varphi_{,s,s} &= tN''(s), \; \varphi_{,t,s} = N'(s) \text{ and } \varphi_{,t,t} = 0, \end{aligned} \tag{3.12}$$

where $\alpha'(s) = T(s)$ is the tangent vector to α.

Using (2.6), the unit isotropic normal vector η_φ of $\varphi(s,t)$ is obtained as

$$\eta_\varphi = \frac{T(s) \wedge N(s) + tN'(s) \wedge N(s)}{w_\varphi}, \tag{3.13}$$

where $w_\varphi = \|T(s) \wedge N(s) + tN'(s) \wedge N(s)\|$.

From equations (3.8) and (2.9), the coefficients of the second fundamental form are given as

$$
\begin{aligned}
(3.14)\qquad L_{\varphi 11} &= \frac{1}{w_\varphi}\langle tN''(s), T(s)\wedge N(s) + tN'(s)\wedge N(s)\rangle \\
L_{\varphi 11} &= \frac{t}{w_\varphi}\langle N''(s), T(s)\wedge N(s)\rangle + \frac{t^2}{w}\langle N''(s), N'(s)\wedge N(s)\rangle, \\
L_{\varphi 12} &= \frac{1}{w_\varphi}\langle N'(s), T(s)\wedge N(s) + tN'(s)\wedge N(s)\rangle \\
L_{\varphi 12} &= \frac{1}{w_\varphi}\langle N'(s), T(s)\wedge N(s)\rangle, \\
L_{\varphi 22} &= 0.
\end{aligned}
$$

Thus, the Gaussian curvature K_φ satisfies

$$
(3.15)\qquad K_\varphi w_\varphi^{\frac{3}{2}} = \langle N'(s), T(s)\wedge N(s)\rangle^2.
$$

Squaring both sides equation (3.15) and writing as polynomial equation, we get a polynomial on t of degree six, this means that

$$
\sum_{n=1}^{6} P_n(s)t^n = 0.
$$

Particularly, $P_n(s) = 0$ for $0 \leq n \leq 6$. Then we can obtain $P_0(s) = K_\varphi^2 = 0$, which implies that $K_\varphi = 0$.

Furthermore we can easily show that

$$
H_\varphi = 0,
$$

which is completed the proof. $\square$

Lemma 3.1. *Let ψ be a connected surface in $\mathbf{G}_3$. If the normal surface φ, which is a minimal surface, constructed along a geodesic α is a constant negative curvature surface or a flat surface, then α is either*

i) an anti Salkowski curve,

ii) a planar curve,

iii) or a line segment and the last case α is a line of curvature of ψ.

Proof. Let α be not a line segment. Then $\kappa > 0$. Since φ is a surface with a negative constant curvature, from equation (3.10), we have

$$
\tau(s) = const.,
$$

for all s, as a result, we have constant torsion but non-constant curvature. This means that α is an anti Salkowski curve [9]. Or, we have

$$
\tau(s) = 0,
$$

this implies α is a planar curve.

In the last case, if α is a line segment, from equation (3.15), we get

$$\langle N'(s), T(s) \wedge N(s) \rangle = 0. \tag{3.16}$$

Taking account of the above equations, we can write

$$T(s) = a(s)N(s) + b(s)N'(s).$$

Since $\langle T(s), N(s) \rangle = \langle N'(s), N(s) \rangle = 0$, we find

$$T(s) = b(s)N'(s),$$

this means that

$$N'(s) = \lambda T(s),$$

from this α is also a line of curvature of the surface. $\square$

Therefore, Lemma 3.3 means that, under the same hypothesis of Theorem 1.1, there exist four geodesics through each point $p \in \psi$ which are the next three types:

i) An anti Salkowski curve (Type 1),

ii) A planar curve (Type 2),

iii) A line segment, which is a line of curvature (Type 3).

