

AN EXAMINATION OF THE CONDITION UNDER WHICH A CONCHOIDAL SURFACE IS A BONNET SURFACE IN THE EUCLIDEAN 3-SPACE

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Abstract. In this study, we examine the condition of the conchoidal surface to be a Bonnet surface in Euclidean 3-space. Especially, we consider the Bonnet conchoidal surfaces which admit an infinite number of isometries. In addition, we study the necessary conditions which have to be fulfilled by the surface of revolution with the rotating curve $c(t)$ and its conchoid curve $c_d(t)$ to be the Bonnet surface in Euclidean 3-space.

Keywords. Conchoidal surface, Bonnet surface, Euclidean 3-space.

1. Introduction

The conchoid of Nicomedes, which is called by the Greek geometer Nicomedes's name, was originally contrived around 200 BC to trisect an angle and duplicate the cube. For any curve and a fixed point, let a straight line, which meets the curve at the point Q , is drawn through the fixed point. If P and R are points on this line such that $RQ = QP = \text{const.}$, then the conchoid of curve with respect to the fixed point is the locus of P and R [12].

The conchoids play an important role in many applications as the construction of buildings, astronomy [9], optics [2], physics [19]. Although the

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conchoidal constructions were extensively mentioned by the ancient Greeks in the seventeenth century, they have been recently addressed by different authors, too. One of these has been put forward by Odehnal. He obtained a generalized conchoid transformation considering a construction with the help of cross ratios [13]. Moreover, Peternel, etc. presented the conchoidal surface of rational ruled surfaces, the conchoidal surfaces of spheres, the conchoids and the pedal surfaces [15, 16, 17].

Surfaces, which admit a one-parameter family of isometries preserving the mean curvature, have been proposed by Bonnet and although Bonnet raised these surfaces [3], the term “Bonnet surface” was firstly used by Lalan [11]. Bonnet showed that all surfaces with the constant mean curvature can be isometrically mapped to each other and the deformable surfaces with the non-constant mean curvature are the isothermic Weingarten surfaces which can be deformable to the revolution surfaces. After that, many mathematicians have contributed these surfaces [18, 10, 7, 1].

Bonnet surfaces may be broken up into three types which is described as follows:

- (i) Surfaces of the constant mean curvature other than the plane or the sphere.
- (ii) Isothermic Weingarten surfaces of the non-constant mean curvature which admit a one parameter family of geometrically distinct non-trivial isometries.
- (iii) Surfaces of the non-constant mean curvature that admit a single non-trivial isometry [10].

In [4], the authors studied the conchoidal surfaces, the surfaces of revolution given with the conchoid curve and their geometrical properties in Euclidean 3-space. In our work, using the geometric properties obtained for conchoidal surfaces in reference [4], we have examined the conditions under which the conchoidal surface and the surface of revolution given with conchoid curve is a Bonnet surface in Euclidean 3- space. According to that, we get the following results:

- (1) If a regular surface M and a conchoidal surface M_d are minimal, then they are the surfaces of the type (i) which can be recognised by an infinite number of isometries preserving the principal curvatures.
- (2) The surfaces M with the radius function $r(u_0, v)$ or $r(u, v_0)$ are the surfaces of the type (ii) which admit an infinite number of isometries. Also, the result is similar for the conchoidal surfaces M_d .
- (3) If a regular surface M and a conchoidal surface M_d , which are the surfaces of revolution generated by the rotating curve and its conchoid curve, are minimal, then they are the surfaces of the type (i) which can be recognised by an infinite number of isometries preserving the principal curvatures.
- (4) If a regular surface M and a conchoidal surface M_d , which are the surfaces of revolution generated by the rotating curve and its conchoid curve with the radius function $r(u_0, v)$ or $r(u, v_0)$, are the surfaces of the type (ii) which admit an infinite number of isometries.

