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ON PROBABILISTIC (ϵ, λ)-LOCAL CONTRACTION MAPPINGS AND A SYSTEM OF INTEGRAL EQUATIONS

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Abstract. In this paper, we consider the concept of probabilistic (ϵ, λ)-local contraction which is a generalization of probabilistic contraction of Sehgal type, and the concept of probabilistic G-metric space, which is a generalization of the Menger probabilistic metric space. Then we prove some new coupled fixed point theorems for uniformly locally contractive mappings on probabilistic metric spaces. Also, we establish some coupled fixed point theorems for contractive mappings in probabilistic G-metric space. The article includes some examples and an application to a system of integral equations which supports of main results.
Keywords: Generalized probabilistic metric space, (ϵ, λ)-local contraction, Coupled fixed point, Uniformly locally contractive.

1. Introduction

In 1942, Menger [9] developed the theory of metric spaces and proposed a generalization of metric spaces called Menger probabilistic metric spaces (briefly, Menger PM-space). After that, the study of contraction mappings defined on probabilistic metric spaces was initiated by Sehgal [15] and Bharucha-Reid [16]. Then different classes of probabilistic contractions have been defined and probabilistic versions of Banach theorem were stated in [6]. Also, Golet and Hedrea [5] discussed local contractions in probabilistic metric spaces, which were formerly introduced by Cain

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and Kasrie [4]. On the other hand, in 2006, Mustafa and Sims [10] introduced a new version of generalized metric spaces, which is called G-metric spaces, and proved some of the fixed point theorems in this space (also, see [2, 11]). In 2014, Zhou et al. [19] defined the probabilistic version of G-metric spaces and obtained new fixed point results.

In 2004, Ran and Reurings [14] considered a partial order to the metric space (X, d) and discussed the existence and uniqueness of fixed points for contractive conditions and for the comparable elements of X. In 2005, Nieto and Rodríguez-López [12] applied this theory for solving ordinary differential equations. After that, Bhaskar and Lakshmikantham [3] defined coupled fixed point and proved some coupled fixed point theorems for a mixed monotone mapping in partially ordered metric spaces. Also, they studied the existence and uniqueness of a solution to a periodic boundary value problem. For more details on coupled, tripled, and n-tupled fixed point theorems in various metric spaces especially in G-metric spaces, we refer to [1, 8, 13, 18] and references therein. On the other hand, Samet and Yazidi [17] introduced the notation of partially ordered ϵ-chainable metric spaces and derived new coupled fixed point theorems for uniformly locally contractive mappings on such spaces.

In the following, we give some preliminary definitions which are needed.

Definition 1.1. [6] A function $f : (-\infty, +\infty) \to [0, 1]$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{x\in\mathbb{R}} f(x) = 0$. In addition if $f(0) = 0$, then f is called a distance distribution function. Furthermore, a distance distribution function f satisfying $\lim_{x\to+\infty} f(x) = 1$ is called a Menger distance distribution function. The set of all Menger distance distribution functions is denoted by D^+.

Definition 1.2. [6] A triangular norm (abbreviated, t-norm) is a binary operation T on $[0, 1]$, which satisfies the conditions: (a) T is associative and commutative; (b) T is continuous; (c) $T(a, 1) = a$ for all $a \in [0, 1]$; (d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 1.3. [6] A triangular norm T is said to be of H-type (Hadzić type) if a family of functions $\{T^n(t)\}$ is equicontinuous at $t = 1$; that is, for each $\varepsilon \in (0, 1)$, there exists $\delta \in (0, 1)$ such that $t > 1 - \delta$ implies that $T^n(t) > 1 - \epsilon$ $(n \geq 1)$, where $T^n : [0, 1] \longrightarrow [0, 1]$ is defined by $T^1(t) = T(t, t)$ and $T^n(t) = T(t, T^{n-1}(t))$ for $n = 2, 3, \cdots$. Obviously, $T^n(t) \leq t$ for all $n \in \mathbb{N}$ and $t \in [0, 1]$.

Definition 1.4. [6] A Menger probabilistic metric space (briefly, Menger PM-space) is a triple (X, F, T), where X is a nonempty set, T is a continuous t-norm and F is a mapping from X^2 in to D^+ such that if $F_{x,y}$ denotes the value of F at the pair (x, y), then the following conditions hold:

(PM1) $F_{x,y}(t) = 1$ for all $t > 0$ if only if $x = y$;

(PM2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t > 0$;

(PM3) $F_{x,z}(t+s) \geq T(F_{x,y}(t), F_{y,z}(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Note that Definition 1.4 is the probabilistic version of metric spaces. Also, for notions such as convergent and Cauchy sequences, completeness and examples in Menger PM-space, we refer to [6].

Definition 1.5. [5] Let $(X, F, T, \preceq)$ be a partially ordered PM-space. The mapping $f : X^2 \to X$ is called an (ϵ, λ)-uniformly local contraction with a constant $k \in (0, 1)$, if $\frac{1}{2}(F_{x,u}(\epsilon) + F_{y,v}(\epsilon)) \geq 1 - \lambda$ for all $t, \epsilon > 0$ and $\lambda \in (0, 1)$ implies that $F_{f(x,y),f(u,v)}(t) \geq \frac{1}{2}(F_{x,u}(\frac{t}{k}) + F_{y,v}(\frac{t}{k}))$ for all $x \succeq u$ and $y \preceq v$.

