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EXISTENCE AND ULAM STABILITY OF SOLUTIONS FOR NONLINEAR CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING TWO FRACTIONAL ORDERS

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Abstract. In this paper, we study existence, uniqueness and Ulam-Hyers stability of solutions for integro-differential equations involving two fractional orders. By using Banach's fixed point theorem, we obtain some sufficient conditions for the existence and uniqueness of solution for the mentioned problem. Furthermore, we derive the Ulam-Hyers stability and the generalized Ulam-Hyers stability of solution. At the end, an illustrative example is discussed.
Key words: differential equations, two fractional orders, stability of solutions.

1. Introduction and Preliminaries

Fractional type differential equations have recently been studied by several scientific researchers due to the fact that they are valuable tools in the modelling of various problems in sciences and engineering such physics, biology, chemistry, economics, signal theory, etc. For more details, see [13, 14, 16, 17, 19, 20] and reference therein. Many studies on differential equations of fractional order, involving different fractional operators such as Riemann-Liouville fractional derivative [6, 9],

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Caputo fractional derivative [8, 24], Hadamard fractional derivative [1, 25],Caputo-Hadamard fractional derivative [15, 22] and Atangana-Baleanu-Caputo fractional derivative [18] have appeared during the past several years. Moreover, by using many classical fixed-point theorems, several authors presented the existence and stability results for various classes of fractional differential equations, see for example [4, 7, 8, 11, 12, 23]. Recently, considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of fractional differential equations. Since then, a large number of papers have been published in connection with various generalizations of Ulam's type stability theory or the Ulam-Hyers stability theory. For the advanced contribution on Ulam's type stability, we refer to [2, 3, 5, 10, 21]and reference therein. In this work, we discuss the existence, uniqueness and the Ulam stability, generalized Ulam-Hyers stability and Ulam-Hyers-Rassias stability for nonlinear fractional differential equation with two Caputo-Hadamard-type fractional derivatives of the form

$$(1.1)\quad \begin{cases} {}^{C}_{H}D^{\beta}[{}^{C}_{H}D^{\alpha}+\lambda]u(t)=\varphi(t,u(t))+^{H}I^{\theta}\psi(t,u(t)), t\in J=[1,e] \\ \\ a^{H}I^{p}u(\eta)=\gamma_1, b^{H}I^{q}u(\xi)=\gamma_2, 1<\eta,\xi<e, \end{cases}$$

where $0<\alpha,\beta\leq 1,\theta,p,q>0$, ${}^{C}_{H}D^{\beta}$ and ${}^{C}_{H}D^{\alpha}$ are the Caputo-Hadamard fractional derivatives, ${}^{H}I^{\rho},\rho\in\{\theta,p,q\}$ are the Hadamard fractional integrals, with a,b,λ,γ_1 and γ_2 are real constants and $f,g:J\times\mathbb{R}\to\mathbb{R}$ are given continuous functions. The operator ${}^{H}I^{\rho}$ is the Hadamard fractional integral given by:

$$ {}^{H}I^{\rho}\phi\left(t\right)=\frac{1}{\Gamma\left(\rho\right)}\int_a^t\left(\log\frac{t}{s}\right)^{\rho-1}\frac{\phi\left(s\right)}{s}ds,\rho>0,$$

where $\Gamma\left(\rho\right)=\int_0^{\infty}e^{-x}x^{\rho-1}dx$. The operator ${}^{C}_{H}D^{\rho}$ is the Caputo-Hadamard fractional derivative defined by:

$$ {}^{C}_{H}D^{\rho}\phi(t)=\frac{1}{\Gamma\left(n-\rho\right)}\int_a^t\left(\log\frac{t}{s}\right)^{n-\rho-1}\delta^{n}\frac{\phi(s)}{s}ds,$$

where $n-1<\rho<n, n=[\rho]+1, \delta=t\frac{d}{dt}, [\rho]$ denotes the integer part of ρ and $\log\left(.\right)=\log_e\left(.\right).$

We recall the following lemma [12, 15].

Lemma 1.1. *Let $x\in C^{n}_{\delta}\left([a,b],\mathbb{R}\right)$. Then*

$$ {}^{H}I^{\rho}\left({}^{CH}D^{\rho}u\right)(t)=u(t)-\sum_{i=0}^{n-1}c_i(\log t)^{i}, c_i\in\mathbb{R},$$

where $C^{n}_{\delta}\left([a,b],\mathbb{R}\right)=\left\{\phi:[a,b]\to\mathbb{R}:\delta^{n-1}\phi\in C\left([a,b],\mathbb{R}\right)\right\}.$

Also, we denote by $W=C\left(J,\mathbb{R}\right)$ the Banach space of all continuous functions from J to $\mathbb{R}$ endowed with the norm defined by $\|u\|=\sup\left\{\left|u\left(t\right)\right|:t\in J\right\}$.

Now, to study the Hyers–Ulam stability of the problem (1.1), we give the following definitions [3].

