

ON WEAKLY SYMMETRIC AND SPECIAL WEAKLY RICCI
SYMMETRIC LP-SASAKIAN MANIFOLDS ADMITTING
SEMI-SYMMETRIC SEMI-METRIC CONNECTION

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Abstract. The aim of this paper is to study the geometric properties of LP-Sasakian manifolds with respect to Levi-Civita connection when they are weakly symmetric, weakly Ricci symmetric and special weakly symmetric with respect to semi-symmetric semi-metric connection. An illustration of three dimensional LP-Sasakian manifold is given.

Keywords: LP-Sasakian manifolds, Levi-Civita connection, weakly Ricci symmetric LP-Sasakian manifolds.

1. Introduction

The concept of an LP-Sasakian manifold was first developed in 1989 by K. Matsumoto [9]. The identical idea was then independently suggested by I. Mihai and R. Rosca [11], who produced multiple results on this manifold. Additionally, Venkatesha and C.S. Bagewadi [19], I. Mihai, A.A. Shaikh and U.C. De [12], A.A. Shaikh [18], C. Ozgur [14] and others have explored the LP-Sasakian manifold.

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Subsequently, numerous geometers have published various works in this field ([8], [4], [15], [16], [6]).

A non-flat Riemannian manifold (M^n, g) ($n > 2$) is called *weakly symmetric* if there exist 1-forms $\alpha, \beta, \gamma, \delta$ and σ such that

$$(1.1) \quad (\nabla_X R)(Y, Z, V, U) = \alpha(X)R(Y, Z, V, U) + \beta(Y)R(X, Z, V, U) \\ + \gamma(Z)R(Y, X, V, U) + \delta(V)R(Y, Z, X, U) \\ + \sigma(U)R(Y, Z, V, X),$$

holds for all vector fields $X, Y, \dots, V \in X(M)$, where R is the Riemannian curvature tensor of (M^n, g) of type $(0, 4)$ and ∇ is the covariant differentiation with respect to the Riemannian metric g . A weakly symmetric manifold is said to be *proper* if $\alpha = \beta = \gamma = \delta = \sigma = 0$ is not the case.

Let $\{e_i\}$, ($i = 1, 2, \dots, n$) be an orthonormal basis of the tangent space at point of the manifold. Then, putting $Y = U = e_i$ in (1.1) and taking summation for $1 \leq i \leq n$, we obtain

$$(1.2) \quad (\nabla_X S)(Z, V) = \alpha(X)S(Z, V) + \gamma(Z)S(X, V) + \delta(V)S(Z, X) \\ + \beta(R(X, Z)V) + \sigma(R(X, V)Z).$$

A Riemannian manifold (M^n, g) ($n > 2$) is called *weakly Ricci-symmetric* if there exist 1-forms ρ, μ, ν such that the relation

$$(1.3) \quad (\nabla_X S)(Y, Z) = \rho(X)S(Y, Z) + \mu(Y)S(X, Z) + \nu(Z)S(X, Y),$$

holds for any vector fields X, Y, Z , where S is the Ricci tensor of type $(0, 2)$ of the manifold M^n . A weakly Ricci-symmetric manifold is said to be *proper* if $\rho = \mu = \nu = 0$ is not the case.

An n -dimensional Riemannian manifold (M^n, g) is called a special weakly Ricci-symmetric $(SWRS)_n$ manifold if

$$(1.4) \quad (\nabla_X S)(Y, Z) = 2\alpha(X)S(Y, Z) + \alpha(Y)S(X, Z) + \alpha(Z)S(X, Y),$$

where α is a 1-form and is defined by

$$(1.5) \quad \alpha(X) = g(X, \rho),$$

where ρ is the associated vector field.

We are the following known result.

Lemma 1.1. [13] *If $M : g = c$ is a surface in R^n , then the gradient vector field is a non-vanishing normal vector field on the entire surface M .*

2. LP-Sasakian manifold

A differentiable manifold of dimensional $n(\text{odd})$ is called LP-Sasakian manifold if it admits a $(1, 1)$ -tensor field ϕ , a contravariant vector field ξ , a covariant vector field η and a Lorentzian metric g which satisfy:

$$(2.1) \quad \phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0,$$

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X),$$

for all $X, Y \in TM$.

Also LP-Sasakian manifold M^n satisfies

$$(2.3) \quad (\nabla_X \phi)Y = \{g(X, Y)\xi + 2\eta(Y)\eta(X)\xi\},$$

$$(2.4) \quad \nabla_X \xi = \phi X,$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g .