Under the conditions of Definition 1.5, the set X is called (ϵ, λ)-chainable if for all $x, y \in X$ with $x \preceq y$, there exists a finite sequence $x = x_0 \preceq x_1 \preceq \cdots \preceq x_n = y$ such that $F_{x_{i+1},x_i}(\epsilon) > 1 - \lambda$ for $i = 0, 1, \cdots, n-1$. Also, the finite sequence $x = x_0 \preceq x_1 \preceq \cdots \preceq x_n = y$ is called (ϵ, λ)-chain joining x and y.

Definition 1.6. [19] A Menger probabilistic G-metric space (shortly, PGM-space) is a triple (X, G, T), where X is a nonempty set, T is a continuous t-norm and G is a mapping from X^3 into D^+ ($G_{x,y,z}$ denotes the value of G at the point (x, y, z)) satisfying the following conditions:

(PG1) $G_{x,y,z}(t) = 1$ for all $x, y, z \in X$ and $t > 0$ if and only if $x = y = z$;

(PG2) $G_{x,x,y}(t) \geq G_{x,y,z}(t)$ for all $x, y \in X$ with $z \neq y$ and $t > 0$;

(PG3) $G_{x,y,z}(t) = G_{x,z,y}(t) = G_{y,x,z}(t) = \cdots$ (symmetry in all three variables);

(PG4) $G_{x,y,z}(t+s) \geq T(G_{x,a,a}(s), G_{a,y,z}(t))$ for all $x, y, z, a \in X$ and $s, t \geq 0$.

Note that Definition 1.6 is the probabilistic version of generalized metric spaces. Also, for notions such as convergent and Cauchy sequences, completeness, and examples in Menger PGM-space, we refer to [19].

Definition 1.7. [19] Let (X, G, T) be a PGM-space and $x_0 \in X$. For any $\epsilon > 0$ and δ with $0 < \delta < 1$, an (ϵ, δ)-neighborhood of x_0 is the set of all $y \in X$ which $G_{x_0,y,y}(\epsilon) > 1 - \delta$ and $G_{y,x_0,x_0}(\epsilon) > 1 - \delta$. We write

$$N_{x_0}(\epsilon, \delta) = \{y \in X : G_{x_0,y,y}(\epsilon) > 1 - \delta, G_{y,x_0,x_0}(\epsilon) > 1 - \delta\}.$$

This means that $N_{x_0}(\epsilon, \delta)$ is the set of all points y in X for which the probability of the distance from x_0 to y being less than ϵ is greater than $1 - \delta$.

Definition 1.8. [3] Let $(X, \preceq)$ be a partially ordered set. The mapping $f : X^2 \to X$ is said to be have the mixed monotone property if f is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument; that is, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $f(x_1, y) \preceq f(x_2, y)$ for each $y \in X$, and for all $y_1, y_2 \in X$, $y_1 \preceq y_2$ implies $f(x, y_1) \succeq f(x, y_2)$ for each $x \in X$.

Definition 1.9. [7] Let $(X, \preceq)$ be an ordered partial metric space. If relation $\sqsubseteq$ is defined on X^2 by $(x, y) \sqsubseteq (u, v)$ iff $x \preceq u$ and $y \succeq v$, then $(X^2, \sqsubseteq)$ is an ordered partial metric space.

2. Coupled Fixed Point Theorems on Local Contractions in Menger PM-space

In this section, we prove some new coupled fixed point theorems for uniformly locally contractive mappings on probabilistic metric spaces.

Theorem 2.1. *Let $(X, F, T, \preceq)$ be a partially ordered complete Menger PM-space with T of Hadzić-type and $f : X^2 \to X$ be a mapping having the mixed monotone property on X. Also, suppose that the following conditions are hold:*

1. *X is (ϵ, λ)-chainable with respect to the partial order "$\preceq$" on X,*
2. *f is continuous,*
3. *f is (ϵ, λ)-uniformly locally contractive mapping,*
4. *there exists $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$.*

Then, f has a coupled fixed point.

Proof. By condition 4, there exists $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$. We define $x_1, y_1 \in X$ as $x_1 = f(x_0, y_0) \succeq x_0$ and $y_1 = f(y_0, x_0) \preceq y_0$. Let $x_2 = f(x_1, y_1)$ and $y_2 = f(y_1, x_1)$. Then we obtain

$$f^2(x_0, y_0) = f(f(x_0, y_0), f(y_0, x_0)) = f(x_1, y_1) = x_2,$$
$$f^2(y_0, x_0) = f(f(y_0, x_0), f(x_0, y_0)) = f(y_1, x_1) = y_2.$$

Now, the mixed monotone property of f implies that

$$x_2 = f^2(x_0, y_0) = f(x_1, y_1) \succeq f(x_0, y_0) = x_1 \succeq x_0,$$
$$y_2 = f^2(y_0, x_0) = f(y_1, x_1) \preceq f(y_0, x_0) = y_1 \preceq y_0.$$

Continuing the above procedure, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_{n+1} \preceq \cdots$$
$$y_0 \succeq y_1 \succeq y_2 \succeq \cdots \succeq y_{n+1} \succeq \cdots$$

for all $n \geq 0$, where

$$x_{n+1} = f^{n+1}(x_0, y_0) = f(f^n(x_0, y_0), f^n(y_0, x_0)),$$
$$y_{n+1} = f^{n+1}(y_0, x_0) = f(f^n(y_0, x_0), f^n(x_0, y_0)).$$