Definition 1.1. The fractional boundary value problem (1.1) is Ulam-Hyers stable if there exists a real number $\mu_{\varphi,\psi} > 0$ such that for each $\vartheta > 0$ and for each solution $v \in W$ of the inequality

$$\left|{}_{H}^{C}D^{\beta}\left({}_{H}^{C}D^{\alpha}+\lambda\right)v\left(t\right)-\varphi\left(t,v\left(t\right)\right)-{}^{H}I^{\theta}\psi\left(t,v\left(t\right)\right)\right|\leq\vartheta,t\in J, \tag{1.2}$$

there exists a solution $u \in W$ of fractional boundary value problem (1.1) with

$$\left|v\left(t\right)-u\left(t\right)\right|\leq\mu_{\varphi,\psi}\vartheta,t\in J.$$

Definition 1.2. The fractional boundary value problem (1.1) is generalized Ulam-Hyers stable if there exists $h_{\varphi,\psi} \in C(\mathbb{R}_{+},\mathbb{R}_{+}), h_{\varphi,\psi}\left(0\right) = 0$, such that for each solution $v \in W$ of the inequality (1.2) there exists a solution $u \in W$ of the fractional boundary value problem (1.1) with

$$\left|v\left(t\right)-u\left(t\right)\right|\leq h_{\varphi,\psi}\left(\vartheta\right),t\in J.$$

Definition 1.3. The fractional boundary value problem (1.1) is Ulam-Hyers-Rassias stable with respect to $g \in W$ if there exists a real number $\mu_{\varphi,\psi} > 0$ such that for each $\vartheta > 0$ and for each solution $v \in X$ of the inequality

$$\left|{}_{H}^{C}D^{\beta}\left({}_{H}^{C}D^{\alpha}+\lambda\right)v\left(t\right)-\varphi\left(t,v\left(t\right)\right)-{}^{H}I^{\theta}\psi\left(t,v\left(t\right)\right)\right|\leq\vartheta g\left(t\right),t\in J, \tag{1.3}$$

there exists a solution $u \in W$ of problem (1.1) with

$$\left|v\left(t\right)-u\left(t\right)\right|\leq\mu_{\varphi,\psi}\vartheta g\left(t\right),t\in J.$$

Definition 1.4. The fractional boundary value problem (1.1) is generalized Ulam-Hyers-Rassias stable with respect to $g \in W$ if there exists a real number $\mu_{\varphi,\psi,g} > 0$ such that for each solution $v \in W$ of the inequality

$$\left|{}_{H}^{C}D^{\beta}\left({}_{H}^{C}D^{\alpha}+\lambda\right)v\left(t\right)-\varphi\left(t,v\left(t\right)\right)-{}^{H}I^{\theta}\psi\left(t,v\left(t\right)\right)\right|\leq g\left(t\right),t\in J, \tag{1.4}$$

there exists a solution $u \in W$ of problem (1.1) with

$$\left|v\left(t\right)-u\left(t\right)\right|\leq\mu_{\varphi,\psi,g}g\left(t\right),t\in J.$$

Remark 1.1. A function $v \in W$ is a solution of the inequality (1.2) if and only if there exists a function $F : [1, e] \to \mathbb{R}$ such that

(i) $\left|F\left(t\right)\right|\leq\vartheta,t\in J.$

(ii) ${}_{H}^{C}D^{\alpha}\left({}_{H}^{C}D^{\beta}+\lambda\right)v\left(t\right)=\varphi\left(t,v\left(t\right)\right)+{}^{H}I^{\theta}\psi\left(t,v\left(t\right)\right)+F\left(t\right),t\in J.$

Remark 1.2. Clearly,

(1) Definition 1.1 $\Rightarrow$ Definition 1.2

(2) Definition 1.3 $\Rightarrow$ Definition 1.4

2. Existence and uniqueness of solution

Lemma 2.1. *Assume that $\Pi \neq 0$. For a given $\varphi \in C([1,e],\mathbb{R})$, the solution of the linear Caputo-Hadamard fractional differential equation*

$$ {}_{H}^{C}D^{\beta}({}_{H}^{C}D^{\alpha}+\lambda)u(t)=f(t), t\in J, 0<\alpha,\beta\leq 1, \tag{2.1}$$

subject to the Hadamard fractional integral conditions

$$ a^{H}I^{p}u(\eta)=\gamma_1, b^{H}I^{q}u(\xi)=\gamma_2, 1<\eta,\xi<e, \tag{2.2}$$

is given by

$$\begin{aligned} u(t) &= \int_1^t \frac{\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(s)\,ds - \lambda\int_1^t \frac{\left(\log\frac{t}{s}\right)^{\alpha-1}}{\Gamma(\alpha)} u(s)\,ds \\ &+ \frac{(\log t)^{\alpha}\Delta_2-\Delta_1\Gamma(\alpha+1)}{\Gamma(\alpha+1)\Pi}\left(\gamma_1 - a\int_1^{\eta}\frac{\left(\log\frac{\eta}{s}\right)^{p+\alpha+\beta-1}}{\Gamma(p+\alpha+\beta)} f(s)\,ds\right. \\ &+ \left.\lambda a\int_1^{\eta}\frac{\left(\log\frac{\eta}{s}\right)^{p+\alpha-1}}{\Gamma(p+\alpha)}u(s)\,ds\right) - \frac{(\log t)^{\alpha}\Lambda_2-\Lambda_1\Gamma(\alpha+1)}{\Gamma(\alpha+1)\Pi} \\ &\times \left(\gamma_2 - b\int_1^{\xi}\frac{\left(\log\frac{\xi}{s}\right)^{q+\alpha+\beta-1}}{\Gamma(q+\alpha+\beta)}f(s)\,ds + \lambda b\int_1^{\xi}\frac{\left(\log\frac{\xi}{s}\right)^{q+\alpha-1}}{\Gamma(q+\alpha)}u(s)\,ds\right), \end{aligned}$$

where

$$\begin{aligned} \Lambda_1 &= \tfrac{a}{\Gamma(p+\alpha+1)}(\log\eta)^{p+\alpha}, \ \Lambda_2=\tfrac{a}{\Gamma(p+1)}(\log\eta)^{p}, \\ \Delta_1 &= \tfrac{a}{\Gamma(q+\alpha+1)}(\log\xi)^{q+\alpha}, \ \Delta_2=\tfrac{a}{\Gamma(q+1)}(\log\xi)^{q}, \end{aligned}$$

and

$$\Pi=\Lambda_1\Delta_2-\Lambda_2\Delta_1.$$

Proof. In view of Lemma 1.1, the solution of the Hadamard (2.1), can be expressed as an equivalent integral equation

$$ u(t)={}^{H}I^{\alpha+\beta}f(t)-\lambda^{H}I^{\alpha}u(t)+\frac{c_0}{\Gamma(\alpha+1)}(\log t)^{\alpha}+c_1, \tag{2.3}$$

where c_0 and c_1 are arbitrary constants. To find the value c_0 and c_1, we apply again the Hadamard fractional integral on both sides of (2.3), we get

$${}^{H}I^{p}u(t)={}^{H}I^{p+\alpha+\beta}f(t)-\lambda^{H}I^{p+\alpha}u(t)+c_0\frac{(\log t)^{p+\alpha}}{\Gamma(p+\alpha)}+c_1\frac{(\log t)^{p}}{\Gamma(p+1)}.$$

