FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 5 (2021), 983- 993
https://doi.org/10.22190/FUMI210322072B
Original Scientific Paper

CONSTRUCTION OF OFFSET SURFACES WITH A GIVEN NON-NULL ASYMPTOTIC CURVE

Ergin Bayram and Fatma Güler

Faculty of Science and Arts, Department of Mathematics,
P. O. Box 55200, Samsun, Turkey

Abstract. In the present work, we study construction of offset surfaces with a given non-null asymptotic curve. Let $\alpha(s)$ be a spacelike or timelike unit speed curve with non-vanishing curvature and $\varphi(s,t)$ be a surface pencil accepting $\alpha(s)$ as a common asymptotic curve. We obtain conditions such that the offset surface possesses the image of $\alpha(s)$ as an asymptotic curve. We validate the method with illustrative examples.
Keywords: Ofset surface, Minkowski 3-space, asymptotic curve.

1. Introduction

Traditional research on curves and surfaces focuses on to find chracteristic curves, such as geodesic curve, asymptotic curve, and principal curve etc. on a present surface. However, the reverse problem, that is finding surfaces possessing a prescribed curve, is much more interesting. The construction of surfaces with a given characteristic curve is a new research area that attracts the interests of many researchers. The first study of this type of construction conducted by Wang et al. [18]. They presented a method for surfaces accepting a given curve as a common geodesic. Inspired by Wang et al. [18], researchers obtained constraints for a prescribed curve to be a specific curve on constructed surfaces [1 - 3, 8, 10, 16, 17].

Offset surfaces have a great importance among surfaces. An offset surface is a surface at a fixed distance along the unit normal vector field of a given surface.

Received March 22, 2021. accepted October 19, 2021.
Communicated by Ljubica Velimirović
Corresponding Author: Ergin Bayram, Faculty of Science and Arts, Department of Mathematics, P. O. Box 55200, Samsun, Turkey | E-mail: erginbayram@yahoo.com
2010 *Mathematics Subject Classification.* Primary 51B20; Secondary 53A35, 53B30

An idea of the value of offset surfaces can be realized from the great volume of literature [7, 9, 11, 12, 14, 15]. Moon [12] presented equivolumetric offset surface. Authors in [14] introduced a new algorithm for the efficient and reliable generation of offset surfaces for polygonal meshes. Hermann [9] showed that a base surface and its offset have the same geometric continuum. Güler et al. [8] obtained necessary constraints such that the image curve is a common asymptotic curve on each offset. The properties of offset surfaces have been examined in [7].

Motivated by the increasing importance of surfaces in mathematical physics, and very restricted knowledge about offset surfaces in Minkowski 3-space, we develop the theory of offset surfaces using non-null curves. We present constraints for a non-null curve to be a common asymptotic on an offset surface pencil. In particular, given a surface pencil with a common asymptotic curve, we give conditions such that the image curve is also a common asymptotic on each offset. The method is illustrated with several examples.

2. Preliminaries

In this section, we review some notions related with curves and surfaces in Minkowski 3-space.

The real vector space IR^3 endowed with the scalar product

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3, \tag{2.1}$$

where $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in IR^3$, is called Minkowski 3-space and denoted by IR^3_1.

A vector $X \in IR^3$ is called spacelike, timelike or null if

$$\begin{cases} \langle X, X\rangle > 0 \;\; or \;\; X = 0, \\ \langle X, X\rangle < 0, \\ \langle X, X\rangle = 0 \;\; and \; X \neq 0, \end{cases} \tag{2.2}$$

respectively [5].

The vectoral product of X and Y is defined as [13]

$$X \times Y = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2,\; x_1y_3 - x_3y_1,\; x_2y_{1-}x_1y_2)\,. \tag{2.3}$$

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha = \alpha(s)$ in Minkowski 3-space, where the vector fields T, N and B are called the tangent, the principal normal and the binormal vector field of α, respectively.

