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LIFTS OF GOLDEN STRUCTURES ON THE TANGENT BUNDLE

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Abstract. The present paper aims to study the complete lift of golden structure on tangent bundles. Integrability conditions for complete lift and third order tangent bundle are established.
Keywords: Golden structure, Complete lift, Nijenhuis tensor, Projection tensors, Tangent bundle.

1. Introduction

The lift of geometric objects on a differentiable manifold is an important tool in the study of differential geometry of tangent bundle. The study of polynomial structure on differentiable manifold was started by Goldberg and Yano in 1970 [4]. Omran et al [1] studied lifts of various structures such as almost product, almost par-contact, para-cantact structures on manifold and integrability conditions of these structures are established. Khan [8] studied complete and horizontal lifts of metallic structures and discussed the integrability of such structures. Several investigators studied lifts of geometric objects in [2, 3, 9, 5, 11, 12, 17]. This paper aims to study the lifts of a golden structure on the tangent bundle and prolongation of a golden structure in third-order tangent bundle.

Suppose M be n-dimensional differentiable manifold. A tensor field F of type (1,1) is said to be the golden structure on M if F satisfies the equation [8]

$$F^2 - F - I = 0, \tag{1.1}$$

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where I is the unit vector field on M and F is of constant rank r everywhere in M.

If g be a Riemannian metric on M such that

$$g(FX,Y) = g(X,FY), \tag{1.2}$$

for all X and Y are vector fields on M. Then a golden structure is said to be a golden Riemannian structure.

Let us introduce the operators l and m

$$\begin{aligned} (a) \quad & l = F^2 - F \\ (b) \quad & m = I - (F^2 - F) \end{aligned} \tag{1.3}$$

The following identities can be easily obtained:

$$\begin{aligned} l + m = 0 \\ l^2 = l, \quad m^2 = m, \quad lm = ml = 0 \\ Fl = lF = F, \quad Fm = mF = 0. \end{aligned} \tag{1.4}$$

Let D_l and D_m of complementary distributions corresponding to the projection tensors l and m respectively in M. If the rank of F is r, then D_l is r-dimensional and D_m is $(n-r)$-dimensional, where $\dim M = r$.

2. The complete lift of a golden structure F on the tangent bundle $T(M)$

Let M be an n-dimensional differentiable manifold and TM its tangent bundle. The set of function, vector field, 1-form and tensor field of type (1,1) are represented by $\wp_0^0(M), \wp_0^1(M), \wp_1^0(M)$ and $\wp_1^1(M)$ respectively in M and $\wp_0^0(TM), \wp_0^1(TM), \wp_1^0(TM)$ and $\wp_1^1(TM)$ respectively in TM [5].

Let $F, G \in \wp_1^1(M)$. It is well known [19]

$$(FG)^C = F^C G^C. \tag{2.1}$$

Setting $F = G$ in above equation (2.1), then

$$(F^2)^C = (F^C)^2. \tag{2.2}$$

and

$$(F+G)^C = F^C + G^C. \tag{2.3}$$

Taking the complete lifts of both sides of the equation (1.1), then the obtained equation is

$$\begin{aligned} (F^2 - F - I)^C = 0 \\ (F^2)^C - F^C - I^C = 0 \end{aligned}$$

Using the equation (2.2) and $I^C = I$, then we have

$$(F^C)^2 - F^C - I = 0 \tag{2.4}$$

By using the equations (1.1), (2.4) and [19], we can easily say that the rank of F^C is $2r$ if and only if the rank of F is r. Therefore, the following theorems have been obtained:

Theorem 2.1. *Let $F \in \wp_1^1(M)$ be a golden structure in M, then its complete lift F^C is also a golden structure in TM.*

Theorem 2.2. *The golden structure F of rank r in M if and only if its complete lift F^C is of rank $2r$ in TM.*

Since F be a golden structure of rank r in M. Then the complete lift l^C of l and m^C of m are complementary projection tensors in TM. Thus, there exists two complementary distributions D_l^C and D_m^C determined by l^C and m^C respectively in TM [2].