2. Preliminaries

Let M be a smooth surface in \mathbb{E}^3 given with the patch $X(u, v)$ for

$(u, v) \in D \subset E^3$. The tangent space to M at an arbitrary point p of M is spanned by $\{X_u, X_v\}$. Let N be the unit normal vector field of the surface M defined by $N = \frac{X_u \times X_v}{\|X_u \times X_v\|}$. The first fundamental form I and the second fundamental form II of the surface M are

$$(2.1) \quad I = edu^2 + 2fdudv + gdv^2, \quad II = ldu^2 + 2mdudv + ndv^2,$$

respectively, where

$$(2.2) \quad e = \langle X_u, X_u \rangle, \quad f = \langle X_u, X_v \rangle, \quad g = \langle X_v, X_v \rangle,$$

and

$$(2.3) \quad l = \langle X_{uu}, N \rangle, \quad m = \langle X_{uv}, N \rangle, \quad n = \langle X_{vv}, N \rangle.$$

In [8], the Gaussian curvature K and the mean curvature H are

$$(2.4) \quad K = \frac{ln - m^2}{eg - f^2}, \quad H = \frac{en - 2fm + gl}{2(eg - f^2)}.$$

A surface M in E^3 is called Weingarten surface if there exists a non-trivial functional relation

$$(2.5) \quad \Omega(K, H) = 0$$

with respect to its Gaussian curvature K and its mean curvature H , where Ω is the Jakobian determinant [14].

If a surface M in E^3 has the coefficients of first fundamental form which satisfy the conditions $e = g, f = 0$, then it is called isothermic [5]. According to [18], the isothermic surface provides the condition

$$(2.6) \quad \frac{\partial^2}{\partial u \partial v} \left(\log \frac{g}{e} \right) = 0.$$

We assume a smooth surface $M \subset E^3$ and a fixed reference point O which can be considered as the origin of a cartesian coordinate system. Let M is described by a polar representation

$$(2.7) \quad X(u, v) = r(u, v)s(u, v)$$

with $\|s(u, v)\| = 1$. Considering $s(u, v) = (\cos u \cos v, \sin u \cos v, \sin v)$ of the unit sphere S^2 , so $s(u, v)$ and $r(u, v)$ are called spherical part and radius function of $X(u, v)$, respectively.

In [17, 15], the one-sided conchoidal surface M_d of M is derived by adding $d \in \mathbb{R}$ to the radius function $r(u, v)$ and thus M_d admits the polar representation

$$(2.8) \quad M_d(u, v) = (r(u, v) + d)s(u, v).$$

Let M be a regular surface given with the parametrization (2.7). Then the coefficients of the first fundamental form of the surface M are

$$(2.9) \quad \begin{aligned} e &= r^2 \cos^2 v + r_u^2, \\ f &= r_u r_v, \\ g &= r^2 + r_v^2. \end{aligned}$$

Additionally, its Gaussian curvature and its mean curvature are

$$(2.10) \quad \begin{aligned} K &= -\frac{1}{r^2 A^2} [rr_{uv} \cos v - 2r_u r_v \cos v + rr_u \sin v]^2 \\ &\quad - \cos^2 v (2r_u^2 + rr_v \sin v \cos v + r^2 \cos^2 v - rr_{uu}) (2r_v^2 + r^2 - rr_{vv}), \end{aligned}$$

and

$$(2.11) \quad \begin{aligned} H &= -\frac{1}{2r^2 A^{3/2}} [\cos v (2r_u^2 + rr_v \sin v \cos v + r^2 \cos^2 v - rr_{uu}) (r^2 + r_v^2) \\ &\quad + \cos v (2r_v^2 + r^2 - rr_{vv}) (r^2 \cos^2 v + r_u^2) \\ &\quad + 2r_u r_v (rr_{uv} \cos v - 2r_u r_v \cos v + rr_u \sin v)], \end{aligned}$$

where $A = (r^2 + r_v^2) \cos^2 v + r_u^2$. Also, if M_d is a conchoidal surface given with the parametrization (2.8), its Gaussian curvature and its mean curvature are