Theorem 2.1. *Let $\alpha = \alpha(s)$ be a spacelike or timelike arclength curve with non vanishing curvature. The Frenet formula of α is given by*

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\varepsilon_1\delta_1\kappa & 0 & \tau \\ 0 & \varepsilon_1\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.4}$$

where $\langle T, T\rangle = \varepsilon_1,\ \langle N, N\rangle = \delta_1$. Also, we have $B = \varepsilon_1\delta_1 (T \times N),\ \kappa = \delta_1 \left\langle T', N\right\rangle$ and $\tau = -\varepsilon_1\delta_1 \left\langle N', B\right\rangle$. The functions κ and τ are called the curvature and torsion of α, respectively.

If $\alpha(s)$ is a non-null curve on a surface, then we have another frame, the so called Darboux frame $\{T, b, n\}$. Here, T is the unit tangent vector field of α, n is the unit normal vector field of the surface and b is a unit vector field given by $b = \varepsilon_1\varepsilon_3 (n \times T)$, where $\langle n, n\rangle = \varepsilon_3$. Because, T is the same in each frame, the other vector fields of these frames lie on the same plane. Thus, we can give the following relation about these frames as:

Let φ be a spacelike surface and $\alpha(s)$ a spacelike curve on φ. We have

$$\begin{bmatrix} T \\ b \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.5}$$

where θ is the hyperbolic angle between the vectors b and N.

Let φ be a timelike surface and $\alpha(s)$ a spacelike or timelike curve on φ.

1) If $\alpha(s)$ is timelike curve, then

$$\begin{bmatrix} T \\ b \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.6}$$

where θ is the angle between the vectors b and N.

2) If $\alpha(s)$ is a spacelike curve, then

$$\begin{bmatrix} T \\ b \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.7}$$

where θ is the hyperbolic angle between the vectors b and N.

Let $\varphi(s, t)$ be a timelike or spacelike surface. We have the following formula for the Darboux frame as

$$\begin{bmatrix} T' \\ b' \\ n' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 k_g & \varepsilon_3 k_n \\ -\varepsilon_1 k_g & 0 & \varepsilon_3 \tau_g \\ -\varepsilon_1 k_n & -\varepsilon_2 \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ b \\ n \end{bmatrix}, \tag{2.8}$$

where $\varepsilon_1 = \langle T, T \rangle$, $\varepsilon_2 = \langle b, b \rangle$, $\varepsilon_3 = \langle n, n \rangle$, $b = -\varepsilon_2 (n \times T)$ and k_g, k_n and τ_g are the geodesic curvature, the normal curvature and the geodesic torsion of $\alpha(s)$, respectively [6].

3. Construction of surfaces with a non-null asymptotic curve

Let $\alpha(s)$ be a spacelike or timelike arclength curve with nonvanishing curvature. Surfaces passing through $\alpha(s)$ are given by

$$\varphi(s,t) = \alpha(s) + x(s,t)\,T(s) + y(s,t)\,N(s) + z(s,t)\,B(s), \tag{3.1}$$

$A_1 \leq s \leq A_2$, $B_1 \leq t \leq B_2$, where $x(s,t)$, $y(s,t)$ and $z(s,t)$ are C^2 marching-scale functions. Assume that $\varphi(s,t_0) = \alpha(s)$ for some $t_0 \in [B_1, B_2]$, so that α becomes a parameter curve on $\varphi(s,t)$.

The normal vector field of $\varphi(s,t)$ is

$$n(s,t) = \frac{\partial \varphi}{\partial s} \times \frac{\partial \varphi}{\partial t} \tag{3.2}$$

and along the curve $\alpha(s)$, one can write it as

$$n(s,t_0) = \phi_1(s,t_0)\,T(s) + \phi_2(s,t)\,N(s) + \phi_3(s,t)\,B(s), \tag{3.3}$$

where

$$\begin{cases} \phi_1(s,t_0) = \left[\frac{\partial z}{\partial s}(s,t_0)\frac{\partial y}{\partial t}(s,t_0) - \frac{\partial y}{\partial s}(s,t_0)\frac{\partial z}{\partial t}(s,t_0)\right]\varepsilon_1, \\ \phi_2(s,t_0) = \left[\left(1+\frac{\partial x}{\partial s}(s,t_0)\right)\frac{\partial z}{\partial t}(s,t_0) - \frac{\partial z}{\partial s}(s,t_0)\frac{\partial x}{\partial t}(s,t_0)\right]\delta_1, \\ \phi_3(s,t_0) = \left[\frac{\partial y}{\partial s}(s,t_0)\frac{\partial x}{\partial t}(s,t_0) - \left(1+\frac{\partial x}{\partial s}(s,t_0)\right)\frac{\partial y}{\partial t}(s,t_0)\right]\delta_2, \end{cases} \tag{3.4}$$

$\varepsilon_1 = \langle T, T \rangle$, $\delta_1 = \langle N, N \rangle$ and $\delta_2 = \langle B, B \rangle$.

