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A STUDY OF THE MATRIX CLASSES (c_0, c) AND $(c_0, c; P)$

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Abstract. In this paper, entries of sequences, infinite series and infinite matrices are real or complex numbers. We prove some interesting properties of the matrix classes (c_0, c) and $(c_0, c; P)$.
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1. Introduction and Preliminaries

We need the following sequence spaces in the sequel:

$$c_0 = \left\{ x = \{x_k\} / \lim_{k\to\infty} x_k = 0 \right\};$$
$$c = \left\{ x = \{x_k\} / \lim_{k\to\infty} x_k \text{ exists} \right\}.$$

We know that c_0 and c are Banach spaces under the norm

$$\|x\| = \sup_{k\geq 0} |x_k|, x = \{x_k\} \in c_0 \text{ or } c.$$

Let $A = (a_{nk})$, $n, k = 0, 1, 2, \ldots$ be an infinite matrix. Then we write $A \in (c_0, c)$ if

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, n = 0, 1, 2, \ldots$$

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is defined and the sequence $A(x) = \{(Ax)_n\} \in c$, whenever $x = \{x_k\} \in c_0$. $A(x)$ is called the A-transform of $x = \{x_k\}$. We write $A \in (c_0, c; P)$ if $A \in (c_0, c)$ and

$$\lim_{n\to\infty}(Ax)_n = \lim_{k\to\infty} x_k = 0, x = \{x_k\} \in c_0.$$

The following results can be easily proved.

Theorem 1.1. [2] $A = (a_{nk}) \in (c_0, c)$ *if and only if*

$$\sup_{n\geq 0}\sum_{k=0}^{\infty}|a_{nk}| < \infty; \tag{1.1}$$

and

$$\lim_{n\to\infty} a_{nk} = \delta_k \text{ exists}, k = 0, 1, 2, \ldots. \tag{1.2}$$

Further, $A \in (c_0, c; P)$ *if and only if (1.1) holds and*

$$\lim_{n\to\infty} a_{nk} = 0, k = 0, 1, 2, \ldots. \tag{1.3}$$

The following definitions are needed ([1]).

Definition 1.1. Given the infinite matrices $A = (a_{nk})$, $B = (b_{nk})$, we define

$$(A * B)_{nk} = \sum_{i=0}^{k} a_{ni}b_{n,k-i}, n, k = 0, 1, 2, \ldots. \tag{1.4}$$

$A * B = ((A * B)_{nk})$ is called the "first convolution" of A and B;

$$(A * *B)_{nk} = \frac{1}{k+1}\sum_{i=0}^{k} a_{ni}b_{n,k-i}, n, k = 0, 1, 2, \ldots. \tag{1.5}$$

$A * *B = ((A * *B)_{nk})$ is called the "second convolution" of A and B.

2. Main Results

We now have

Theorem 2.1. (c_0, c) *is a Banach space under the norm*

$$\|A\| = \sup_{n\geq 0}\sum_{k=0}^{\infty}|a_{nk}|, A = (a_{nk}) \in (c_0, c). \tag{2.1}$$

Proof. We can check that $\|\cdot\|$, defined by (2.1), is indeed a norm. We will prove that (c_0, c) is complete with respect to the norm defined by (2.1). To this end, let $\{A^{(n)}\}$ be a Cauchy sequence in (c_0, c), where

$$A^{(n)} = (a_{ij}^{(n)}), i, j = 0, 1, 2, \ldots; n = 0, 1, 2, \ldots.$$

Since $\{A^n\}$ is Cauchy, for $\epsilon > 0$, there exists a positive integer n_0 such that

$$\|A^{(m)} - A^{(n)}\| < \epsilon, m, n \geq n_0,$$

$$\text{i.e., } \sup_{i\geq 0} \sum_{j=0}^{\infty} |a_{ij}^{(m)} - a_{ij}^{(n)}| < \epsilon, m, n \geq n_0. \tag{2.2}$$