If $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then f has a coupled fixed point. Otherwise, let $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \geq 0$; that is, we assume that either $x_{n+1} = f(x_n, y_n) \neq x_n$ or

$y_{n+1} = f(y_n, x_n) \neq y_n$. Since X is ϵ-chainable, there exists $\alpha_0, \alpha_1, \cdots, \alpha_n \in X$ and $\beta_0, \beta_1, \cdots, \beta_n \in X$ such that

$$x_i = \alpha_0 \preceq \alpha_1 \preceq \cdots \preceq \alpha_n = x_{i+1}$$
$$y_i = \beta_0 \succeq \beta_1 \succeq \cdots \succeq \beta_n = y_{i+1}$$

for all $i = 1, 2, \cdots, n$. Hence, we have $F_{x_i, x_{i+1}}(\epsilon) \geq 1 - \lambda$ and $F_{y_i, y_{i+1}}(\epsilon) \geq 1 - \lambda$. Using condition 3, we have

$$F_{f(x_i,y_i),f(x_{i+1},y_{i+1})}(t) \geq \frac{1}{2}(F_{x_i,x_{i+1}}(\frac{t}{k}) + F_{y_i,y_{i+1}}(\frac{t}{k})).$$

Now, for all $i \geq 0$, one can show by induction that

$$F_{f(x_i,y_i),f(x_{i+1},y_{i+1})}(t) = F_{x_i,x_{i+1}}(t) \geq \frac{1}{2}(F_{x_1,x_0}(\frac{t}{k^i}) + F_{y_1,y_0}(\frac{t}{k^i})),$$
$$F_{f(y_i,x_i),f(y_{i+1},x_{i+1})}(t) = F_{y_i,y_{i+1}}(t) \geq \frac{1}{2}(F_{y_1,y_0}(\frac{t}{k^i}) + F_{x_1,x_0}(\frac{t}{k^i})).$$

Hence, we have $\frac{1}{2}(F_{x_1,x_0}(\frac{t}{k^i}) + F_{y_1,y_0}(\frac{t}{k^i})) \to 1$ and $\frac{1}{2}(F_{y_1,y_0}(\frac{t}{k^i}) + F_{x_1,x_0}(\frac{t}{k^i})) \to 1$ as $i \to \infty$, so

$$F_{x_i,x_{i+1}}(t) \geq 1 - \lambda \quad \text{and} \quad F_{y_i,y_{i+1}}(t) \geq 1 - \lambda \tag{2.1}$$

for all $i \in \mathbb{N}$ and any $t > 0$. Now, we show by induction that for any $k \geq 0$, $n \geq 1$ and $t > 0$,

$$F_{x_n,x_{n+k}}(t) \geq T^k(F_{x_n,x_{n+1}}(t - \lambda t)). \tag{2.2}$$

For $k = 0$, since $T(a, b)$ is a real number, $T^0(a, b) = 1$ for all $a, b \in [0, 1]$. Hence, $F_{x_n,x_n}(t) = T^0(F_{x_n,x_{n+1}}(t - \lambda t)) = 1$, which implies that (2.2) holds for $k = 0$. Assume that (2.2) holds for some $k \geq 1$. Then, since T is monotone, it follows from **(PM3)** that

$$\begin{aligned} F_{x_n,x_{n+k+1}}(t) &= F_{x_n,x_{n+k+1}}(t - \lambda t + \lambda t) \\ &\geq T(F_{x_n,x_{n+1}}(t - \lambda t), F_{x_{n+1},x_{n+k+1}}(\lambda t)) \\ &\geq T(F_{x_n,x_{n+1}}(t - \lambda t), F_{x_n,x_{n+k}}(\lambda t)) \\ &\geq T(F_{x_n,x_{n+1}}(t - \lambda t), T^k(F_{x_n,x_{n+1}}(t - \lambda t))) \\ &= T^{k+1}(F_{x_n,x_{n+1}}(t - \lambda t)). \end{aligned} \tag{2.3}$$

Thus, (2.2) is hold. Now, we show that $\{x_n\}$ is a Cauchy sequence in X, i.e., $\lim_{m,n\to\infty} F_{x_n,x_m}(t) = 1$ for any $t > 0$. To this end, by hypothesis of the t-norm T is H-type we have $\{T^n : n \geq 1\}$ is equicontinuous at 1; that is, there exists $\delta > 0$ such that

$$T^n(a) \geq 1 - \epsilon \tag{2.4}$$

for all $n \geq 1$ and any $a \in (1 - \delta, 1]$. On the other hand, it follows from (2.1) that $\lim_{n\to\infty} F_{x_n,x_{n+1}}(t - \lambda t) = 1$. Hence, there exists $n_0 \in \mathbb{N}$ such that $F_{x_n,x_{n+1}}(t - \lambda t) \in (1 - \delta, 1]$ for all $n \geq n_0$. By (2.3) and (2.4), we conclude that $F_{x_n,x_{n+k}}(t) > 1 - \epsilon$

for any $k \geq 1$. This shows $\lim_{n,m\to\infty} F_{x_n,x_m}(t) = 1$ for any $t > 0$; that is $\{x_n\}$ is a Cauchy sequence in X. Similarly, $\{y_n\}$ is a Cauchy sequence. Since X is a complete space, there exists $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Now, since $x_{n+1} = f(x_n, y_n)$ and f is continuous, and by taking the limit as $n \to \infty$, we have $f(x, y) = x$. Similarly, $f(y, x) = y$. Thus, (x, y) is a coupled fixed point of f. □