Theorem 3.1. ([14]) *A connected surface in R^3 with the property that there exist two proper helical geodesics through each point of the surface is an open of a right circular cylinder.*

Now our aim is to give the proof of Theorem 1.1. Considering Theorem 3.4 and Proposition 3.2, we can give the following claim:

Claim 3.1. *Let $p \in \psi$ a non-umbilic point. In a neighborhood A_φ of p, there are two proper helical geodesics which is a curve that is both a proper circular helix and a geodesic on ψ, through any point of A_φ.*

By Lemma 3.3, if $A_\varphi \subset \psi$ is an open set around p formed by non-umbilic points, then there exist four tangent directions at $q \in A_\varphi$ such that the corresponding geodesic refers to one of the above three Types 1, 2, 3. Because the point is not umbilic there are at most two geodesics of Type 2 or Type 3. As there are four geodesics of Types either 1, 2, or 3, we have two geodesics which are an anti salkowski curve, i.e. of Type 1. In particular, If we get $\kappa = const.$, then the anti salkowski curve turns out to be a circular helix. This proves the claim.

Let us denote ψ_1 is the set of umbilic points of ψ. This set is closed on ψ.

i) If $\psi - \psi_1 \neq \emptyset$, then $\psi - \psi_1$ is contained in circular hyperboloid . In particular, we can write $K_\varphi = -\dfrac{1}{r^2} < 0$ and $H_\varphi = 0$ on $\psi - \psi_1$. Thus we can define closed set in

ψ such that $\psi_2 = \left\{ p \in \psi : K_\varphi(p) = -\dfrac{1}{r^2} < 0,\ H_\varphi(p) = 0 \right\}$. From connectedness, we get proved that $\psi - \psi_1 \subset \psi_2$. Since $\psi_2 \cap \psi_1 = \emptyset$, we easily say that $\psi_2 \subset \psi - \psi_1$, which proves that $\psi_2 = \psi - \psi_1$. As ψ_2 is both an open and closed set of ψ, $\psi_2 = \psi$ by connectedness, proving that ψ is an open set of a circular hyperboloid.

ii) If $\psi - \psi_1 = \emptyset$, then ψ is an umbilic surface. Then ψ is an open of a plane since we have a flat and a minimal surface.

Then we finish the proof of Theorem 1.1.

4. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2 in different steps as the proof of Theorem 1.1.

Now, our starting point is to give the definition of the binormal surface as follows:

Definition 4.1. Let ψ be a surface in $\mathbf{G}_3$. The binormal surface ϕ through α is the surface whose the base curve is α and the ruling are the straightlines orthogonal to ϕ through α.

The binormal surface ϕ along α is a regular surface at least around α. Then ϕ is specifed by

$$\phi(s,t) = \alpha(s) + tB(s), \tag{4.1}$$

where $s \in I$ and $t \in R$. Then we get

$$\phi_{,s} = \alpha'(s) + t(dB)_{\alpha(s)}(\alpha'(s)) \tag{4.2}$$

and

$$\phi_{,t} = B(s). \tag{4.3}$$

If we consider $s_0 \in I$, then we get $\left\| \phi_{,s} \wedge \phi_{,t} \right\| (s_0, 0) = \left\| \alpha'(s_0) \wedge B(\alpha'(s_0)) \right\| \neq 0$. Thus, from the inverse function theorem, $\phi(s,t)$ is an immersion.

Our first step is to prove Theorem 1.2 and to get the Gaussian curvature of the binormal surface constructed along a geodesic. A negative constant curvature or zero curvature and should be minimal.

Proposition 4.1. *Let ψ be a connected surface in $\mathbf{G}_3$. If the binormal surface ϕ constructed along a geodesic of ψ is a surface with constant Gaussian curvature, then ϕ is either a constant negative curvature surface or flat surface and ϕ should be a minimal surface.*

Proof. Assume first that α is not a straightline. Then its curvature is defined. If we consider equations (4.2) and (4.3),

$$\phi_{,s} \wedge \phi_{,t} = (0, -1, 0), \tag{4.4}$$

then we have

$$\|\phi_{,s} \wedge \phi_{,t}\| = 1. \tag{4.5}$$

From (2.6), the unit isotropic normal vector η_ϕ of $\phi(s,t)$ is found as

$$\eta_\phi = (0, -1, 0). \tag{4.6}$$

On the other hand, from equations (4.6) and (2.1), it is easy to calculate that

$$\delta_\phi = (0, 0, -1).$$

Since η_ϕ is the isotropic vector, using the Galilean frame, we can get $g_{\phi 1} = 1$, $g_{\phi 2} = 0$.