Example of LP-Sasakian manifold: Consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3; z \neq 0\}$, where (x, y, z) are the standard co-ordinates in R^3 . Let $\{E_1, E_2, E_3\}$ be linearly independent global frame field on M^n given by

$$(2.5) \quad E_1 = \frac{e^z}{x} \frac{\partial}{\partial x}, \quad E_2 = \frac{e^{z-ax}}{y} \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(E_1, E_2) &= g(E_1, E_3) = g(E_2, E_3) = 0, \\ g(E_1, E_1) &= g(E_2, E_2) = 1 \quad \text{and} \quad g(E_3, E_3) = -1. \end{aligned}$$

The (ϕ, ξ, η) is given by

$$\begin{aligned} \eta &= -dz, \quad \xi = E_3 = \frac{\partial}{\partial z}, \\ \phi E_1 &= -E_1, \quad \phi E_2 = -E_2, \quad \phi E_3 = 0. \end{aligned}$$

The linearity property of ϕ and g yields that

$$\begin{aligned} \eta(E_3) &= -1, \quad \phi^2 U = U + \eta(U)E_3, \\ g(\phi U, \phi W) &= g(U, W) + \eta(U)\eta(W), \quad g(U, \xi) = \eta(U), \end{aligned}$$

for any vector fields U, W on M . By definition of Lie bracket, we have

$$(2.6) \quad [E_1, E_2] = -\frac{ae^z}{x} E_2, \quad [E_1, E_3] = -E_1, \quad [E_2, E_3] = -E_2.$$

The Levi-Civita connection with respect to above metric g is given by Koszula formula

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Then we have,

$$\begin{aligned} \nabla_{E_1} E_1 &= -E_3, & \nabla_{E_1} E_2 &= 0, & \nabla_{E_1} E_3 &= -E_1, \\ \nabla_{E_2} E_1 &= \frac{ae^z}{x} E_2, & \nabla_{E_2} E_2 &= -\frac{ae^z}{x} E_1 - E_3, & \nabla_{E_2} E_3 &= -E_2, \\ \nabla_{E_3} E_1 &= 0, & \nabla_{E_3} E_2 &= 0, & \nabla_{E_3} E_3 &= 0. \end{aligned}$$

The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3 , i.e., $X = a_1 E_1 + a_2 E_2 + a_3 E_3$ and $Y = b_1 E_1 + b_2 E_2 + b_3 E_3$, where a_i and b_j are scalars. Clearly (ϕ, ξ, η, g) and X, Y satisfy equations (2.1), (2.2), (2.3) and (2.4). Thus M^n is LP-Sasakian manifold.

Also, in LP-Sasakian manifold M^n the following relations hold:

$$(2.7) \quad \eta(R(X, Y)Z) = \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\},$$

$$(2.8) \quad R(X, Y)\xi = \{\eta(Y)X - \eta(X)Y\},$$

$$(2.9) \quad R(\xi, X)Y = \{g(X, Y)\xi - \eta(Y)X\},$$

$$(2.10) \quad R(\xi, X)\xi = \{\eta(X)\xi + X\},$$

$$(2.11) \quad S(X, \xi) = (n-1)\eta(X),$$

$$(2.12) \quad Q\xi = (n-1)\xi,$$

for any vector fields X, Y, Z , where $R(X, Y)Z$ is the curvature tensor and S is the Ricci tensor.

3. Semi-symmetric semi-metric connection

A. Friedmann and J.A. Schouten [5] introduced the idea of a semi-symmetric linear connection. A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor T is of the form

$$(3.1) \quad T(X, Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form. Motivated by studies of author in [1], introduced the notion of semi-symmetric semi-metric connection $\tilde{\nabla}$ on a contact metric manifold and it is defined as

$$(3.2) \quad \tilde{\nabla}_X Y = \nabla_X Y - \eta(X)Y + g(X, Y)\xi,$$

where ∇ is Levi-Civita connection. A study on semi-symmetric connections and their properties can be found in [20, 3, 5, 7]. More recently, Mobin Ahmad and M. Danish Siddiqui [1] have studied a nearly Sasakian manifold with a semi-symmetric

semi-metric connection, proving the results of integrability conditions of distribution of semi-invariant submanifolds of an approximately Sasakian manifold, inspired by research done by the author in [1]. Our focus is on LP-Sasakian manifolds that exhibit weakly symmetry.