Example 2.1. Let $X = [0, \infty)$, " $\preceq$ " be a partially ordered on X (note that we consider the same ordinary order on real numbers) and $T(a, b) = \min\{a, b\}$. Define $F : X^2 \to D^+$ by $F_{x,y}(t) = 1$ if $x = y$ and otherwise, $F_{x,y}(t) = \exp(-t)$. Clearly, F satisfies in **(PM1)-(PM4)**. Define the mapping $f : X^2 \to X$ by $f(a, b) = ab$. We have

$$F_{f(x,y),f(u,v)}(t) \geq \frac{1}{2}(F_{x,u}(\frac{t}{k}) + F_{y,v}(\frac{t}{k}))$$

for $k \in (0, 1)$. Therefore, f is (ϵ, λ)-uniformly locally contractive mapping. Also, f is continuous, $[0, \infty)$ is (ϵ, λ)-chainable, and there exists $x_0 = 0$ and $y_0 = 1$ such that $0 = x_0 \preceq f(x_0, y_0) = x_0y_0$ and $1 = y_0 \succeq f(y_0, x_0) = y_0x_0$. Therefore, all the hypothesis of Theorem 2.1 are satisfied and f has a coupled fixed point.

Theorem 2.2. *Suppose that the assumptions of Theorem 2.1 is true. If we replace the assumption the continuity of f by the following conditions:*

1. *if a non-decreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \preceq x$ for all n,*
2. *if a non-increasing sequence $\{y_n\}$ converges to $y \in X$, then $y_n \succeq y$ for all n,*

then f has a coupled fixed point.

Proof. As in the proof of Theorem 2.1, we construct $\{x_n\}$ and $\{y_n\}$. Then, by conditions 1 and 2, we have $x_n \preceq x$ and $y_n \succeq y$ for all $n \geq 0$. Let $x_n = x$ and $y_n = y$ for some n. Then, due to the structure of both sequences, we have $x_{n+1} = x$ and $y_{n+1} = y$. Hence, (x, y) is a coupled fixed point. Now, we assume either $x_n \neq x$ or $y_n \neq y$. Since $x_n \to x$ and $y_n \to y$, for given $\epsilon_1, \epsilon_2, \lambda_1, \lambda_2 > 0$, there exists $k_1, k_2 \in \mathbb{N}$ such that $F_{x_{n_1},x}(\epsilon_1) \geq 1 - \lambda_1$ and $F_{y_{n_2},y}(\epsilon_2) \geq 1 - \lambda_2$ for all $n_1 \geq k_1$ and $n_2 \geq k_2$, respectively. Let $k = \max\{k_1, k_2\}$, $\lambda = \max\{\lambda_1, \lambda_2\}$ and $\epsilon = \max\{\epsilon_1, \epsilon_2\}$. Then, by conditions 1 and 2, we have $\frac{1}{2}(F_{x_n,x}(\epsilon) + F_{y_n,y}(\epsilon)) \geq 1 - \lambda$ for all $n \geq k$. Since f is (ϵ, λ)-uniformly locally contractive, by conditions 1 and 2, we have

$$F_{f(x_n,y_n),f(x,y)}(t) \geq \frac{1}{2}(F_{x_n,x}(\frac{t}{k}) + F_{y_n,y}(\frac{t}{k})).$$

Now, by letting $n \to \infty$ by $x_{n+1} = f(x_n, y_n)$, we have $x = f(x, y)$. Similarly, one can show that $y = f(y, x)$. This completes the proof. □

Theorem 2.3. *Adding the following property to the hypotheses of Theorem 2.1 (Theorem 2.2). Then the coupled fixed point of f is unique.*

(H) for all $(x, y), (x_1, y_1) \in X^2$, there exists $(z_1, z_2) \in X^2$ such that is comparable with (x, y) and (x_1, y_1).

Proof. Let (x_1, y_1) be another coupled fixed point of f. We consider two cases.

Case 1. suppose that (x, y) and (x_1, y_1) are comparable with respect to the partial ordering $\sqsubseteq$ in X^2. Without loss of the generality, we can assume that $x \preceq x_1$ and $y \succeq y_1$. Applying the procedure of Theorem 2.1, by X is (ϵ, λ)-chainable, we have $F_{x,x_1}(\epsilon) \geq 1 - \lambda$ and $F_{y,y_1}(\epsilon) \geq 1 - \lambda$. Since f is (ϵ, λ)-uniformly locally contractive, we have

$$F_{f^n(x,y),f^n(x_1,y_1)}(t) \geq \frac{1}{2}(F_{x,x_1}(\frac{t}{k^n}) + F_{y,y_1}(\frac{t}{k^n}))$$

for all $n \in \mathbb{N}$. Now, by letting $n \to \infty$, we have $x = x_1$. Similarly, $y = y_1$.