Thus, for all $i, j = 0, 1, 2, \ldots$,

$$|a_{ij}^{(m)} - a_{ij}^{(n)}| < \epsilon, m, n \geq n_0. \tag{2.3}$$

So, $\{a_{ij}^{(n)}\}_{n=0}^{\infty}$ is a Cacuhy sequence of real (or complex) numbers. Since the field of real (or complex) numbers is complete,

$$a_{ij}^{(n)} \to a_{ij}, n \to \infty,$$

where a_{ij} is a real (or complex) number, $i, j = 0, 1, 2, \ldots$. Consider the infinite matrix $A = (a_{ij})$. From (2.2), we get, for all $i = 0, 1, 2, \ldots$,

$$\sum_{j=0}^{J} |a_{ij}^{(m)} - a_{ij}^{(n)}| < \epsilon, m, n \geq n_0, J = 0, 1, 2, \ldots. \tag{2.4}$$

Now, for all $n \geq n_0$, allowing $m \to \infty$ in (2.4), we get

$$\sum_{j=0}^{J} |a_{ij} - a_{ij}^{(n)}| \leq \epsilon, n \geq n_0, i, J = 0, 1, 2, \ldots,$$

from which we have

$$\sum_{j=0}^{\infty} |a_{ij} - a_{ij}^{(n)}| \leq \epsilon, n \geq n_0, i = 0, 1, 2, \ldots,$$

$$\begin{aligned} &\text{i.e.,} \quad \sup_{i\geq 0} \sum_{j=0}^{\infty} |a_{ij} - a_{ij}^{(n)}| \leq \epsilon, n \geq n_0, \qquad (2.5)\\ &\text{i.e.,} \quad \|A^{(n)} - A\| \leq \epsilon, n \geq n_0,\\ &\text{i.e.,} \quad A^{(n)} \to A, n \to \infty. \end{aligned}$$

We now claim that $A \in (c_0, c)$. In view of (2.5),

$$\sup_{i\geq 0} \sum_{j=0}^{\infty} |a_{ij} - a_{ij}^{(n_0)}| \leq \epsilon. \tag{2.6}$$

Since $A^{(n_0)} = (a_{ij}^{(n_0)}) \in (c_0, c)$,

$$\sup_{i\geq 0} \sum_{j=0}^{\infty} |a_{ij}^{(n_0)}| = M < \infty \tag{2.7}$$

and

$$\lim_{i\to\infty} a_{ij}^{(n_0)} = \delta_j^{(n_0)} \text{ exists}, j = 0, 1, 2, \ldots. \tag{2.8}$$

Now, for all $i = 0, 1, 2, \ldots$,

$$\begin{aligned}
\sum_{j=0}^{\infty} |a_{ij}| &= \sum_{j=0}^{\infty} |\{a_{ij} - a_{ij}^{(n_0)}\} + a_{ij}^{(n_0)}| \\
&\leq \sum_{j=0}^{\infty} |a_{ij} - a_{ij}^{(n_0)}| + \sum_{j=0}^{\infty} |a_{ij}^{(n_0)}| \\
&\leq \sup_{i\geq 0} \sum_{j=0}^{\infty} |a_{ij} - a_{ij}^{(n_0)}| + \sup_{i\geq 0} \sum_{j=0}^{\infty} |a_{ij}^{(n_0)}| \\
&\leq \epsilon + M, \text{ using (2.6) and (2.7)} \\
&< \infty,
\end{aligned}$$

so that

$$\sup_{i\geq 0} \sum_{j=0}^{\infty} |a_{ij}| < \infty.$$

Next, we claim that $\{a_{ij}\}_{i=0}^{\infty}$ is a Cauchy sequence of real (or complex) numbers, $j = 0, 1, 2, \ldots$. To this end,

$$\begin{aligned}
|a_{uj} - a_{vj}| &= |\{a_{uj} - a_{uj}^{(n_0)}\} + \{a_{vj}^{(n_0)} - a_{vj}\} \\
&\qquad + \{a_{uj}^{(n_0)} - a_{vj}^{(n_0)}\}| \\
&\leq |a_{uj} - a_{uj}^{(n_0)}| + |a_{vj}^{(n_0)} - a_{vj}| \\
&\qquad + |a_{uj}^{(n_0)} - a_{vj}^{(n_0)}| \\
&\leq 2\epsilon + |a_{uj}^{(n_0)} - a_{vj}^{(n_0)}|, \text{ using (2.6)}. \qquad (2.9)
\end{aligned}$$