$$(2.12) \quad \begin{aligned} \tilde{K} &= -\frac{1}{(r \pm d)^2 A^2} [(r \pm d)r_{uv} \cos v - 2r_u r_v \cos v + (r \pm d)r_u \sin v]^2 \\ &\quad - \cos^2 v (2r_u^2 + (r \pm d)r_v \sin v \cos v \\ &\quad + (r \pm d)^2 \cos^2 v - (r \pm d)r_{uu}) (2r_v^2 + (r \pm d)^2 - (r \pm d)r_{vv}), \end{aligned}$$

and

$$(2.13) \quad \begin{aligned} \tilde{H} &= -\frac{1}{2(r \pm d)^2 A^{3/2}} [\cos v (2r_u^2 + (r \pm d)r_v \sin v \cos v \\ &\quad + (r \pm d)^2 \cos^2 v - (r \pm d)r_{uu}) ((r \pm d)^2 + r_v^2) \\ &\quad + \cos v (2r_v^2 + (r \pm d)^2 - (r \pm d)r_{vv}) ((r \pm d)^2 \cos^2 v + r_u^2) \\ &\quad + 2r_u r_v ((r \pm d)r_{uv} \cos v - 2r_u r_v \cos v + (r \pm d)r_u \sin v)], \end{aligned}$$

where $A = ((r \pm d)^2 + r_v^2) \cos^2 v + r_u^2$ [4].

Let M be a surface of revolution generated by the rotating curve $c(t)$. The surface is given with the surface patch

$$(2.14) \quad X(t, s) = (r(t) \cos t, r(t) \sin t \cos s, r(t) \sin t \sin s),$$

where $c(t) = r(t)(\cos t, \sin t)$. The coefficients of the first fundamental form of the surface M hold:

$$(2.15) \quad \begin{aligned} e &= r^2 + (r')^2, \\ f &= 0, \\ g &= r^2 \sin^2 t. \end{aligned}$$

The Gaussian and mean curvatures of the surface M are as follows:

$$(2.16) \quad K = \frac{(r' \cos t - r \sin t)(rr'' - 2(r')^2 - r^2)}{r \sin t(r^2 + (r')^2)^3},$$

and

$$(2.17) \quad H = \frac{r \sin t(rr'' - 2(r')^2 - r^2) + (r^2 + (r')^2)(r' \cos t - r \sin t)}{2r \sin t(r^2 + (r')^2)^{3/2}},$$

respectively. Let M_d be a surface of revolution generated by the conchoid curve $c_d(t)$. The surface is parametrized by

$$(2.18) \quad \tilde{X}(t, s) = ((r(t) \pm d) \cos t, (r(t) \pm d) \sin t \cos s, (r(t) \pm d) \sin t \sin s),$$

where $c_d(t) = (r(t) \pm d)(\cos t, \sin t)$. The coefficients of the first fundamental form of the surface M_d are calculated as

$$(2.19) \quad \begin{aligned} \tilde{e} &= (r(t) \pm d)^2 + (r')^2, \\ \tilde{f} &= 0, \\ \tilde{g} &= (r(t) \pm d)^2 \sin^2 t. \end{aligned}$$

The Gaussian and mean curvatures of the surface M_d become

$$(2.20) \quad \tilde{K} = \frac{(r' \cos t - (r(t) \pm d) \sin t)((r(t) \pm d)r'' - 2(r')^2 - (r(t) \pm d)^2)}{(r(t) \pm d) \sin t((r(t) \pm d)^2 + (r')^2)^3},$$

$$(2.21) \quad \begin{aligned} \tilde{H} &= \frac{(r(t) \pm d) \sin t((r(t) \pm d)r'' - 2(r')^2 - (r(t) \pm d)^2)}{2(r(t) \pm d) \sin t((r(t) \pm d)^2 + (r')^2)^{3/2}} \\ &+ \frac{((r(t) \pm d)^2 + (r')^2)(r' \cos t - (r(t) \pm d) \sin t)}{2(r(t) \pm d) \sin t((r(t) \pm d)^2 + (r')^2)^{3/2}}, \end{aligned}$$

respectively [4].