Case 2. assume that (x, y) and (x_1, y_1) are not comparable. From (H), there exists $(z_1, z_2) \in X^2$ that is comparable to (x, y) and (x_1, y_1). Without loss of the generality, we can suppose that $x \preceq z_1$, $y \succeq z_2$, $x_1 \preceq z_1$ and $y_1 \succeq z_2$. Similar to the Case 1, we have

$$F_{f^n(x,y),f^n(z_1,z_2)}(t) \geq \frac{1}{2}(F_{x,z_1}(\frac{t}{k^n}) + F_{y,z_2}(\frac{t}{k^n})),$$

which by letting $n \to \infty$ implies that $\lim_{n\to\infty} f^n(x, y) = \lim_{n\to\infty} f^n(z_1, z_2)$. Similarly, we have $\lim_{n\to\infty} f^n(y, x) = \lim_{n\to\infty} f^n(z_2, z_1)$, $\lim_{n\to\infty} f^n(x_1, y_1) = \lim_{n\to\infty} f^n(z_1, z_2)$ and $\lim_{n\to\infty} f^n(y_1, x_1) = \lim_{n\to\infty} f^n(z_2, z_1)$. Thus, we obtain $F_{x,x_1}(t) = F_{f^n(x,y),f^n(x_1,y_1)}(t)$ and $F_{y,y_1}(t) = F_{f^n(y,x),f^n(y_1,x_1)}(t)$, which by letting $n \to \infty$ implies that $x = x_1$ and $y = y_1$.

Consequently, the coupled fixed point of f is unique in both cases. □

Theorem 2.4. *In addition of the hypotheses of Theorem* 2.1 (*Theorem* 2.2)*, suppose that every pair of elements of X has an upper or a lower bound in X. Then $x = y$.*

Proof. Case 1. suppose that x and y are comparable. Without loss of the generality, we can assume that $x \preceq y$ and $y \succeq y$. Then similar to the proof of Theorem 2.3, we have $x = y$

Case 2. suppose x is not comparable to y. Then, there exists an upper bound or lower bound of x and y; that is, there exists $z \in X$ comparable with x and y. For example, we can suppose that $x \preceq z$ and $y \succeq z$. Similar to the proof of Theorem 2.3, we have $(x, y) = (z, z)$. Thus, $x = y$. □

3. Coupled Fixed Point Theorems in Menger PGM-spaces

In this section, we establish some coupled fixed point theorems in probabilistic G-metric spaces.

Theorem 3.1. *Let $(X, G, T, \preceq)$ be a partially ordered complete Menger PGM-space with T of Hadzić-type and $f : X^2 \to X$ be a continuous mapping having the mixed monotone property. Assume that there exists $k \in [0, 1)$ such that*

$$G_{f(x,y),f(u,v),f(w,z)}(t) \geq \frac{1}{2}(G_{x,u,w}(\frac{t}{k}) + G_{y,v,z}(\frac{t}{k})) \tag{3.1}$$

for all $x, y, z, u, v, w \in X$ with $x \succeq u \succeq w$ and $y \preceq v \preceq z$, where either $u \neq w$ or $v \neq z$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$, then f has a coupled fixed point in X.

Proof. Construct $\{x_n\}$ and $\{y_n\}$ as in the proof of Theorem 2.1. If $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then f has a coupled fixed point. Otherwise, let $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \geq 0$; that is, we assume that either $x_{n+1} = f(x_n, y_n) \neq x_n$ or $y_{n+1} = f(y_n, x_n) \neq y_n$. Now, one can show by induction that

$$G_{x_{n+1},x_{n+1},x_n}(t) \geq \frac{1}{2}(G_{x_1,x_1,x_0}(\frac{t}{k^n}) + G_{y_1,y_1,y_0}(\frac{t}{k^n})),$$
$$G_{y_{n+1},y_{n+1},y_n}(t) \geq \frac{1}{2}(G_{y_1,y_1,y_0}(\frac{t}{k^n}) + G_{x_1,x_1,x_0}(\frac{t}{k^n}))$$

for all $n \geq 0$. Since X is a Menger PGM-space, we have

$$\lim_{n\to\infty} G_{x_1,x_1,x_0}(\frac{t}{k^n}) = 1 \quad \text{and} \quad \lim_{n\to\infty} G_{y_1,y_1,y_0}(\frac{t}{k^n}) = 1, \tag{3.2}$$

which imply that

$$\lim_{n\to\infty} G_{x_{n+1},x_{n+1},x_n}(t) = 1 \quad \text{and} \quad \lim_{n\to\infty} G_{y_{n+1},y_{n+1},y_n}(t) = 1$$

for any $t > 0$. Now, by induction, we show that for any $k \geq 0$, $n \geq 1$ and $t > 0$,

$$G_{x_n,x_{n+k},x_{n+k}}(t) \geq T^k(G_{x_n,x_{n+1},x_{n+1}}(t - \lambda t)). \tag{3.3}$$