By using the boundary conditions (2.2), we have

$$\begin{aligned} c_0\Lambda_1+c_1\Lambda_2 &= \gamma_1 - a^{H}I^{p+\alpha+\beta}h(\eta)+a\lambda^{H}I^{p+\alpha}x(\eta), \\ c_0\Delta_1+c_1\Delta_2 &= \gamma_2 - b^{H}I^{q+\alpha+\beta}\varphi(\xi)+b\lambda^{H}I^{q+\alpha}x(\xi). \end{aligned}$$

Thus

$$\begin{aligned} c_0 &= \frac{\Delta_2}{\Pi}\left[\gamma_1 - a^H I^{p+\alpha+\beta}\varphi(\eta) + a\lambda^H I^{p+\alpha} x(\eta)\right] \\ &- \frac{\Lambda_2}{\Pi}\left[\gamma_2 - b^H I^{q+\alpha+\beta}\varphi(\xi) + b\lambda^H I^{q+\alpha} x(\xi)\right], \end{aligned}$$

and

$$\begin{aligned} c_1 &= \frac{\Lambda_1}{\Pi}\left[\gamma_2 - b^H I^{q+\alpha+\beta}\varphi(\xi) + b\lambda^H I^{q+\alpha} x(\xi)\right] \\ &- \frac{\Delta_1}{\Pi}\left[\gamma_1 - a^H I^{p+\alpha+\beta}\varphi(\eta) + a\lambda^H I^{p+\alpha} x(\eta)\right]. \end{aligned}$$

Substituting the value of c_0 and c_1 in (2.3), we obtain the solution . □

In view of Lemma 2.1, we define an operator $P : W \to W$ as

$$\begin{aligned} Pu(t) &= \int_1^t \frac{\left(\log \frac{t}{s}\right)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \varphi(s, x(s)) \frac{ds}{s} \\ &+ \int_1^t \frac{\left(\log \frac{t}{s}\right)^{\alpha+\beta+\theta-1}}{\Gamma(\alpha+\beta+\theta)} \psi(s, x(s)) \frac{ds}{s} - \lambda \int_1^t \frac{\left(\log \frac{t}{s}\right)^{\alpha-1}}{\Gamma(\alpha)} u(s) \frac{ds}{s} \\ &+ \frac{\Delta_2 (\log t)^\alpha - \Delta_1 \Gamma(\alpha+1)}{\Gamma(\alpha+1)\Pi} \left(\gamma_1 - a\int_1^\eta \frac{\left(\log \frac{\eta}{s}\right)^{p+\alpha+\beta-1}}{\Gamma(p+\alpha+\beta)} \varphi(s, x(s)) \frac{ds}{s}\right. \\ &- \left. a\int_1^\eta \frac{\left(\log \frac{\eta}{s}\right)^{p+\alpha+\beta+\theta-1}}{\Gamma(p+\alpha+\beta+\theta)} \psi(s, x(s)) \frac{ds}{s} + \lambda a \int_1^\eta \frac{\left(\log \frac{\eta}{s}\right)^{p+\alpha-1}}{\Gamma(p+\alpha)} u(s) \frac{ds}{s}\right) \\ &- \frac{\Lambda_2 (\log t)^\alpha - \Lambda_1 \Gamma(\alpha+1)}{\Gamma(\alpha+1)\Pi} \left(\gamma_2 - b\int_1^\xi \frac{\left(\log \frac{\xi}{s}\right)^{q+\alpha+\beta-1}}{\Gamma(q+\alpha+\beta)} \varphi(s, x(s)) \frac{ds}{s}\right. \\ &- \left. b\int_1^\xi \frac{\left(\log \frac{\xi}{s}\right)^{q+\alpha+\beta+\theta-1}}{\Gamma(q+\alpha+\beta+\theta)} \psi(s, x(s)) \frac{ds}{s} + \lambda b \int_1^\xi \frac{\left(\log \frac{\xi}{s}\right)^{q+\alpha-1}}{\Gamma(q+\alpha)} u(s) \frac{ds}{s}\right). \end{aligned}$$

For computational convenience, we set

$$\begin{aligned} \Theta_1 &:= \frac{1}{\Gamma(\alpha+\beta+1)} + \frac{1}{\Gamma(\alpha+\beta+\theta+1)} \\ &+ \frac{|\Delta_2| + |\Delta_1|\Gamma(\alpha+1)}{\Gamma(\alpha+1)|\Pi|}\left(\frac{|a|(\log\eta)^{p+\alpha+\beta}}{\Gamma(p+\alpha+\beta+1)} + \frac{|a|(\log\eta)^{p+\alpha+\beta+\theta}}{\Gamma(p+\alpha+\beta+\theta+1)}\right) \\ &+ \frac{|\Lambda_2| + |\Lambda_1|\Gamma(\alpha+1)}{\Gamma(\alpha+1)|\Pi|}\left(\frac{|b|(\log\xi)^{q+\alpha+\beta}}{\Gamma(q+\alpha+\beta+1)} + \frac{|b|(\log\xi)^{q+\alpha+\beta+\theta}}{\Gamma(q+\alpha+\beta+\theta+1)}\right), \end{aligned}$$