3. Some theorems on integrability of golden structure on the tangent bundle

Let N be the Nijenhuis tensor of golden structure F in M and N^C be the Nijenhuis tensor of F^C in TM. Then we have [19]

$$N(X,Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y]. \tag{3.1}$$

and

$$\begin{aligned} N^C(X^C, Y^C) &= [F^C X^C, F^C Y^C] - F^C[F^C X^C, Y^C] \\ &- F^C[X^C, F^C Y^C] + (F^2)^C[X^C, Y^C]. \end{aligned} \tag{3.2}$$

Let X and Y be vector fields and F tensor field of type (1,1) in M, then

$$\begin{aligned} [X^C, Y^C] &= [X, Y]^C \\ (X+Y)^C &= X^C + Y^C \\ F^C X^C &= (FX)^C. \end{aligned} \tag{3.3}$$

Using the equations (1.4) and (3.5), we have

$$\begin{aligned} F^C l^C &= (Fl)^C = F^C \\ F^C m^C &= (Fm)^C = 0. \end{aligned} \tag{3.4}$$

Theorem 3.1. *The following identities hold:*

$$N^C(m^C X^C, m^C Y^C) = (F^C)^C[m^C X^C, m^C Y^C], \tag{3.5}$$

$$m^C N^C(X^C, Y^C) = m^C[F^C X^C, F^C Y^C], \tag{3.6}$$

$$m^C(l^C X^C, l^C Y^C) = m^C[F^C X^C, F^C Y^C], \tag{3.7}$$

$$m^C N^C((F^2 - \alpha F)^C X^C, (F^2 - F)^C Y^C) = m^C N^C(l^C X^C, l^C Y^C). \tag{3.8}$$

Proof: The proof of the equations (3.5) to (3.8) follow by using the equations (1.4), (3.4) and (3.1).

Theorem 3.2. *Let X and Y be vector fields and F tensor field of type (1,1) in M, the following conditions are equivalent*

$$(a)\quad m^C N^C(X^C, Y^C) = 0$$
$$(b)\quad m^C N^C(l^C X^C, l^C Y^C) = 0$$
$$(c)\quad m^C N^C((F^2 - F)^C X^C, (F^2 - F)^C Y^C) = 0.$$

Proof: Making use of the equation (3.8), we get

$$N^C(l^C X^C, l^C Y^C) = 0 \leftrightarrow N^C((F^2 - F)^C X^C, (F^2 - F)^C Y^C) = 0$$

Since the right sides of the the equations (3.6), (3.7) are equal and using the last equation which shows that conditions (a), (b), and (c) are equivalent.

Theorem 3.3. *The complete lift D_m^C in TM of a distribution D_m in M is integral if D_m is integrable in M.*

Proof: The distribution D_m is integral if and only if [19]

$$l[mX, mY] = 0 \tag{3.9}$$

for all $X, Y \in \wp_0^1(M)$, where $l = I - m$.

Taking complete lift of both sides and using (3.5), we have

$$l^C[m^C X^C, m^C Y^C] = 0 \tag{3.10}$$

for all $X, Y \in \wp(M)$, where $l^C = (I - m)^C = I - m^C$ is the projection tensor complementary to m^C. Thus the condition (3.9) implies (3.10).

Theorem 3.4. *The complete lift D_m^C in TM of a distribution D_m in M is integral if $l^C N^C(m^C X^C, m^C Y^C) = 0$, or equivalently $N^C(m^C X^C, m^C Y^C) = 0$, for all $X, Y \in \wp(M)$.*

Proof: The distribution D_m is integral in M if and only if [19]

$$N(mX, mY) = 0$$

for all $X, Y \in \wp(M)$. By virtue of condition (3.5), we have

$$N^C(m^C X^C, m^C Y^C) = (F^2)^C(m^C X^C, m^C Y^C)$$

Multiplying throughout by l^C, we get

$$l^C N^C(m^C X^C, m^C Y^C) = (F^2)^C l^C(m^C X^C, m^C Y^C)$$

Using the equation (3.10), the above relation becomes

$$l^C N^C(m^C X^C, m^C Y^C) = 0 \tag{3.11}$$

and

$$m^C N^C(m^C X^C, m^C Y^C) = 0 \tag{3.12}$$

Adding the equations (3.11) and (3.12), we have

$$(l^C + m^C)N^C(m^C X^C, m^C Y^C) = 0$$

Since $l^C + m^C = I^C = I$, we get

$$N^C(m^C X^C, m^C Y^C) = 0.$$

Theorem 3.5. *Let the distribution D_l be integrable in M, that is $mN(X,Y) = 0$ for all $X, Y \in \wp_0^1(M)$.Then the distribution D_l^C is integrable in TM if and only if the one of the conditions of Theorem (3.2) is satisfied.*