For $k = 0$, since $T(a, b)$ is a real number, $T^0(a, b) = 1$ for all $a, b \in [0, 1]$. Hence,

$$G_{x_n,x_n,x_n}(t) \geq T^0(G_{x_n,x_{n+1},x_{n+1}}(t - \lambda t)),$$

which implies that (3.3) holds for $k = 0$. Assume that (3.3) holds for some $k \geq 1$. Since T is monotone, it follows from **(PG4)** that

$$\begin{aligned}
G_{x_n,x_{n+k+1},x_{n+k+1}}(t) &= G_{x_n,x_{n+k+1},x_{n+k+1}}(t - \lambda t + \lambda t) \\
&\geq T(G_{x_n,x_{n+1},x_{n+1}}(t - \lambda t), G_{x_{n+1},x_{n+k+1},x_{n+k+1}}(\lambda t)) \\
&\geq T(G_{x_n,x_{n+1},x_{n+1}}(t - \lambda t), G_{x_n,x_{n+k},x_{n+k}}(t)) \\
&\geq T(G_{x_n,x_{n+1},x_{n+1}}(t - \lambda t), T^k(G_{x_n,x_{n+1},x_{n+1}}(t - \lambda t))) \\
&= T^{k+1}(G_{x_n,x_{n+1},x_{n+1}}(t - \lambda t)).
\end{aligned}$$

Thus, (3.3) is hold. Now, we show that $\{x_n\}$ is a Cauchy sequence in X, i.e., $\lim_{m,n,l\to\infty} G_{x_n,x_m,x_l}(t) = 1$ for all $t > 0$. To this end, we first prove

$$\lim_{n,m\to\infty} G_{x_n,x_m,x_m}(t) = 1$$

for any $t > 0$. By hypothesis of the t-norm T is H-type we have $\{T^n : n \geq 1\}$ is equicontinuous at 1; that is, there exists $\delta > 0$ such that $T^n(a) \geq 1 - \epsilon$ for all $a \in (1-\delta, 1]$, $\epsilon > 0$ and $n \geq 1$. From (3.2), it follows that $\lim_{n\to\infty} G_{x_n,x_{n+1},x_{n+1}}(t-\lambda t) = 1$. Hence, there exists $n_0 \in \mathbb{N}$ such that $G_{x_n,x_{n+1},x_{n+1}}(t-\lambda t) \in (1-\delta, 1]$ for any $n \geq n_0$. Thus, by (3.2) and (3.3), we conclude that $G_{x_n,x_{n+k},x_{n+k}}(t) > 1 - \epsilon$ for any $k \geq 1$. This shows $\lim_{n,m\to\infty} G_{x_n,x_m,x_m}(t) = 1$ for any $t > 0$, similarly $\lim_{n,l\to\infty} G_{x_n,x_l,x_l}(t) = 1$ for any $t > 0$. By **(PG4)**, we have

$$G_{x_n,x_m,x_l}(t) \geq T(G_{x_n,x_n,x_m}(\frac{t}{2}), G_{x_n,x_n,x_l}(\frac{t}{2})),$$
$$G_{x_n,x_n,x_m}(\frac{t}{2}) \geq T(G_{x_n,x_m,x_m}(\frac{t}{4}), G_{x_n,x_m,x_m}(\frac{t}{4})),$$
$$G_{x_n,x_n,x_l}(\frac{t}{2}) \geq T(G_{x_n,x_l,x_l}(\frac{t}{4}), G_{x_n,x_l,x_l}(\frac{t}{4})).$$

Therefore, by the continuity of T, we conclude that $\lim_{m,n,l\to\infty} G_{x_n,x_m,x_l}(t) = 1$ for any $t > 0$. Hence, $\{x_n\}$ is a Cauchy sequence in X. Similarly, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exist $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Now, we show that f has a coupled fixed point in X. From $x_{n+1} = f(x_n, y_n)$, take the limit as $n \to \infty$. Since f is continuous, we have $f(x, y) = x$. Similarly, we have $f(y, x) = y$. $\square$

Example 3.1. Consider X, " $\preceq$ " and $T(a, b)$ as in Example 2.1. Define $G : X^3 \to \mathbb{R}^+$ by

$$G_{x,y,z}(t) = \frac{t}{t + G^*(x, y, z)},$$

where $G^*(x, y, z) = |x - y| + |x - z| + |y - z|$ for all $x, y, z \in X$. Clearly, G satisfies in **(PG1)-(PG4)** (see [19]). Define the mapping $f : X^2 \to X$ by $f(x, y) = 1$. Then, for all $t > 0$ and $k \in [0, 1)$, we have

$$G_{f(x,y),f(u,v),f(w,z)}(t) = G_{1,1,1}(t) = 1 \geq \frac{1}{2}(G_{x,u,w}(\frac{t}{k}) + G_{y,v,z}(\frac{t}{k}))$$

for all $x, y, z, u, v, w \in X$ with $x \succeq u \succeq w$ and $y \preceq v \preceq z$, where either $u \neq w$ or $v \neq z$. Also, there exist $x_0 = 0$ and $y_0 = 1$ such that $0 = x_0 \preceq f(x_0, y_0) = 1$ and $1 = y_0 \succeq f(y_0, x_0) = 1$. Therefore, all the hypothesis of Theorem 3.1 are satisfied. Thus, f has a coupled fixed point.

Theorem 3.2. *Assume that the assumptions of Theorem* 3.1 *are hold and replace the assumption the continuity of f by the following conditions:*

1. *if a non-decreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \preceq x$ for all n;*
2. *if a non-increasing sequence $\{y_n\}$ converges to $y \in X$, then $y_n \succeq y$ for all n.*

Then f has a coupled fixed point.