3. Discussion and Conclusion

3.1. An examination of the condition of the conchoidal surface to be a Bonnet surface in E^3

In this section, we will examine condition which is the conchoidal surface to be a Bonnet surface in Euclidean 3-space. Especially, we will deal with the conchoidal surfaces admitting an infinite number of isometries. Thus, it will be sufficient to determine: (a) the conchoidal surfaces of the constant mean curvature and (b) the isothermic Weingarten conchoidal surfaces.

(a) **The conchoidal surfaces of the constant mean curvature**

Let M be a regular surface given with the parametrization (2.7). It is possible that the mean curvature H given by (2.11) is equal to a non-zero constant when the radius function $r(u, v)$ is a constant. This means that the surface M is a sphere.

Example 3.1. Let the radius function be a constant. For $r(u, v) = 3$ and $d = 1$, the conchoidal surface M_d is given by the parametrization

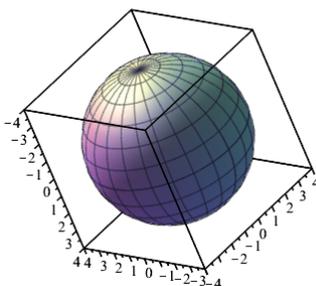


Figure 3.1: Conchoidal surface with $r(u, v) = 3$ and $d = 1$

$$(3.1) \quad X_d(u, v) = (4 \cos u \cos v, 4 \sin u \cos v, 4 \sin v).$$

It denotes a sphere as given in Figure 3.1.

The mean curvature is a constant when the surface M is minimal, except that the radius function is a constant. In this case, considering [4], if u -parameter radius function is

$$(3.2) \quad r(u) = \pm \frac{\sqrt{\cos v}}{\sqrt{c_1 \sin(2u \cos v) - c_2 \cos(2u \cos v)}}$$

or if v -parameter radius function is

$$(3.3) \quad r(v) = \frac{1}{c_1 \sin v},$$

where c_1, c_2 are constants, then M is the minimal surface. So, the surfaces M determined by (3.2) and (3.3) are the surfaces of the type (i) which can be recognised by an infinite number of isometries preserving the principal curvature.

Similar results for conchoidal surface M_d are obtained as follows:

If the radius function is a constant, the mean curvature \tilde{H} of the conchoidal surface is equal to $\frac{1}{r \pm d}$. This means that the surface M_d is a sphere. If u -parameter

radius function is

$$(3.4) \quad r(u) = \pm \frac{\sqrt{\cos v}}{\sqrt{c_1 \sin(2u \cos v) - c_2 \cos(2u \cos v)}} \pm d$$

or if v -parameter radius function is

$$(3.5) \quad r(v) = \mp d + \frac{1}{c_1 \sin v},$$

where c_1, c_2 are constants, then the surface M_d is minimal. So, the conchoidal surfaces M_d determined by (3.4) and (3.5) are the conchoidal surfaces of the type (i) which can be recognised by an infinite number of isometries preserving the principal curvature.

Example 3.2. Let the radius function is given by

$$(3.6) \quad r(u) = \frac{\sqrt{\cos v}}{\sqrt{\sin(2u \cos v) - \cos(2u \cos v)}}$$

and $d = -1$. Then, the conchoidal surface M_d is parametrized by

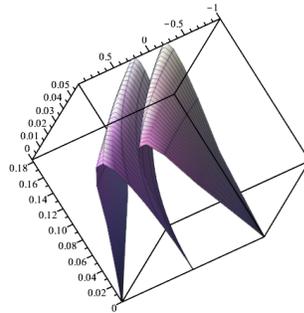


Figure 3.2: Conchoidal surface with $r(u)$ and $d = -1$

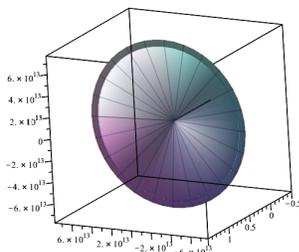
$$(3.7) \quad X_d(u, v) = (r(u) - 1)(\cos u \cos v, \sin u \cos v, \sin v).$$

It is shown as given in Figure 3.2.