Proof: The distribution D_l is integral in M if and only if

$$mN(lX, lY) = 0$$

Thus distribution D_l^C is integrable in TM if and only if

$$m^C N^C(l^C X^C, l^C Y^C) = 0,$$

Hence the theorem follows by using of the equation (3.8).

Theorem 3.6. *Let complete lift F^C of a golden structure F in M is partially integrable in TM if and only if F is partially integrable in M.*

Proof: The golden structure F in M is partially integrable if and only if

$$N(lX, lY) = 0, \forall X, Y \in \wp_0^1(M). \tag{3.13}$$

Using the equations (1.4) and (3.1), we have

$$N^C(l^C X^C, l^C Y^C) = (N(lX, lY))^C$$

which implies

$$N^C(l^C X^C, l^C Y^C) = 0 \Leftrightarrow N(lX, lY) = 0$$

and from Theorem (3.2), $N^C(l^C X^C, l^C Y^C) = 0$ is equivalent to

$$N^C((F^2 - \alpha F)^C X^C, (F^2 - \alpha F)^C Y^C) = 0.$$

Theorem 3.7. *The complete lift F^C of a golden structure F in M is partially integrable in TM if and only if F is partially integrable in M.*

Proof: A necessary and sufficient condition for a golden structure in M to be integrable is that

$$(N(X,Y)) = 0 \tag{3.14}$$

for all $X, Y \in \wp_0^1(M)$.

Using the equation (3.1), we get

$$N^C(X^C, Y^C) = (N(X,Y))^C.$$

Therefore, using the equation (3.14) we obtain the result.

4. Prolongation of a golden structure in third-order tangent bundle T_3M

Let M be n-dimensional differentiable manifold and T_3M its third order tangent bundle over M. Let F^{III} be the third lift on F in T_3M. If X be vector field and F, G be tensor field of type (1,1), then

$$\begin{aligned}(G^{III}F^{III})X^{III} &= (G^{III}(F^{III}X^{III}) \\ &= (G^{III}(FX)^{III}) \\ &= (G(FX))^{III} \\ &= (GF)^{III}X^{III} \qquad (4.1)\end{aligned}$$

Thus,

$$G^{III}F^{III} = (GF)^{III}$$

Theorem 4.1. *Let $F \in \wp_1^1(M)$ be a golden structure in M, then the third lift F^{III} is also a golden structure in T_3M.*

Proof: If $P(t)$ is a polynomial in one variable t, then we get [19]

$$(P(F))^{III} = P(F^{III}) \tag{4.2}$$

for all $F \in \wp_1^1(M)$.

Taking the third lifts of both sides of the equation (1.1), we get

$$(F^2 - F - I)^{III} = 0$$
$$(F^2)^{III} - F^{III} - I^{III} = 0$$

Using the equation (4.2) and $I^{III} = I$, we have

$$(F^{III})^2 - F^{III} - I = 0 \tag{4.3}$$

which shows that F^{III} is a golden structure in T_3M.

Theorem 4.2. *The third lift F^{III} is integrable in T_3M if and only if F is integrable in M.*

Proof: Let N^{III} and N be Nijenhuis tensors of F^{III} and F respectively. Then we have

$$N^{III}(X,Y) = (N(X,Y))^{III}. \tag{4.4}$$

since golden structure is integrable in M if and only if $N(X,Y) = 0$. then from (4.4), we get

$$N^{III}(X,Y) = 0. \tag{4.5}$$

Thus F^{III} is integrable if and only if F is integrable in M.

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