Theorem 3.1. *A non-null curve $\alpha(s)$ is a common asymptotic curve on the surface pencil $\varphi(s,t)$ [16] if*

$$x(s,t_0)=y(s,t_0)=z(s,t_0)=\frac{\partial z}{\partial t}(s,t_0)\equiv 0. \tag{3.5}$$

To obtain regular surfaces one need $\frac{\partial y}{\partial t}(s,t_0)\neq 0$ as an extra condition.

Definition 3.1. Let $\varphi(s,t)$ be a parametric surface with unit normal vector field $\widehat{n}(s,t)$. A parametric offset surface is defined by

$$\overline{\varphi}(s,t)=\varphi(s,t)+r\widehat{n}(s,t), \tag{3.6}$$

r being a non zero real constant [19].

Using Eqn. (3.1) offset surface pencil has the form

$$\overline{\varphi}(s,t)=\alpha(s)+r\widehat{n}(s,t)+x(s,t)T(s)+y(s,t)N(s)+z(s,t)B(s), \tag{3.7}$$

$\beta(s)=\alpha(s)+r\widehat{n}(s,t)$ being the image of $\alpha(s)$ on $\overline{\varphi}(s,t)$.

Theorem 3.2. *Let $\alpha(s)$ be a non-null regular curve on the surface pencil $\varphi(s,t)$. Then*

$$\begin{aligned}\overline{k_g}^r&=-\frac{1}{v^3}\left[-k_gv^2-r\varepsilon_3\left(r\tau_gk_n'+\tau_g'(1+r\varepsilon_1k_n)\right)\right]\\ \overline{k_n}^r&=\frac{1}{v^2}\left[k_n(1+r\varepsilon_1k_n)+r\varepsilon_2\tau_g^2\right]\\ \overline{\tau_g}^r&=-\frac{1}{v^2}\left[r\varepsilon_1\varepsilon_2k_n\tau_g-\varepsilon_2\tau_g(1+r\varepsilon_1k_n)\right],\end{aligned} \tag{3.8}$$

for the image curve $\beta(s)$ on the offset surface pencil $\overline{\varphi}(s,t)$, respectively, where

$$v=\|\beta'(s)\|=\left|(1+r\varepsilon_1k_n)^2\varepsilon_1+\varepsilon_2r^2\tau_g^2\right|^{1/2}, \tag{3.9}$$

and k_g, k_n, τ_g are the geodesic, the normal curvature and the geodesic torsion of $\alpha(s)$, respectively.

This result also exists in [4] for spacelike surfaces.

Theorem 3.3. *Let $\left\{\overline{T}^r,\overline{N}^r,\overline{B}^r\right\}$ be the Frenet frame of the image curve $\beta(s)$ on $\overline{\varphi}(s,t)$ and $\{T,b,n\}$ the Darboux frame of $\alpha(s)$ on $\varphi(s,t)$. Then we have*

$$\begin{cases}\overline{T}^r=\frac{1}{v}\left[(1+r\varepsilon_1k_n)T+r\varepsilon_2\tau_gb\right]\\ \overline{N}^r=\frac{1}{v^4\sqrt{\left(\overline{k_g}^r\right)^2-\left(\overline{k_n}^r\right)^2}}\left[-rv^3\tau_g\overline{k_g}^rT+\varepsilon_1v^3\overline{k_g}^r(1+r\varepsilon_1k_n)b-\varepsilon_3\overline{k_n}^rv^4n\right]\\ \overline{B}^r=\frac{1}{v^3\sqrt{\left(\overline{k_g}^r\right)^2-\left(\overline{k_n}^r\right)^2}}\left[rv^2\tau_g\overline{k_n}^rT-\varepsilon_1v^2\overline{k_n}^r(1+r\varepsilon_1k_n)b+v^3\varepsilon_3\overline{k_g}^rn\right],\end{cases} \tag{3.10}$$

where $v=\|\beta'(s)\|=\left|(1+r\varepsilon_1k_n)^2\varepsilon_1+\varepsilon_2r^2\tau_g^2\right|^{1/2}$, $\overline{k_g}^r$, $\overline{k_n}^r$ are the geodesic curvature and the normal curvature of the image curve $\beta(s)$ and k_g, k_n, τ_g are the geodesic, the normal curvature and the geodesic torsion of $\alpha(s)$, respectively.