Example 3.3. Let the radius function is given by $r(v) = \frac{1}{2 \sin v}$ and $d = -1$. Then, the conchoidal surface M_d is parametrized by

$$(3.8) \quad X_d(u, v) = \left(\frac{1}{2 \sin v} - 1 \right) (\cos u \cos v, \sin u \cos v, \sin v).$$

It is shown as given in Figure 3.3.

Figure 3.3: Conchoidal surface with $r(v)$ and $d = -1$

(b) The isothermic Weingarten conchoidal surfaces of the non-constant mean curvature

Firstly, let's calculate the condition which is satisfied by the surface M to be an isothermal surface. When the curves of an orthogonal system have the constant geodesic curvature, the system is an isothermal [6]. For this, we assume that the parameter curves of the surface M constitute the orthogonal system, namely, $\langle X_u, X_v \rangle = 0$. When the surface is assigned by these parametric curves and the linear element is written $ds^2 = edu^2 + gdv^2$, from [6], the condition that the geodesic curvature is a constant becomes $\frac{\partial^2}{\partial u \partial v} \left(\log \frac{g}{e} \right) = 0$.

When the parameter curves are orthogonal, $\langle X_u, X_v \rangle = r_u r_v = 0$. This means that $r_u = 0$ or $r_v = 0$. Therefore the parametric curves of the conchoidal surface M_d are orthogonal. Thus, when the surface M is isothermal, the obtained cases are valid for the conchoidal surface M_d . So, we have the following cases:

Case 1: We assume that $r_u = 0$ and $r_v \neq 0$. In order to examine whether the surface M with the radius function $r(u_0, v)$ is a Bonnet surface, we will work the isothermic Weingarten surfaces.

Using (2.9) into (2.6), then we obtain as follows:

$$(3.9) \quad \frac{\partial^2}{\partial u \partial v} \left(\log \frac{r^2 + r_v^2}{r^2 \cos^2 v} \right) = 0.$$

From (3.9), we conclude that the surface M with the radius function $r(u_0, v)$ is the isothermal surface.

Secondly, we investigate the necessary conditions for the surface M to be a Weingarten surface. Differentiating (2.10) and (2.11) with respect to u and considering $r_u = 0$, then we find $\frac{\partial K}{\partial u} = 0$ and $\frac{\partial H}{\partial u} = 0$. Hence, the surface M with the radius function $r(u_0, v)$ is the Weingarten surface. Additionally, from (2.11), we see that the mean curvature of the surface M with the radius function $r(u_0, v)$ is the non-constant.

As a result, since the surface M is both the isothermal and Weingarten surface with the non-constant mean curvature, then it has an infinite number of the Bonnet nets. Thus, the following theorem is given.

Theorem 3.1. *The surface M with the radius function $r(u_0, v)$ is a surface of the type (ii) which admits an infinite number of isometries. So, this surface is a Bonnet surface.*

Let M_d be a conchoidal surface of M given with the parametrization (2.8). If the radius function $r(u, v)$ is a v -parameter function, then the coefficients of the first fundamental form of the surface M_d are

$$(3.10) \quad \begin{aligned} \tilde{e} &= (r \pm d)^2 \cos^2 v, \\ \tilde{f} &= 0, \\ \tilde{g} &= (r \pm d)^2 + r_v^2. \end{aligned}$$

Considering these coefficients, the conchoidal surface M_d of M with the radius function $r(u_0, v)$ is the isothermic surface, since we get

$$(3.11) \quad \frac{\partial^2}{\partial u \partial v} \left(\log \frac{(r \pm d)^2 + r_v^2}{(r \pm d)^2 \cos^2 v} \right) = 0.$$

To determine the necessary condition to be a Weingarten surface of M_d , we have (2.12) and (2.13) for $r_u = 0$. From $\frac{\partial \tilde{K}}{\partial u} = 0$ and $\frac{\partial \tilde{H}}{\partial u} = 0$, the conchoidal surface M_d of M with the radius function $r(u_0, v)$ is the Weingarten surface. From (2.13), it is easily seen that $\tilde{H} \neq const$. Therefore, the following theorem is given for the conchoidal surface M_d .