Considering the projection of $\phi_{,s}$ and $\phi_{,t}$ onto the Euclidean $yz-$plane, we get

$$h_{\phi 22} = 1. \tag{4.7}$$

The coefficients of the first fundamental form of the surface in Galilean space are found as

$$g^*_{\phi 11} = 1, \quad g^*_{\phi 12} = 0, \quad g^*_{\phi 22} = 1.$$

To calculate the second fundamental form of $\phi(s,t)$, we have to compute the following:

$$\begin{aligned} \phi_{,s,s} &= (\kappa - t\tau')\, N - t\tau^2 B \\ \phi_{,t,s} &= -\tau N, \\ \phi_{,t,t} &= 0. \end{aligned} \tag{4.8}$$

From equations (4.8) and (2.9), the coefficients of the second fundamental form are found as

$$L_{\phi 11} = -\kappa + t\tau', \ L_{\phi 12} = \tau, \ L_{\phi 22} = 0. \tag{4.9}$$

Thus, K_ϕ and H_ϕ are computed as

$$K_\phi = -\tau^2 < 0, \tag{4.10}$$

$$H_\phi = 0. \tag{4.11}$$

Secondly, suppose that α is a straightline. Then the similar calculations as first case can be done as follows:

$$\begin{aligned} \phi_{,s} &= T(s) + tB'(s), \ \phi_{,t} = B(s), \\ \phi_{,s,s} &= tB''(s), \ \phi_{,t,s} = B'(s) \text{ and } \phi_{,t,t} = 0, \end{aligned} \tag{4.12}$$

where $\alpha'(s) = T(s)$ is the tangent vector to α.

Using (2.6), the unit isotropic normal vector η_ϕ of $\phi(s,t)$ is obtained as

$$\eta_\phi = \frac{T(s)\wedge B(s) + tB'(s)\wedge B(s)}{w_\phi}, \tag{4.13}$$

where $w_\phi = \|T(s)\wedge B(s) + tB'(s)\wedge B(s)\|$.

From equations (4.8) and (2.9), the coefficients of the second fundamental form are given as

$$\begin{aligned}
L_{\phi 11} &= \frac{1}{w_\phi}\langle tB''(s), T(s)\wedge B(s) + tB'(s)\wedge B(s)\rangle \\
L_{\phi 11} &= \frac{t}{w_\phi}\langle B''(s), T(s)\wedge B(s)\rangle + \frac{t^2}{w}\langle B''(s), B'(s)\wedge B(s)\rangle, \\
L_{\phi 12} &= \frac{1}{w_\phi}\langle B'(s), T(s)\wedge B(s) + tB'(s)\wedge B(s)\rangle \\
L_{\phi 12} &= \frac{1}{w_\phi}\langle B'(s), T(s)\wedge B(s)\rangle, \\
L_{\phi 22} &= 0.
\end{aligned} \tag{4.14}$$

Thus, the Gaussian curvature K_ϕ satisfies

$$K_\phi w_\phi^{\frac{3}{2}} = \langle B'(s), T(s)\wedge B(s)\rangle^2. \tag{4.15}$$

Squaring both sides equation (4.15) and writing as polynomial equation, we obtain a polynomial on t of degree six, this means that

$$\sum_{n=1}^{6} Q_n(s)t^n = 0.$$

Particularly, $Q_n(s) = 0$ for $0 \leq n \leq 6$. Then we can obtain $Q_0(s) = K_\phi^2 = 0$, which implies that $K_\phi = 0$.

Moreover we can easily show that

$$H_\phi = 0.$$

Hence this completes the proof. $\square$

Lemma 4.1. *Let ψ be a connected surface in G_3. If the binormal surface ϕ, which is a minimal surface, constructed along a geodesic α is a constant negative curvature surface or a flat surface, then α is either*

i) an anti Salkowski curve,

ii) a planar curve,

iii) or a line segment.

Proof. This proof can be done in a similar way to the proof of Lemma 3.3. $\square$

Therefore, Lemma 4.3 implies that, under the same hypothesis of Theorem 1.2, there exist four geodesics through each point $p \in \psi$ which are the next three types:

i) An anti Salkowski curve (Type 1),

ii) A planar curve (Type 2),

iii) A line segment (Type 3).

Claim 4.1. *Let $p \in \psi$ a non-umbilic point. In a neighborhood A_ϕ of p, there are two proper helical geodesics which is a curve that is both a proper circular helix and a geodesic on ψ, through any point of A_ϕ.*

Then we finish the proof of Theorem 1.2.

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