A relation between the curvature tensor of M^n with respect to the semi-symmetric semi-metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ is given by

$$(3.3) \quad \begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + 2[\eta(Y)g(X, Z) - \eta(X)g(Y, Z)]\xi + [g(X, \phi Y) \\ &- g(Y, \phi X)]Z + [g(Y, Z)\phi X - g(X, Z)\phi Y], \end{aligned}$$

where \tilde{R} and R are the Riemannian curvatures of the connections $\tilde{\nabla}$ and ∇ respectively. From (3.3), it follows that

$$(3.4) \quad \tilde{S}(Y, Z) = S(Y, Z) + 2\eta(Y)\eta(Z) + 2g(Y, Z) - g(Z, \phi Y) + Tg(Y, Z),$$

where $T = \text{trace}\phi = g(\phi e_i, e_i)$, \tilde{S} and S are the Ricci tensors of the connections $\tilde{\nabla}$ and ∇ respectively.

Taking Z instead of ξ , the above expression becomes

$$(3.5) \quad \tilde{S}(Y, \xi) = [(n - 1) + T]\eta(Y).$$

4. Weakly symmetric LP-Sasakian manifold admitting semi-symmetric semi-metric connection

Let \tilde{M}^n denote LP-Sasakian manifold admitting semi-symmetric semi-metric connection. Let \tilde{M}^n be weakly symmetric. Then equation (1.2) may be written as

$$(4.1) \quad \begin{aligned} (\tilde{\nabla}_X \tilde{S})(Z, V) &= \alpha(X)\tilde{S}(Z, V) + \gamma(Z)\tilde{S}(X, V) + \delta(V)\tilde{S}(Z, X) \\ &+ \beta(\tilde{R}(X, Z)V) + \sigma(\tilde{R}(X, V)Z). \end{aligned}$$

Taking covariant differentiation of the Ricci tensor \tilde{S} with respect to X , we have

$$(4.2) \quad (\tilde{\nabla}_X \tilde{S})(Z, V) = \tilde{\nabla}_X \tilde{S}(Z, V) - \tilde{S}(\tilde{\nabla}_X Z, V) - \tilde{S}(Z, \tilde{\nabla}_X V).$$

Putting $V = \xi$ in (4.2) and by virtue of (2.1), (2.4), (2.11), (3.2), (3.4), we find

$$(4.3) \quad \begin{aligned} (\tilde{\nabla}_X \tilde{S})(Z, \xi) &= (n - 1)\eta(\nabla_X Z) - (n - 1)\eta(X)\eta(Z) - (n - 1)g(X, Z) \\ &+ (n - 1)g(Z, \phi X) + X(T)\eta(Z) + T\eta(\nabla_X Z) - T\eta(X)\eta(Z) \\ &- Tg(X, Z) + Tg(Z, \phi X) + (n - 1)g(Z, \phi X) - S(Z, \phi X) \\ &- 2g(Z, \phi X) + g(Z, X) + \eta(X)\eta(Z). \end{aligned}$$

On the other hand replacing V with ξ in (4.1) and use (2.1), (2.11), (3.3), (3.4), (3.5), we immediately obtain

$$(4.4) \quad (\tilde{\nabla}_X \tilde{S})(Z, \xi) = [(n - 1) + T]\alpha(X)\eta(Z) + [(n - 1) + T]\gamma(Z)\eta(X)$$

$$\begin{aligned}
& + \delta(\xi)S(Z, X) + 2\delta(\xi)\eta(Z)\eta(X) + 2\delta(\xi)g(Z, X) \\
& - \delta(\xi)g(X, \phi Z) + T\delta(\xi)g(Z, X) + \eta(Z)\beta(X) \\
& - \eta(X)\beta(Z) + g(X, \phi Z)\beta(\xi) - g(Z, \phi X)\beta(\xi) \\
& + \eta(Z)\beta(\phi X) - \eta(X)\beta(\phi Z) + \eta(Z)\sigma(X) - g(X, Z)\sigma(\xi) \\
& - 2g(X, Z)\sigma(\xi) - 2\eta(X)\eta(Z)\sigma(\xi) + \eta(Z)\sigma(\phi X).
\end{aligned}$$