Theorem 3.2. *The conchoidal surface M_d with the radius function $r(u_0, v)$ is a surface of the type (ii) which admits an infinite number of isometries. So, this surface is a Bonnet surface.*

Corollary 3.1. *There is no surfaces M and M_d that admits a single non-trivial isometry with the non-constant mean curvature.*

Example 3.4. Let the radius function is given by $r(v) = \frac{1}{\cos v}$ and $d = 2$. Then, the conchoidal surface M_d is parametrized by

$$(3.12) \quad X_d(u, v) = \left(\frac{1}{\cos v} + 2 \right) (\cos u \cos v, \sin u \cos v, \sin v).$$

It is a Bonnet surface and shown as given in Figure 3.4.

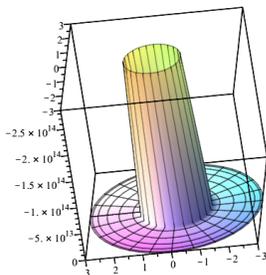


Figure 3.4: Conchoidal surface with $r(v) = \frac{1}{\cos v}$ and $d = 2$

Case 2: We assume that $r_v = 0$ and $r_u \neq 0$. In order to examine whether the surface M with the radius function $r(u, v_0)$ is a Bonnet surface, we will study this kind of surface to be the isothermic Weingarten surface.

Using (2.9) into (2.6), then we obtain as follows:

$$(3.13) \quad \frac{\partial^2}{\partial u \partial v} \left(\log \frac{r^2}{r^2 \cos^2 v + r_u^2} \right) = \frac{2rr_u \sin 2v (r_u^2 - rr_{uu})}{(r^2 \cos^2 v + r_u^2)^2}.$$

For $\frac{\partial^2}{\partial u \partial v} \left(\log \frac{g}{e} \right) = 0$, there exists $r_u^2 - rr_{uu} = 0$ from (3.13), that is, the surface M admitting $r_u^2 - rr_{uu} = 0$ is an isothermic surface. When we solve this differential equation, we find $r(u) = e^{c_1 u} c_2$, where c_1, c_2 are constants. Thus, the following theorem can be written.

Theorem 3.3. *The surface M with the radius function $r(u, v_0)$ is an isothermic surface if and only if it is parametrized by*

$$(3.14) \quad X(u, v) = e^{c_1 u} c_2 (\cos u \cos v, \sin u \cos v, \sin v).$$

Let M_d be a conchoidal surface of M given with the parametrization (2.8). If the radius function $r(u, v)$ is a u -parameter function, then the coefficients of the first fundamental form of the surface M_d are

$$(3.15) \quad \begin{aligned} \tilde{e} &= (r \pm d)^2 \cos^2 v + r_u^2, \\ \tilde{f} &= 0, \\ \tilde{g} &= (r \pm d)^2. \end{aligned}$$

Considering these coefficients for the conchoidal surface M_d of M with the radius function $r(u, v_0)$, we get

$$(3.16) \quad \frac{\partial^2}{\partial u \partial v} \left(\log \frac{(r \pm d)^2}{(r \pm d)^2 \cos^2 v + r_u^2} \right) = \frac{2(r \pm d)r_u \sin 2v (r_u^2 - (r \pm d)r_{uu})}{((r \pm d)^2 \cos^2 v + r_u^2)^2}.$$

For $\frac{\partial^2}{\partial u \partial v} \left(\log \frac{g}{e} \right) = 0$, there exists $r_u^2 - (r \pm d)r_{uu} = 0$ from (3.16), that is, the surface M admitting $r_u^2 - (r \pm d)r_{uu} = 0$ is an isothermic surface. Solving this differential equation, then we obtain $r(u) = e^{c_1 u} c_2 \mp d$, where c_1, c_2 are constants. Thus, the following theorem can be written.