Proof. As in the proof of Theorem 2.1, we construct $\{x_n\}$ and $\{y_n\}$. Then, by conditions 1 and 2, we have $x_n \preceq x$ and $y_n \succeq y$ for all $n \geq 0$. Let $x_n = x$ and $y_n = y$ for some n. Then, due to the structure of both sequences, we have $x_{n+1} = x$ and $y_{n+1} = y$. Hence, (x, y) is a coupled fixed point. Now, we assume that either $x_n \neq x$ or $y_n \neq y$. Then we have

$$\begin{aligned} G_{f(x,y),x,x}(2t) &\geq T(G_{f(x,y),f(x_n,y_n)f(x_n,y_n)}(t), G_{f(x_n,y_n),x,x}(t)) \\ &\geq T(\frac{1}{2}(G_{x,x_n,x_n}(\frac{t}{k}) + G_{y,y_n,y_n}(\frac{t}{k})), G_{x_{n+1},x,x}(t)). \end{aligned}$$

Now, taking $n \to \infty$, we obtain $G_{f(x,y),x,x}(2t) = 1$; that is, $f(x, y) = x$. Similarly, we have $f(y, x) = y$. This completes the proof of the theorem. □

Theorem 3.3. *Let $(X, G, T, \preceq)$ be a partially ordered complete Menger PGM-space with T of Hadzić-type and $f : X^2 \to X$ be a continuous mapping having the mixed monotone property on X, and $f(x, y) \preceq f(y, x)$ whenever $x \preceq y$. Assume that there exists $k \in [0, 1)$ such that*

$$G_{f(x,y),f(u,v),f(w,z)}(t) \geq \frac{1}{2}(G_{x,u,w}(\frac{t}{k}) + G_{y,v,z}(\frac{t}{k}))$$

for all $x, y, z, u, v, w \in X$ with $x \succeq u \succeq w$ and $y \preceq v \preceq z$, where either $u \neq w$ or $v \neq z$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq y_0$, $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$, then f has a coupled fixed point in X.

Proof. By the last assumption of the theorem, there exist $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$. We define $x_1, y_1 \in X$ as $x_1 = f(x_0, y_0) \succeq x_0$ and $y_1 = f(y_0, x_0) \preceq y_0$. Since $x_0 \preceq y_0$ and by another assumption of the theorem, we have $f(x_0, y_0) \preceq f(y_0, x_0)$. Hence, $x_0 \preceq x_1 = f(x_0, y_0) \preceq f(y_0, x_0) = y_1 \preceq y_0$. Continuing the above procedure, we have two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$x_n \preceq f(x_n, y_n) = x_{n+1} \preceq y_{n+1} = f(y_n, x_n) \preceq y_n$$

for all $n \geq 0$. Now, if $x_n = y_n = c$ for some n, then $c \preceq f(c, c) \preceq f(c, c) \preceq c$. Thus, $c = f(c, c)$ and (c, c) is a coupled fixed point. Hence, we assume that $x_n \preceq y_n$ for all $n \geq 0$. Further, for the same reason as stated in Theorem 3.1, we assume that $(x_n, y_n) \neq (x_{n+1}, y_{n+1})$. Then, for all $n \geq 0$, (3.1) will hold with $x = x_{n+2}, u = x_{n+1}, w = x_n, y = y_n, v = y_{n+1}$ and $z = y_{n+2}$. The rest of the proof is obtained by repeating the same steps as in Theorem 3.1. □

Theorem 3.4. *Suppose that the assumptions of Theorem 3.3 are true and replace the assumption the continuity of f by the following conditions:*

1. *if a non-decreasing sequence* $\{x_n\}$ *converges to* $x \in X$*, then* $x_n \preceq x$ *for all* n*;*
2. *if a non-increasing sequence* $\{y_n\}$ *converges to* $y \in X$*, then* $y_n \succeq y$ *for all* n*.*

Then f *has a coupled fixed point.*

Proof. The proof is similar to the proof of Theorem 3.2. □

Remark 3.1. (i) All the previous results can be considered if instead "mixed monotone property" we suppose so-called only "monotone property" as in 1 and 2. It is well known that this property has an advantage under the mixed monotone property.

(ii) Some authors think that the notion of coupled fixed point is not still such actual for research. But Soleimani Rad et al. [18] only showed that some of the results in coupled fixed point theory can be obtained from fixed point theory and conversely (also, see [1, 13]).

4. Application to a System of Integral Equations

Consider the following system of integral equations:

$$\begin{cases} x(t) = \int_a^b M(t,s)K(s,x(s),y(s))ds \\ \\ y(t) = \int_a^b M(t,s)K(s,y(s),x(s))ds \end{cases} \tag{4.1}$$

for all $t \in I = [a,b]$, where $b > a$, $M \in C(I \times I, [0,\infty))$ and $K \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Let $C(I,\mathbb{R})$ be the Banach space of all real continuous functions defined on I with the sup norm $||x||_\infty = \max_{t\in I} |x(t)|$ for all $x \in C(I,\mathbb{R})$ and $C(I \times I \times C(I,\mathbb{R}),\mathbb{R})$ be the space of all continuous functions defined on $I \times I \times C(I,\mathbb{R})$ and the induced G^*-metric be defined by $G^*(x,y,z) = ||x-y|| + ||x-z|| + ||y-z||$ for all $x,y,z \in C(I,\mathbb{R})$. Now, suppose that $G : C(I,\mathbb{R}) \times C(I,\mathbb{R}) \to D^+$ is defined by $G_{x,y,z}(t) = \chi(\frac{t}{2} - G^*(x,y,z))$ for all $x,y,z \in C(I,\mathbb{R})$ and $t > 0$, where

$$\chi(t) = \begin{cases} 0 & \text{if} \quad t \leq 0, \\ 1 & \text{if} \quad t > 0. \end{cases}$$

The space $(C(I,\mathbb{R}), G, T)$ with $T(a,b) = \min\{a,b\}$ is a complete Menger PGM-space. Also, we define the partial order relation " $\preceq$ " on $C(I,\mathbb{R})$ by $x \preceq y$ iff $||x||_\infty \leq ||y||_\infty$ for all $x,y \in C(I,\mathbb{R})$. Thus, $(C(I,\mathbb{R}), F, T, \preceq)$ is a partially ordered complete probabilistic G-metric space.