$$\begin{aligned}
\Theta_2 &:= \frac{|\lambda|}{\Gamma(\alpha+1)} + \frac{|\Delta_2| + |\Delta_1| \Gamma(\alpha+1)}{\Gamma(\alpha+1)|\Pi|} \left(\frac{|\lambda| |a| (\log \eta)^{p+\alpha}}{\Gamma(p+\alpha+1)} \right), \\
&+ \frac{(|\Lambda_2| + |\Lambda_1| \Gamma(\alpha+1))}{\Gamma(\alpha+1)|\Pi|} \left(\frac{|\lambda| |b| (\log \xi)^{q+\alpha}}{\Gamma(q+\alpha+1)} \right), \\
\Theta_3 &:= \frac{|\Delta_2| + |\Delta_1| \Gamma(\alpha+1)}{\Gamma(\alpha+1)|\Pi|} |\gamma_1| + \frac{|\Lambda_2| + |\Lambda_1| \Gamma(\alpha+1)}{\Gamma(\alpha+1)|\Pi|} |\gamma_2|.
\end{aligned}$$

The following notations and assumptions are considered throughout the rest of this paper. (A_1) : there exists $M_1 > 0$ such that

$$|\varphi(t,x) - \varphi(t,y)| \leq M_1 |x-y| \text{ for } t \in J \text{ and } (x,y) \in \mathbb{R}^2.$$

(A_2) : there exists $M_2 > 0$ such that

$$|\psi(t,x) - \psi(t,y)| \leq M_2 |x-y| \text{ for } t \in J \text{ and } (x,y) \in \mathbb{R}^2.$$

(A_3) : Assume that

$$\widetilde{M} = \frac{M_1}{\Gamma(\alpha+\beta+1)} + \frac{M_2}{\Gamma(\alpha+\beta+\theta+1)} + \frac{|\lambda|}{\Gamma(\alpha+1)} < 1$$

Theorem 2.1. *Assume that (A_1) and (A_2) hold. If there exists a constant $M > 0$ such that*

$$\Theta_1 M < 1 - \Theta_2, \tag{2.4}$$

where $M = \max\{M_i, i = 1,2\}$, then the fractional boundary value problem (1.1) has a unique solution.

Proof. Let $L = \max\{L_i, i = 1,2\}$, where L_i are finite numbers given by

$$L_1 = \sup_{t\in[1,e]} |f(t,0)| \text{ and } L_2 = \sup_{t\in[1,e]} |g(t,0)|.$$

Selecting

$$r \geq \frac{L\Theta_2 + \Theta_3}{1 - M\Theta_1 - \Theta_2},$$

we show that $PB_r \subset B_r$, $B_r = \{x \in X : \|x\| \leq r\}$. Using (A_1) and (A_2), we can write

$$\begin{aligned}
|f(s,x(s))| &\leq |f(s,x(s)) - f(s,0)| + |f(s,0)| \leq M_1 \|u\| + L_1 \leq M_1 r + L_1, \\
|g(s,x(s))| &\leq |g(s,x(s)) - g(s,0)| + |g(s,0)| \leq M_2 \|u\| + L_2 \leq M_2 r + L_2,
\end{aligned}$$

for $x \in B_r$, we can show that

$$\|Px\| \leq \left\{ M \left[\left(\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{1}{\Gamma(\alpha+\beta+\theta+1)} \right) \right. \right.$$

$$
\begin{aligned}
&+ \quad \frac{|\Delta_2| + |\Delta_1| \Gamma(\alpha+1)}{\Gamma(\alpha+1)|\Pi|} \left(\frac{|a| (\log \eta)^{p+\alpha+\beta}}{\Gamma(p+\alpha+\beta+1)} + \frac{|a| (\log \eta)^{p+\alpha+\beta+\theta}}{\Gamma(p+\alpha+\beta+\theta+1)} \right) \\
&+ \quad \frac{|\Lambda_2| + |\Lambda_1| \Gamma(\alpha+1)}{\Gamma(\alpha+1)|\Pi|} \left(\frac{|b| (\log \xi)^{q+\alpha+\beta}}{\Gamma(q+\alpha+\beta+1)} + \frac{|b| (\log \xi)^{q+\alpha+\beta+\theta}}{\Gamma(q+\alpha+\beta+\theta+1)} \right) \Bigg] \\
&+ \quad \frac{|\lambda|}{\Gamma(\alpha+1)} + \frac{(|\Delta_2| + |\Delta_1| \Gamma(\alpha+1))}{\Gamma(\alpha+1)|\Pi|} \left(\frac{|\lambda| |a| (\log \eta)^{p+\alpha}}{\Gamma(p+\alpha+1)} \right) \\
&+ \quad \frac{(|\Lambda_2| + |\Lambda_1| \Gamma(\alpha+1))}{\Gamma(\alpha+1)|\Pi|} \left(\frac{|\lambda| |b| (\log \xi)^{q+\alpha}}{\Gamma(q+\alpha+1)} \right) \Bigg\} r \\
&+ \quad L \Bigg[\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{1}{\Gamma(\alpha+\beta+\theta+1)} \\
&+ \quad \frac{|\Delta_2| + |\Delta_1| \Gamma(\alpha+1)}{\Gamma(\alpha+1)|\Pi|} \left(\frac{|a| (\log \eta)^{p+\alpha+\beta}}{\Gamma(p+\alpha+\beta+1)} + \frac{|a| (\log \eta)^{p+\alpha+\beta+\theta}}{\Gamma(p+\alpha+\beta+\theta+1)} \right) \\
&+ \quad \frac{|\Lambda_2| + |\Lambda_1| \Gamma(\alpha+1)}{\Gamma(\alpha+1)|\Pi|} \left(\frac{|b| (\log \xi)^{q+\alpha+\beta}}{\Gamma(q+\alpha+\beta+1)} + \frac{|b| (\log \xi)^{q+\alpha+\beta+\theta}}{\Gamma(q+\alpha+\beta+\theta+1)} \right) \Bigg] \\
&+ \quad \frac{|\Delta_2| + |\Delta_1| \Gamma(\alpha+1)}{\Gamma(\alpha+1)|\Pi|} |\gamma_1| + \frac{|\Lambda_2| + |\Lambda_1| \Gamma(\alpha+1)}{\Gamma(\alpha+1)|\Pi|} |\gamma_2|, \\
&\leq \quad (M\Theta_1 + \Theta_2) r + L\Theta_2 + \Theta_3 \leq r.
\end{aligned}
$$