Since $\{a_{uj}^{(n_0)}\}_{u=0}^{\infty}$ converges, $A^{(n_0)} \in (c_0, c)$, it is a Cauchy sequence and so, for $\epsilon > 0$, there exists a positive integer L such that

$$|a_{uj}^{(n_0)} - a_{vj}^{(n_0)}| < \epsilon, u, v \geq L. \tag{2.10}$$

In view of (2.9) and (2.10), we have

$$|a_{uj} - a_{vj}| < 2\epsilon + \epsilon, u, v \geq L.$$

Consequently, $\{a_{ij}\}_{i=0}^{\infty}$ is a Cauchy sequence of real (or complex) numbers and so it converges, i.e.,

$$\lim_{i\to\infty} a_{ij} \;\text{ exists}, j = 0, 1, 2, \ldots.$$

Hence $A = (a_{ij}) \in (c_0, c)$, completing the proof of the theorem. □

Theorem 2.2. (c_0, c) *is a commutative Banach algebra with identity under the first convolution* $*$.

Proof. It suffices to prove closure under $*$ and the submultiplicative property of the norm. Let $A = (a_{nk})$, $B = (b_{nk}) \in (c_0, c)$ and $C = (c_{nk}) = A * B$. Now, for $k = 0, 1, 2, \ldots$,

$$\begin{aligned} c_{nk} &= (A * B)_{nk} \\ &= \sum_{i=0}^{k} a_{ni} b_{n,k-i} \\ &\to \sum_{i=0}^{k} a_i b_{k-i}, n \to \infty, \end{aligned}$$

where, $\lim_{n\to\infty} a_{nk} = a_k$, $\lim_{n\to\infty} b_{nk} = b_k$, $k = 0, 1, 2, \ldots$.
For $n = 0, 1, 2, \ldots$,

$$\begin{aligned} \sum_{k=0}^{\infty} |c_{nk}| &= \sum_{k=0}^{\infty} \left| \sum_{i=0}^{k} a_{ni} b_{n,k-i} \right| \\ &\leq \sum_{k=0}^{\infty} \sum_{i=0}^{k} |a_{ni}||b_{n,k-i}| \\ &= \left(\sum_{k=0}^{\infty} |a_{nk}| \right) \left(\sum_{k=0}^{\infty} |b_{nk}| \right) \\ &\leq \left(\sup_{n\geq 0} \sum_{k=0}^{\infty} |a_{nk}| \right) \left(\sup_{n\geq 0} \sum_{k=0}^{\infty} |b_{nk}| \right) \\ &= \|A\|\|B\|, \end{aligned}$$

so that

$$\sup_{n\geq 0} \sum_{k=0}^{\infty} |c_{nk}| \leq \|A\|\|B\|,$$

$$i.e., \|A * B\| \leq \|A\|\|B\|,$$

completing the proof of the theorem. □

Theorem 2.3. *(c_0, c) is a Banach space, which is a commutative, non-associative algebra without identity, under the second convolution $**$, with norm defined by (2.1).*

Proof. Let $A = (a_{nk})$, $B = (b_{nk}) \in (c_0, c)$. Then

$$(A * *B)_{nk} = \frac{1}{k+1} \sum_{i=0}^{k} a_{ni} b_{n,k-i}, \text{ by (1.5).}$$

We first claim that (c_0, c) is closed under the second convolution $**$. For $k = 0, 1, 2, \ldots$,

$$(A * *B)_{nk} \to \frac{1}{k+1} \sum_{i=0}^{k} a_i b_{k-i}, n \to \infty,$$

where $\lim_{n\to\infty} a_{nk} = a_k$, $\lim_{n\to\infty} b_{nk} = b_k$, $k = 0, 1, 2, \ldots$.
Also, for $n = 0, 1, 2, \ldots$,

$$\begin{aligned} \sum_{k=0}^{\infty} |(A * *B)_{nk}| &\le \sum_{k=0}^{\infty} \sum_{i=0}^{k} |a_{ni}||b_{n,k-i}| \\ &= \left(\sum_{k=0}^{\infty} |a_{nk}| \right) \left(\sum_{k=0}^{\infty} |b_{nk}| \right) \\ &\le \|A\| \|B\|. \end{aligned}$$