Theorem 3.4. *The conchoidal surface M_d with the radius function $r(u, v_0)$ is an isothermic surface if and only if it is parametrized by*

$$(3.17) \quad X_d(u, v) = (e^{c_1 u} c_2 \mp d)(\cos u \cos v, \sin u \cos v, \sin v).$$

Secondly, we investigate the necessary condition for the surface M to be a Weingarten surface, namely $\frac{\partial K}{\partial u} \frac{\partial H}{\partial v} - \frac{\partial K}{\partial v} \frac{\partial H}{\partial u} = 0$. Differentiating (2.10), (2.11) and considering $r_v = 0$, then we get

$$(3.18) \quad \frac{\partial K}{\partial u} \frac{\partial H}{\partial v} - \frac{\partial K}{\partial v} \frac{\partial H}{\partial u} = \frac{2c_1^3 \sin v (-\cos^4 v + 2 \cos^2 v + c_1^2)}{c_2^3 e^{3c_1 u} (\cos^2 v + c_1^2)^{7/2}}.$$

If (3.18) is equal to zero, then $(\cos^2 v - 1)^2 = c_1^2 + 1$. Thus, $\cos v$ is a constant and this contradicts with M , which is defined (3.14), being a surface. There is no surface M given by (3.14) that is a Weingarten surface and so, the surface M with the radius function $r(u, v_0)$ is not a Bonnet surface. When we examine the conchoidal surface M_d , we get similar results. There is no surface M_d given by (3.17) that is a Weingarten surface and so, the surface M_d with the radius function $r(u, v_0)$ is not a Bonnet surface.

Example 3.5. Let the radius function is given by $r(u) = 2e^u$ and $d = 1$. Then, the conchoidal surface M_d is parametrized by

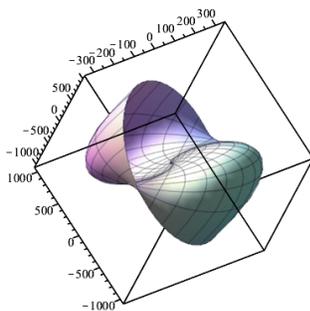


Figure 3.5: Conchoidal surface with $r(u) = 2e^u$ and $d = 1$

$$(3.19) \quad X_d(u, v) = (2e^u + 1)(\cos u \cos v, \sin u \cos v, \sin v).$$

It is the isothermic surface, however it is not the Weingarten surface. Thus, it is not a Bonnet surface and it is shown as given in Figure 3.5.

3.2. An examination of the condition of the surface of revolution given with conchoid curve to be a Bonnet surface in \mathbb{E}^3

In this section, we will examine condition which is the surface of revolution given with the rotating curve $c(t)$ and its the conchoid curve $c_d(t)$ to be a Bonnet surface.

(a) The surfaces of revolution of the constant mean curvature

Assume that M and M_d are the surfaces of revolution generated by the rotating curve $c(t)$ and its conchoid curve $c_d(t)$ parametrized by (2.14) and (2.18). It is possible that the mean curvature H given by (2.17) is equal to a non-zero constant when the radius function $r(t)$ is a constant. This means that the surfaces M and M_d are the spheres.

Example 3.6. Let M_d be a surface of revolution generated by the conchoid curve $c_d(t) = 5$. Then, its parametrization is given by

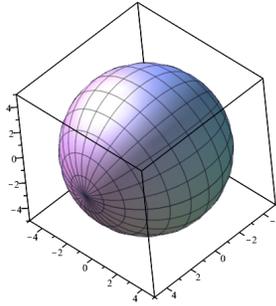


Figure 3.6: Surface of revolution with a constant radius function

$$(3.20) \quad X_d(t, s) = (5 \cos t, 5 \sin t \cos s, 5 \sin t \sin s).$$

It denotes a sphere and it is shown as given in Figure 3.6.