which implies that $PB_r \subset B_r$. Now, for $x, y \in B_r$, we obtain

$$
\begin{aligned}
& \quad \|Px - Py\| \\
\leq & \quad \sup_{t \in [1,e]} \Bigg\{ \int_1^t \frac{\left(\log \frac{t}{s}\right)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(s, x(s)) - f(s, y(s))| \frac{ds}{s} \\
+ & \quad \int_1^t \frac{\left(\log \frac{t}{s}\right)^{\alpha+\beta+\theta-1}}{\Gamma(\alpha+\beta+\theta)} |g(s, x(s)) - g(s, y(s))| \frac{ds}{s} \\
+ & \quad |\lambda| \int_1^t \frac{\left(\log \frac{t}{s}\right)^{\alpha-1}}{\Gamma(\alpha)} |x(s) - y(s)| \frac{ds}{s} \\
+ & \quad \frac{\Delta_2 (\log t)^{\alpha} - \Delta_1 \Gamma(\alpha+1)}{\Gamma(\alpha+1)\Pi} \cdot \\
& \quad \Bigg(\gamma_1 - a \int_1^{\eta} \frac{\left(\log \frac{\eta}{s}\right)^{p+\alpha+\beta-1}}{\Gamma(p+\alpha+\beta)} |f(s, x(s)) - f(s, y(s))| \frac{ds}{s} \\
& \quad -a \int_1^{\eta} \frac{\left(\log \frac{\eta}{s}\right)^{p+\alpha+\beta+\theta-1}}{\Gamma(p+\alpha+\beta+\theta)} |g(s, x(s)) - g(s, y(s))| \frac{ds}{s} \\
& \quad +\lambda a \int_1^{\eta} \frac{\left(\log \frac{\eta}{s}\right)^{p+\alpha-1}}{\Gamma(p+\alpha)} x(s) \frac{ds}{s} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{\Lambda_2 (\log t)^{\alpha} - \Lambda_1 \Gamma(\alpha+1)}{\Gamma(\alpha+1)\Pi} \cdot \\
& \left(\gamma_2 - b \int_1^{\xi} \frac{\left(\log \frac{\xi}{s}\right)^{q+\alpha+\beta-1}}{\Gamma(q+\alpha+\beta)} \left| f(s,x(s)) - f(s,y(s)) \right| \frac{ds}{s} \right. \\
& -b \int_1^{\xi} \frac{\left(\log \frac{\xi}{s}\right)^{q+\alpha+\beta+\theta-1}}{\Gamma(q+\alpha+\beta+\theta)} \left| g(s,x(s)) - g(s,y(s)) \right| ds \\
& \left. \left. +\lambda b \int_1^{\xi} \frac{\left(\log \frac{\xi}{s}\right)^{q+\alpha-1}}{\Gamma(q+\alpha)} \left| x(s) - y(s) \right| ds \right) \right\} \\
\leq \; & M \left\{ \left[\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{1}{\Gamma(\alpha+\beta+\theta+1)} \right. \right. \\
+ \; & \frac{|\Delta_2| + |\Delta_1| \Gamma(\alpha+1)}{\Gamma(\alpha+1)|\Pi|} \left(\frac{|a| (\log \eta)^{p+\alpha+\beta}}{\Gamma(p+\alpha+\beta+1)} + \frac{|a| (\log \eta)^{p+\alpha+\beta+\theta}}{\Gamma(p+\alpha+\beta+\theta+1)} \right) \\
+ \; & \left. \frac{|\Lambda_2| + |\Lambda_1| \Gamma(\alpha+1)}{\Gamma(\alpha+1)|\Pi|} \left(\frac{|b| (\log \xi)^{q+\alpha+\beta}}{\Gamma(q+\alpha+\beta+1)} + \frac{|b| (\log \xi)^{q+\alpha+\beta+\theta}}{\Gamma(q+\alpha+\beta+\theta+1)} \right) \right] \\
+ \; & \frac{|\lambda|}{\Gamma(\alpha+1)} + \frac{(|\Delta_2| + |\Delta_1| \Gamma(\alpha+1))}{\Gamma(\alpha+1)|\Pi|} \left(\frac{|\lambda| |a| (\log \eta)^{p+\alpha}}{\Gamma(p+\alpha+1)} \right) \\
+ \; & \left. \frac{(|\Lambda_2| + |\Lambda_1| \Gamma(\alpha+1))}{\Gamma(\alpha+1)|\Pi|} \left(\frac{|\lambda| |b| (\log \xi)^{q+\alpha}}{\Gamma(q+\alpha+1)} \right) \right\} \|x-y\| \\
= \; & (M\Theta_1 + \Theta_2) \|x-y\| .
\end{aligned}
$$

Using (2.4), we can see that P is a contraction. Consequently, by the contraction mapping principle, problem (1.1) has a uniqueness solution. $\square$