Thus,

$$\sup_{n \ge 0} \left(\sum_{k=0}^{\infty} |(A * *B)_{nk}| \right) \le \|A\| \|B\|,$$

so that $A * *B \in (c_0, c)$ and

$$\|A * *B\| \le \|A\| \|B\|.$$

Commutativity can be easily checked. Non-associativity can be established as follows: Let

$$A = B = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

Note that $A, B, C \in (c_0, c)$, using Theorem 1.1. Simple computation shows that

$$((A * *B) * *C)_{11} = \frac{1}{2}$$

and

$$(A**(B**C))_{11} = \frac{1}{4},$$

which proves that

$$(A**B)**C \neq A**(B**C),$$

i.e., (c_0, c) is non-associative. Again (c_0, c) does not have an identity under $**$. Suppose an identity $E = (e_{nk})$ exists. Then

$$A**E = A, \text{ for all } A = (a_{nk}) \in (c_0, c).$$

Consider

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \in (c_0, c).$$

Simple computation shows that

$$e_{11} = 1. \tag{2.11}$$

Again, consider

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \in (c_0, c).$$

Again, simple computation shows that

$$e_{11} = 0. \tag{2.12}$$

(2.11) and (2.12) lead to a contradiction, proving that (c_0, c) has no identity. By Theorem 2.1, (c_0, c) is a Banach space under the norm defined by (2.1). This completes the proof of the theorem. □

As noted in ([1], p. 183), the set S of all infinite matrices is a groupoid under the second convolution $**$, i.e., S is closed under $**$. Also S is commutative, non-associative and S has no identity. We now have

Theorem 2.4. *$(c_0, c; P)$ is a subgroupoid of S under the second convolution $**$.*

Proof. Let $A = (a_{nk})$, $B = (b_{nk}) \in (c_0, c; P)$. Let $C = (c_{nk}) = A**B$. We already know that $A**B \in (c_0, c)$.
Now,

$$\lim_{n\to\infty} a_{nk} = \lim_{n\to\infty} b_{nk} = 0, k = 0, 1, 2, \dots.$$

$$\begin{aligned} c_{nk} &= \frac{1}{k+1}\left[a_{n0}b_{nk} + a_{n1}b_{n,k-1} + \dots + a_{nk}b_{n0}\right] \\ &\to 0, n \to \infty, k = 0, 1, 2, \dots. \end{aligned}$$

Thus, $A**B \in (c_0, c; P)$, completing the proof. □

Let $(c_0, c)'$ denote the subclass of (c_0, c) consisting of all $A = (a_{nk}) \in (c_0, c)$ such that

$$a_{nk} \to 0, k \to \infty, n = 0, 1, 2, \ldots .$$

Theorem 2.5. $(c_0, c)'$ *is an ideal of* (c_0, c) *under the second convolution* $**$.

Proof. Let $A = (a_{nk}) \in (c_0, c)$ and $B = (b_{nk}) \in (c_0, c)'$. We claim that $A ** B \in (c_0, c)'$. We know that (c_0, c) is commutative under the second convolution $**$. We already know that $A ** B \in (c_0, c)$. Now,

$$(A ** B)_{nk} = \frac{1}{k+1}\left(\sum_{i=0}^{k} a_{ni} b_{n,k-i}\right),$$

$$\begin{aligned}
|(A ** B)_{nk}| &\leq \frac{1}{k+1}\left(\sum_{i=0}^{k} |a_{ni}||b_{n,k-i}|\right) \\
&\leq \frac{1}{k+1}\|A\|\|B\| \\
&\to 0, k \to \infty, n = 0, 1, 2, \ldots .
\end{aligned}$$

Consequently, $A ** B \in (c_0, c)'$, completing the proof. □

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