Their mean curvatures are constants when the surfaces M and M_d are the minimal surfaces. According to that, considering [4], if the radius function is $r(t) = \frac{c}{\cos t}$, the surface M is a minimal and if the radius function is $r(t) = \pm d + \frac{c}{\cos t}$, the surface M_d is a minimal. So, the surfaces M and M_d are the surfaces of the type (i) which can be recognised by an infinite of isometries preserving the principal curvatures where M is determined by (2.14) with $r(t) = \frac{c}{\cos t}$ and M_d is determined by (2.18) with $r(t) = \pm d + \frac{c}{\cos t}$.

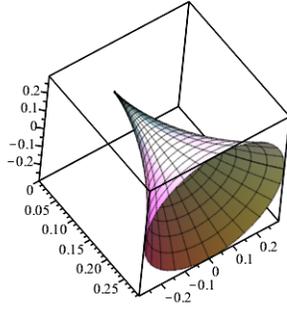


Figure 3.7: Surface of revolution with $c_d(t) = \left(\frac{1}{\cos t} - 1\right) (\cos t, \sin t)$

Example 3.7. Let M_d be a surface of revolution generated by the conchoid curve $c_d(t) = \left(\frac{1}{\cos t} - 1\right) (\cos t, \sin t)$. Then, its parametrization is given by

$$(3.21) \quad X_d(t, s) = \left(\frac{1}{\cos t} - 1\right) (\cos t, \sin t \cos s, \sin t \sin s).$$

It is shown as given in Figure 3.7.

(b) The isothermic Weingarten surface of revolution of the non-constant mean curvature

According to (2.15), from $f = 0$, we see that the parameter curves of the surface M constitute the orthogonal system. Similarly, from $\tilde{f} = 0$, the parameter curves of the surface of revolution M_d are the orthogonal system.

Firstly, we consider the surface providing the condition $\frac{\partial^2}{\partial t \partial s} \left(\log \frac{g}{e}\right) = 0$ since every Bonnet surface is an isothermic surface. For the surface M , using (2.15), then we have $\frac{\partial^2}{\partial t \partial s} \left(\log \frac{r^2 \sin^2 t}{r^2 + (r')^2}\right) = 0$.

Then, we need to show the necessary condition for the surface of revolution M to be a Weingarten surface. From (2.5), (2.16) and (2.17), we find $\frac{\partial K}{\partial s} = 0$ and $\frac{\partial H}{\partial s} = 0$. So, the surface of revolution M is the isothermic Weingarten surface.

Using (2.17), we realize that the mean curvature of the surface M is a non-constant. Hence, the surface of revolution M generated by the rotating curve $c(t)$ with the non-constant mean curvature is the Bonnet surface since it is the isothermic Weingarten surface. Also, if we study the surface of revolution M_d generated by the conchoid curve $c_d(t)$ with the help of the above calculations, then we conclude that the surface M_d is the Bonnet surface.

Theorem 3.5. *The surface of revolution M parametrized by (2.14) and the surface of revolution M_d parametrized by (2.18) are the surfaces of the type (ii) which admit an infinite number of isometries. So, the surfaces of revolution M and M_d are the Bonnet surfaces.*

Corollary 3.2. *There is no surface of revolution given with the conchoid curve that permits a single non-trivial isometry with the non-constant mean curvature.*

Example 3.8. Let M_d be a surface of revolution generated by the conchoid curve $c_d(t) = (2 \sin t + 2) (\cos t, \sin t)$. Then, its parametrization is given by

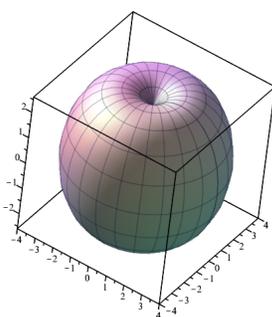


Figure 3.8: Surface of revolution with $c_d(t) = (2 \sin t + 2) (\cos t, \sin t)$

$$(3.22) \quad X_d(t, s) = (2 \sin t + 2) (\cos t, \sin t \cos s, \sin t \sin s).$$

It is shown as given in Figure 3.8 and it is a Bonnet surface.

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