3. Ulam-Hyers stability

Theorem 3.1. *Assume that the assumptions $(A_1), (A_2)$ and (A_3) hold, then problem (1.1) is Ulam-Hyers stable and consequently, generalized Ulam-Hyers stable.*

Proof. Let $v \in W$ be a solution of the inequality (1.2), i.e.

$$\left| {}_{H}^{C}D^{\beta} \left({}_{H}^{C}D^{\alpha} + \lambda \right) v(t) - \varphi(t, v(t)) - I^{\theta} \psi(t, v(t)) \right| \leq \vartheta, t \in J,$$

and let us denote by $u \in W$ the unique solution of the fractional boundary value problem

$$
\begin{cases}
{}_{H}^{C}D^{\beta} \left({}_{H}^{C}D^{\alpha} + \lambda \right) u(t) = \varphi(t, u(t)) + I^{\theta} \psi(t, u(t)), t \in J, 0 < \alpha, \beta < 1 \\
{}^{H}I^{p} u(\eta) = {}^{H}I^{p} v(\eta), {}^{H}I^{q} u(\xi) = {}^{H}I^{q} v(\xi), 1 < \eta, \xi < e
\end{cases}
$$

By using Lemma 1.1 we have

$$u(t) =^H I^{\alpha+\beta} f_u(t) - \lambda^H I^{\alpha} u\,(t) + \frac{c_0}{\Gamma\,(\alpha+1)}\,(\log t)^{\alpha} + c_1$$

and by integration of the inequality (1.2), we obtain

$$\left| v\,(t) -^H I^{\alpha+\beta} f_v(t) + \lambda^H I^{\alpha} v\,(t) - \frac{c_2}{\Gamma\,(\alpha+1)}\,(\log t)^{\alpha} - c_3 \right| \leq \frac{\varepsilon}{\Gamma\,(\alpha+1)}\,(\log t)^{\alpha}.$$

On the other hand, if ${}^H I^p u\,(\eta) = {}^H I^p v\,(\eta)\,, {}^H I^q u\,(\xi) = {}^H I^q v\,(\xi)$, then

$$c_0 = c_2 \text{ and } c_1 = c_3.$$

For any $t \in J$, we have

$$\begin{aligned} v\,(t) - u\,(t) &= v\,(t) -^H I^{\alpha+\beta} f_v(t) + \lambda^H I^{\alpha} v\,(t) - \frac{c_2}{\Gamma\,(\alpha+1)}\,(\log t)^{\alpha} - c_3 \\ &+ {}^H I^{\alpha+\beta}\,(f_v(t) - f_u(t)) - \lambda^H I^{\alpha}\,(v\,(t) - u\,(t))\,, \end{aligned}$$

where

$$f_v(t) = \varphi\,(t, v\,(t)) + {}^H I^{\theta} \psi\,(t, v\,(t))\,,$$

and

$$f_u(t) = \varphi\,(t, u\,(t)) + {}^H I^{\theta} \psi\,(t, u\,(t))\,,$$

then

$$\begin{aligned} & {}^H I^{\alpha+\beta}\,(f_v(t) - f_u(t)) \\ = \; & {}^H I^{\alpha+\beta}[\varphi(s, v(s)) - \varphi(s, u(s))] \\ + \; & {}^H I^{\alpha+\beta+\theta}[\psi(s, v(s)) - \psi(s, u(s))] \\ = \; & \frac{1}{\Gamma(\alpha+\beta)} \int_1^t (\log \frac{t}{s})^{\alpha+\beta-1} [\varphi(s, v(s)) - \varphi(s, u(s))] \frac{ds}{s} \\ + \; & \frac{1}{\Gamma(\alpha+\beta+\theta)} \int_1^t (\log \frac{t}{s})^{\alpha+\beta+\theta-1} [\psi(s, v(s)) - \psi(s, u(s))] \frac{ds}{s}. \end{aligned}$$

Using (A_1) and (A_2) we get

$$\begin{aligned} \left| {}^H I^{\alpha+\beta}\,(\varphi_y(t) - \varphi_x(t)) \right| &\leq \frac{M_1}{\Gamma(\alpha+\beta)} \int_1^t (\log \frac{t}{s})^{\alpha+\beta-1} |v(s) - u(s)| \frac{ds}{s} \\ &+ \frac{M_2}{\Gamma(\alpha+\beta+\theta)} \int_1^t (\log \frac{t}{s})^{\alpha+\beta+\theta-1} |v(s) - u(s)| \frac{ds}{s}. \end{aligned}$$

This yields that

$$|v\,(t) - u\,(t)| \leq \left| v\,(t) -^H I^{\alpha+\beta} f_v(t) + \lambda^H I^{\alpha} v\,(t) - \frac{c_2}{\Gamma\,(\alpha+1)}\,(\log t)^{\alpha} - c_3 \right|$$

$$\begin{aligned} &+ \frac{M_1}{\Gamma(\alpha+\beta)} \int_1^t (\log \frac{t}{s})^{\alpha+\beta-1} |v(s)-u(s)| \frac{ds}{s} \\ &+ \frac{M_2}{\Gamma(\alpha+\beta+\theta)} \int_1^t (\log \frac{t}{s})^{\alpha+\beta+\theta-1} |v(s)-u(s)| \frac{ds}{s} \\ &+ \frac{|\lambda|}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} |v(s)-u(s)| \frac{ds}{s}. \end{aligned}$$

Then

$$\begin{aligned} |v(t)-u(t)| &\leq \frac{\vartheta}{\Gamma(\alpha+1)} + \frac{M_1}{\Gamma(\alpha+\beta)} \int_1^t (\log \frac{t}{s})^{\alpha+\beta-1} \|v(s)-u(s)\| \frac{ds}{s} \\ &+ \frac{M_2}{\Gamma(\alpha+\beta+\theta)} \int_1^t (\log \frac{t}{s})^{\alpha+\beta+\theta-1} \|v(s)-u(s)\| \frac{ds}{s} \\ &+ \frac{|\lambda|}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} \|v(s)-u(s)\| \frac{ds}{s}. \end{aligned}$$