Hence, comparing the right hand side of the equations (4.3) and (4.4), we get

$$\begin{aligned}
(4.5) \quad & (n-1)\eta(\nabla_X Z) - (n-1)\eta(X)\eta(Z) - (n-1)g(X, Z) + (n-1)g(Z, \phi X) \\
& + X(T)\eta(Z) + T\eta(\nabla_X Z) - T\eta(X)\eta(Z) - Tg(X, Z) + Tg(Z, \phi X) \\
& + (n-1)g(Z, \phi X) - S(Z, \phi X) - 2g(Z, \phi X) + g(Z, X) + \eta(X)\eta(Z) \\
& = [(n-1) + T]\alpha(X)\eta(Z) + [(n-1) + T]\gamma(Z)\eta(X) + \delta(\xi)S(Z, X) \\
& + 2\delta(\xi)\eta(Z)\eta(X) + 2\delta(\xi)g(Z, X) - \delta(\xi)g(X, \phi Z) + T\delta(\xi)g(Z, X) \\
& + \eta(Z)\beta(X) - \eta(X)\beta(Z) + g(X, \phi Z)\beta(\xi) - g(Z, \phi X)\beta(\xi) \\
& + \eta(Z)\beta(\phi X) - \eta(X)\beta(\phi Z) + \eta(Z)\sigma(X) - g(X, Z)\sigma(\xi) \\
& - 2g(X, Z)\sigma(\xi) - 2\eta(X)\eta(Z)\sigma(\xi) + \eta(Z)\sigma(\phi X).
\end{aligned}$$

Plugging $Z = \xi$ in (4.5) and using these equations (2.1), (2.4), (2.11), we get the equation

$$\begin{aligned}
(4.6) \quad -X(T) & = -[(n-1) + T]\alpha(X) + [(n-1) + T]\gamma(\xi)\eta(X) \\
& + [(n-1) + T]\delta(\xi)\eta(X) - \beta(X) - \eta(X)\beta(\xi) \\
& - \beta(\phi X) - \sigma(X) - \eta(X)\sigma(\xi) - \sigma(\phi X).
\end{aligned}$$

At this stage we can't give any geometric meaning to this equation. If we take $X = \xi$, then

$$\begin{aligned}
(4.7) \quad \xi(T) & = [(n-1) + T][\alpha(\xi) + \gamma(\xi) + \delta(\xi)], \\
\text{i.e., } \text{grad}T.\xi & = [(n-1) + T][\alpha(\xi) + \gamma(\xi) + \delta(\xi)].
\end{aligned}$$

Since $[(n-1) + T] \neq 0$, we have $\text{grad}T$ is normal to ξ if and only if $[\alpha(\xi) + \gamma(\xi) + \delta(\xi)] = 0$.

Thus by Lemma 1.1 we can state the following:

Theorem 4.1. *Let \widetilde{M}^n be weakly symmetric LP-Sasakian manifold with respect to semi-symmetric semi-metric connection. Then the sum of 1-forms α , γ and δ on vanish on the characteristic vector field ξ if and only if the gradient of trace of the endomorphism ϕ is normal to M^n along ξ .*

5. On special weakly Ricci-symmetric LP-Sasakian manifold admitting semi-symmetric semi-metric connection

Let \widetilde{M}^n be special weakly Ricci-symmetric LP-Sasakian manifold. Then (1.4) may be written as

$$(5.1) \quad (\widetilde{\nabla}_X \widetilde{S})(Y, Z) = 2\alpha(X)\widetilde{S}(Y, Z) + \alpha(Y)\widetilde{S}(X, Z) + \alpha(Z)\widetilde{S}(X, Y).$$

Taking cyclic sum of (5.1). This implies that

$$(5.2) \quad (\tilde{\nabla}_X \tilde{S})(Y, Z) + (\tilde{\nabla}_Y \tilde{S})(Z, X) + (\tilde{\nabla}_Z \tilde{S})(X, Y) = 4[\alpha(X)\tilde{S}(Y, Z) + \alpha(Y)\tilde{S}(Z, X) + \alpha(Z)\tilde{S}(X, Y)].$$

Let \tilde{M}^n admit a cyclic Ricci tensor. Then (5.2) reduces to

$$(5.3) \quad 0 = \alpha(X)\tilde{S}(Y, Z) + \alpha(Y)\tilde{S}(Z, X) + \alpha(Z)\tilde{S}(X, Y).$$

Now setting $Z = \xi$ in (5.3) and yield (2.1), (3.4), (3.5), we get

$$(5.4) \quad 0 = [(n - 1) + T]\alpha(X)\eta(Y) + [(n - 1) + T]\alpha(Y)\eta(X) + \alpha(\xi)S(X, Y) + 2\alpha(\xi)\eta(X)\eta(Y) + 2\alpha(\xi)g(X, Y) - \alpha(\xi)g(Y, \phi X) + T\alpha(\xi)g(X, Y).$$

Again setting $Y = \xi$ in (5.4) and employ (1.5) and (2.1), we obtain

$$(5.5) \quad 2\eta(\rho)\eta(X) = \alpha(X).$$

Changing X to ξ in (5.5) and make use of (1.5) and (2.1), it follows that

$$(5.6) \quad \eta(\rho) = 0.$$

By virtue of (5.6) in (5.5), we procure $\alpha(X) = 0$, for all X .