Theorem 4.1. *Let* $(C(I,\mathbb{R}), G, T, \preceq)$ *be the partially ordered complete probabilistic* G*-metric space and* $f : C(I,\mathbb{R}) \times C(I,\mathbb{R}) \to C(I,\mathbb{R})$ *be an operator defined by* $f(x,y)t = \int_a^b M(t,s)K(s,x(s),y(s))ds$*, where* $M \in C(I \times I, [0,\infty))$ *and* $K \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *are two operators satisfying the following conditions:*

(i) $||K||_\infty = \sup_{s\in I,\, x,y\in C(I,\mathbb{R})} |K(s,x(s),y(s))| < \infty,$

(ii) for all $x, y \in C(I, \mathbb{R})$ *and all* $t, s \in I$ *we have*

$$||K(s, x(s), y(s)) - K(s, u(s), v(s))|| \leq \tfrac{1}{4}(\max|x(s) - u(s)| + \max|y(s) - v(s)|),$$

(iii) $\sup_{t \in I} \int_a^b G(t,s)ds < 1.$

Then, the system of integral equations (4.1) *has a solution in* $C(I, \mathbb{R}) \times C(I, \mathbb{R})$.

Proof. For all $x, y, z \in C(I, \mathbb{R})$, we consider

$$G^*(x, y, z) = \max_{t \in I}(|x(t) - y(t)|) + \max_{t \in I}(|x(t) - z(t)|) + \max_{t \in I}(|y(t) - z(t)|).$$

Therefore, for all $x, y, z, u, v, w \in C(I, \mathbb{R})$ with $x \succeq u \succeq w$ and $y \preceq v \preceq z$, where either $u \neq w$ or $v \neq z$, we get

$$\begin{aligned}
& G^*(f(x,y), f(u,v), f(w,z)) \\
\leq\ & \max_{t \in I} \int_a^b M(t,s)|K(s, x(s), y(s)) - K(s, u(s), v(s))|ds \\
& + \max_{t \in I} \int_a^b M(t,s)|K(s, x(s), y(s)) - K(s, w(s), z(s))|ds \\
& + \max_{t \in I} \int_a^b M(t,s)|K(s, u(s), v(s)) - K(s, w(s), z(s))|ds \\
\leq\ & \max(\frac{1}{4}(|x(s) - u(s)| + |y(s) - v(s)|)) \max_{t \in I} \int_a^b M(t,s)ds \\
& + \max(\frac{1}{4}(|x(s) - w(s)| + |y(s) - z(s)|)) \max_{t \in I} \int_a^b M(t,s)ds \\
& + \max(\frac{1}{4}(|u(s) - w(s)| + |v(s) - z(s)|)) \max_{t \in I} \int_a^b M(t,s)ds \\
\leq\ & \max(\frac{1}{4}(|x(s) - u(s)| + |y(s) - v(s)|)) \\
& + \max(\frac{1}{4}(|x(s) - w(s)| + |y(s) - z(s)|)) \\
& + \max(\frac{1}{4}(|u(s) - w(s)| + |v(s) - z(s)|)),
\end{aligned}$$

which implies that

$$\begin{aligned}
G_{f(x,y), f(u,v), f(w,z)}(t) &= \chi(\frac{t}{2} - G^*(f(x,y), f(u,v), f(w,z)) \\
&\geq \chi(\frac{t}{2} - (\max(\frac{1}{4}(|x(s) - u(s)| + |y(s) - v(s)|)) \\
&\quad + \max(\frac{1}{4}(|x(s) - w(s)| + |y(s) - z(s)|))
\end{aligned}$$

$$
\begin{aligned}
& + \max(\frac{1}{4}(|u(s) - w(s)| + |v(s) - z(s)|))) \\
= \;& \chi(\frac{1}{2}(t - \frac{1}{2}(\max(|x(s) - u(s)| + |x(s) - w(s)| \\
& + |u(s) - w(s)|) + \max(|y(s) - v(s)| + |y(s) - z(s)| \\
& + |v(s) - z(s)|)))) \\
\geq \;& \frac{1}{2}\chi(t - (\max(|x(s) - u(s)| + |x(s) - w(s)| \\
& + |u(s) - w(s)|))) + \frac{1}{2}\chi(t - (\max(|y(s) - v(s)| \\
& + |y(s) - z(s)| + |v(s) - z(s)|))) \\
= \;& \frac{1}{2}(G_{x,u,w}(2t) + G_{y,v,z}(2t)).
\end{aligned}
$$

Therefore, all the hypotheses of Theorem 3.1 are held with $k = \frac{1}{2}$ and the operator f has a coupled fixed point which is the solution of the system of the integral equations. □

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