Thus

$$|v(t)-u(t)| \leq \frac{\vartheta}{\Gamma(\alpha+1)} + \widetilde{M} \|v(s)-u(s)\|.$$

Then

$$\|v(s)-u(s)\| \left(1-\widetilde{M}\right) \leq \frac{\vartheta}{\Gamma(\alpha+1)}.$$

Then, for each $t \in [1,e]$

$$|u(t)-v(t)| \leq \frac{\vartheta}{\left(1-\widetilde{M}\right)\Gamma(\alpha+1)} = \mu_{\varphi,\psi}\vartheta.$$

So, the fractional boundary value problem (1.1) is Ulam–Hyers stable. By putting $g(\vartheta) = \mu\vartheta, g(0) = 0$ yields that the fractional boundary value problem (1.1) generalized Ulam-Hyers stable. $\square$

Theorem 3.2. *Assume that the assumptions $(A_1), (A_2)$ and (A_3) hold. In addition, the following assumption holds*

(A_4) : *There exists an function $g \in C([1,e], \mathbb{R}_+)$ and there exists $v_g > 0$ such that for any $t \in J$*

$$\frac{1}{\Gamma(\alpha+\beta)} \int_1^t (\log \frac{t}{s})^{\alpha+\beta-1} g(s) \frac{ds}{s} \leq v_g g(t). \tag{3.1}$$

Then the fractional boundary value problem (1.1) *is Ulam-Hyers-Rassias stable.*

Proof. Let $v \in W$ be a solution of the inequality (1.3), i.e.

$$\left|{}_H^C D^\beta \left({}_H^C D^\alpha + \lambda\right) v(t) - \varphi(t, x(t)) - I^\theta \psi(t, v(t))\right| \leq \vartheta g(t), t \in J,$$

and let us denote by $u \in W$ the unique solution of the fractional boundary value problem

$$\begin{cases} {}_H^C D^{\beta}\left({}_H^C D^{\alpha}+\lambda\right) u\left(t\right)=\varphi\left(t,u\left(t\right)\right)+I^{\theta}\psi\left(t,u\left(t\right)\right), t\in J, 0<\alpha,\beta<1 \\ {}^H I^p u\left(\eta\right)= {}^H I^p v\left(\eta\right), {}^H I^q u\left(\xi\right)= {}^H I^q v\left(\xi\right), 1<\eta,\xi<e \end{cases}$$

By applying Lemma 1.1, we have

$$u(t) = {}^H I^{\alpha+\beta} f_u(t) - \lambda {}^H I^{\alpha} u\left(t\right) + \frac{c_0}{\Gamma\left(\alpha+1\right)}\left(\log t\right)^{\alpha} + c_1,$$

and by integration of the inequality (1.3), we obtain

$$\begin{aligned} &\left| v\left(t\right) - {}^H I^{\alpha+\beta} f_v(t) + \lambda {}^H I^{\alpha} v\left(t\right) - \frac{c_2}{\Gamma\left(\alpha+1\right)}\left(\log t\right)^{\alpha} - c_3 \right| \\ \leq\ & \frac{\vartheta}{\Gamma(\alpha+\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1} g(s)\frac{ds}{s}. \end{aligned}$$

Now, using (A_1) and (A_2), we can write

$$\begin{aligned} \left|v\left(t\right)-u\left(t\right)\right| &\leq \left| v\left(t\right) - {}^H I^{\alpha+\beta} f_v(t) + \lambda {}^H I^{\alpha} v\left(t\right) - \frac{c_2}{\Gamma\left(\alpha+1\right)}\left(\log t\right)^{\alpha} - c_3 \right| \\ &+ \frac{M_1}{\Gamma(\alpha+\beta)}\int_1^t (\log\frac{t}{s})^{\alpha+\beta-1}|v(s)-u(s)|\frac{ds}{s} \\ &+ \frac{M_2}{\Gamma(\alpha+\beta+\theta)}\int_1^t (\log\frac{t}{s})^{\alpha+\beta+\theta-1}|v(s)-u(s)|\frac{ds}{s} \\ &+ \frac{|\lambda|}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}|v(s)-u(s)|\frac{ds}{s}. \end{aligned}$$

Then by (A_4)

$$\begin{aligned} \left|v\left(t\right)-u\left(t\right)\right| &\leq \frac{\vartheta v_g g(t)}{\Gamma(\alpha+\beta)} + \frac{M_1}{\Gamma(\alpha+\beta)}\int_1^t (\log\frac{t}{s})^{\alpha+\beta-1}\left\|v(s)-u(s)\right\|\frac{ds}{s} \\ &+ \frac{M_2}{\Gamma(\alpha+\beta+\theta)}\int_1^t (\log\frac{t}{s})^{\alpha+\beta+\theta-1}\left\|v(s)-u(s)\right\|\frac{ds}{s} \\ &+ \frac{|\lambda|}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}\left\|v(s)-u(s)\right\|\frac{ds}{s}. \end{aligned}$$

Thus

$$\left|v\left(t\right)-u\left(t\right)\right| \leq \frac{\vartheta v_g g(t)}{\Gamma(\alpha+\beta)} + \widetilde{M}\left\|v(s)-u(s)\right\|.$$

Then

$$\left\|v(s)-u(s)\right\|\left(1-\widetilde{M}\right) \leq \frac{\vartheta v_g g(t)}{\Gamma(\alpha+\beta)}$$

Then, for each $t \in [1, e]$

$$|v(t) - u(t)| \leq \frac{\vartheta \upsilon_g}{\left(1 - \widetilde{M}\right) \Gamma(\alpha + \beta)} g(t).$$