This lead us to the following

Theorem 5.1. *Let \tilde{M}^n be special weakly Ricci-symmetric LP-Sasakian manifold M^n with respect to semi-symmetric semi-metric connection and admits a cyclic Ricci tensor. Then the 1-form α must vanish on M^n . However the converse holds trivially.*

Next setting $Z = \xi$ in (5.1), we have the following

$$(5.7) \quad (\tilde{\nabla}_X \tilde{S})(Y, \xi) = 2\alpha(X)\tilde{S}(Y, \xi) + \alpha(Y)\tilde{S}(X, \xi) + \alpha(\xi)\tilde{S}(X, Y).$$

The left hand side can be written in the form

$$(5.8) \quad (\tilde{\nabla}_X \tilde{S})(Y, \xi) = \tilde{\nabla}_X \tilde{S}(Y, \xi) - \tilde{S}(\tilde{\nabla}_X Y, \xi) - \tilde{S}(Y, \tilde{\nabla}_X \xi).$$

By view of (1.5), (2.1), (2.11), (3.2), (3.4), (3.5), we infer that

$$(5.9) \quad (n - 1)\eta(\nabla_X Y) - (n - 1)\eta(X)\eta(Y) - (n - 1)g(X, Y) + (n - 1)g(Y, \phi X) + X(T)\eta(Y) + T\eta(\nabla_X Y) - T\eta(X)\eta(Y) - Tg(X, Y) + Tg(Y, \phi X) + (n - 1)g(Y, \phi X) - S(Y, \phi X) - 2g(Y, \phi X) + g(Y, X) + \eta(X)\eta(Y) = 2[(n - 1) + T]\alpha(X)\eta(Y) + [(n - 1) + T]\alpha(Y)\eta(X) + \eta(\rho)\{S(X, Y) + 2\eta(X)\eta(Y) + 2g(X, Y) - g(Y, \phi X) + Tg(X, Y)\}.$$

Choosing $Y = \xi$ in (5.9) and utilize (1.5) and (2.1), (2.4), (2.11), gives

$$(5.10) \quad -X(T) = -2[(n-1) + T]\alpha(X) + 2[(n-1) + T]\eta(\rho)\eta(X),$$

$$(5.11) \quad \text{i.e., } X(T) = 2[\eta(\rho)\eta(X) - \alpha(X)][(n-1) + T].$$

We know that $X(T) = \text{grad}T \cdot X$. Since $[(n-1) + T] \neq 0$, $\text{grad}T$ is normal to M^n if and only if $\eta(\rho)\eta(X) = \alpha(X)$.

Hence we state the following lemma 1.1.

Theorem 5.2. *Let \widetilde{M}^n be special weakly Ricci-symmetric LP-Sasakian manifold M^n with respect to semi-symmetric semi-metric connection. Then the gradient of the trace of the endomorphism of T is normal to M^n if and only if $\eta(\rho)\eta(X) = \alpha(X)$.*

If we put $X = \xi$ in $\eta(\rho)\eta(X) = \alpha(X)$, then $\eta(\rho) = 0$. Thus $\alpha(X) = 0$.

Hence we can restate the Theorem 5.2 as follows:

Corollary 5.1. *Let \widetilde{M}^n be special weakly symmetric LP-Sasakian manifold with respect to semi-symmetric semi-metric connection. Then the gradient of the trace of the endomorphism of T is normal to M^n along ξ if and only if 1-form vanish on the whole space M^n .*

We conclude from the above results:

Conclusion: If \widetilde{M}^n is weakly symmetric LP-Sasakian manifold, then the sum of the 1-forms vanish along the characteristic vector field ξ if and only if the trace of endomorphism of ϕ is normal to M^n along ξ , whereas if \widetilde{M}^n is special weakly Ricci-symmetric then the 1-form vanishes for every vector field if and only if trace of endomorphism ϕ is normal to M^n along ξ . If \widetilde{M}^n admits cyclic Ricci tensor then the 1-form vanish the whole manifold M^n without any endomorphism.

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