So, the fractional boundary value problem (1.1) is Ulam-Hyers-Rassias stable. $\square$

4. Application

Consider the following nonlinear fractional differential equation with Hadamard-Caputo type fractional derivatives

$$\text{(4.1)} \quad \begin{cases} {}^{C}_{H}D^{\frac{3}{5}} \left({}^{C}_{H}D^{\frac{7}{11}} + \frac{1}{32} \right) x(t) = \frac{x(t)-1}{2\ln(t)+21} + {}^{H} I^{\frac{1}{2}} \left(\frac{x(t)e^{1-t}-3}{t+39} \right), t \in [1, e], \\ 3I^{\frac{1}{3}} x\left(\frac{3}{2}\right) = \frac{9}{2} (\ln 3)^{\frac{2}{3}}, -5I^{\frac{9}{10}} x\left(\frac{8}{3}\right) = -\frac{20}{3} (\ln 5)^{\frac{5}{3}} \end{cases}$$

Here $\alpha = \frac{7}{11}, \beta = \frac{3}{5}, \lambda = \frac{1}{32}, \theta = \frac{1}{2}, p = \frac{1}{3}, q = \frac{9}{10}, a = 3, b = -5, \eta = \frac{3}{2}$, $\xi = \frac{8}{3}, f(t, x) = \frac{x(t)-1}{2\ln(t)+21}$ and $g(t, x) = \frac{x(t)e^{1-t}-3}{t+39}, t \in [1, e], x \in \mathbb{R}$.

For each $x, y \in \mathbb{R}$ and $t \in [1, e]$, we have

$$|f(t, x) - f(t, y)| \leq \frac{1}{21} |x - y| \text{ and } |g(t, x) - g(t, y)| \leq \frac{1}{40} |x - y|,$$

then the conditions $(A_1), (A_2)$ are satisfied with

$$M = \max\left\{\frac{1}{21}, \frac{1}{40}\right\} = \frac{1}{21}.$$

We can find that

$$\begin{aligned} \Lambda_1 &= \frac{a}{\Gamma(p+\alpha+1)} (\log \eta)^{p+\alpha} \approx 0.55682, \Lambda_2 = \frac{a}{\Gamma(p+1)} (\ln \eta)^p \approx 1.883036, \\ \Delta_1 &= \frac{b}{\Gamma(q+\alpha+1)} (\ln \xi)^{q+\alpha} \approx -0.987821, \Delta_2 = \frac{b}{\Gamma(q+1)} (\ln \xi)^q \approx -2.411796, \\ \Pi &= \Lambda_1 \Delta_2 - \Lambda_2 \Delta_1 = 0.517168, \end{aligned}$$

$$\begin{aligned} \Theta_1 \ := &\ \frac{1}{\Gamma(\alpha+\beta+1)} + \frac{1}{\Gamma(\alpha+\beta+\theta+1)} \\ + &\ \frac{|\Delta_2| + |\Delta_1| \Gamma(\alpha+1)}{\Gamma(\alpha+1) |\Pi|} \left(\frac{|a| (\log \eta)^{p+\alpha+\beta}}{\Gamma(p+\alpha+\beta+1)} + \frac{|a| (\log \eta)^{p+\alpha+\beta+\theta}}{\Gamma(p+\alpha+\beta+\theta+1)} \right) \\ + &\ \frac{|\Lambda_2| + |\Lambda_1| \Gamma(\alpha+1)}{\Gamma(\alpha+1) |\Pi|} \left(\frac{|b| (\log \xi)^{q+\alpha+\beta}}{\Gamma(q+\alpha+\beta+1)} + \frac{|b| (\log \xi)^{q+\alpha+\beta+\theta}}{\Gamma(q+\alpha+\beta+\theta+1)} \right) \\ \approx &\ 8.712578, \end{aligned}$$

$$\Theta_2 := \frac{|\lambda|}{\Gamma(\alpha+1)} + \frac{(|\Delta_2| + |\Delta_1| \Gamma(\alpha+1))}{\Gamma(\alpha+1)|\Pi|}\left(\frac{|\lambda||a|(\log\eta)^{p+\alpha}}{\Gamma(p+\alpha+1)}\right)$$
$$+ \frac{(|\Lambda_2| + |\Lambda_1| \Gamma(\alpha+1))}{\Gamma(\alpha+1)|\Pi|}\left(\frac{|\lambda||b|(\log\xi)^{q+\alpha}}{\Gamma(q+\alpha+1)}\right) \approx 0.318327.$$

Therefor, we have

$$\Theta_1 M \approx 0.414\,88 < 1 - \Theta_2 \approx 0.681\,67.$$

Moreover condition A_3

$$\widetilde{M} = \frac{M1}{\Gamma(\alpha+\beta+1)} + \frac{M2}{\Gamma(\alpha+\beta+\theta+1)} + \frac{\lambda}{\Gamma(\alpha+1)} = 9.2865 \times 10^{-2} < 1,$$

is satisfied Hence, by the Theorem 2.1, problem (P) has a unique solution on $[1, e]$, and by Theorem 3.1 problem (P) is Ulam-Hyers stable.

Also, the hypothesis (A_4) is satisfied with $g(t) = \mu, \mu > 0$ and $\upsilon_g = \frac{1}{\Gamma(\alpha+\beta+1)}$ Indeed, for each $t \in [1, e]$, we get

$${}^H I^{\alpha+\beta} g(t) = {}^H I^{\alpha+\beta}(\mu) = \frac{\mu}{\Gamma(\alpha+\beta+1)}(\log t)^{\alpha+\beta} \leq \frac{\mu}{\Gamma(\alpha+\beta+1)} = \upsilon_g g(t).$$

Consequently, Theorem 3.2 implies that (P) is Ulam-Hyers-Rassias stable.

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