Now, suppose that $\alpha(s)$ is a common spacelike asymptotic and parameter curve with timelike binormal on the spacelike surface pencil. Our objective is to find sufficient constraints for the curve $\beta(s)$ to be both an asymptotic curve and parameter curve on the offset surface pencil $\overline{\varphi}(s,t)$.

Observe that, by Eqn. (3.7), $\beta(s)$ is a parameter curve on each offset.

The necessary and sufficient condition forthe image curve $\beta(s)$ to be an asymptotic curve on the offset surface $\overline{\varphi}(s,t)$ is

$$\left\langle \frac{\partial \overline{n}^r}{\partial s}(s,t_0), \overline{T}^r(s) \right\rangle = 0, \tag{3.11}$$

where $\overline{T}^r(s)$ is the tangent vector field of the image curve $\beta(s)$ and $\overline{n}^r(s,t_0)$ is the unit normal vector field of $\overline{\varphi}(s,t)$ through the image curve. According to [19], we have $\overline{n}^r(s,t_0) = \pm n(s,t_0)$. Now, we have the following equivalent asymptotic requirement

$$\left\langle \frac{\partial n}{\partial s}(s,t_0), \overline{T}^r(s) \right\rangle = 0, \tag{3.12}$$

where $n(s,t_0)$ is the normal vector field of $\varphi(s,t)$. By the asymptotic requirement of $\alpha(s)$, we have

$$n(s,t_0) = \frac{\partial y}{\partial s}(s,t_0)\, B(s). \tag{3.13}$$

With the help of Eqns. (2.4), (2.7), (3.10) and (3.12) we obtain

$$\tau(s)\,\tau_g(s)\,\frac{\partial y}{\partial t}(s,t_0)\, ch\theta(s) = \tau_g(s)\,\frac{\partial^2 y}{\partial s \partial t}(s,t_0)\, sh\theta(s), \tag{3.14}$$

for $\beta(s)$ to be an asymptotic curve on every spacelike offset surface pencil $\overline{\varphi}(s,t)$.

Note that, if $\alpha(s)$ is a line of curvature, i.e $\tau_g(s) \equiv 0$, then Eqn. (3.14) is satisfied and $\beta(s)$ be an asymptotic curve on the spacelike offset surface pencil $\overline{\varphi}(s,t)$.

Theorem 3.4. *Let $\varphi(s,t)$ be a spacelike surface pencil with a common spacelike parametric and asymptotic curve $\alpha(s)$ with timelike binormal. The image curve $\beta(s)$ of $\alpha(s)$ is a common asymptotic curve on the spacelike offset surface pencil $\overline{\varphi}(s,t)$, if*

$$\begin{cases} x(s,t_0) = y(s,t_0) = z(s,t_0) \equiv 0. \\ y(s,t) = e^{\int \tau(s)\coth\theta(s)ds} \int \psi(t)\,dt + \xi(s), \end{cases} \tag{3.15}$$

where $A_1 \leq s \leq A_2,\ \ B_1 \leq t \leq B_2,\ \psi \in C^2,\ \xi \in C^1$.

Proof. Since the $\alpha(s)$ curve is a parameter curve on the surface $\varphi(s,t)$, we have

$$x(s,t_0) = y(s,t_0) = z(s,t_0) \equiv 0.$$

For the image curve $\beta(s)$ of $\alpha(s)$ to be a common asymptotic curve on the spacelike offset surface pencil $\overline{\varphi}(s,t)$, we can use Eqn. (3.12). If Eqns. (3.4), (3.10) and (2.7) are written in Eqn. (3.12), then we obtain a second- order linear partial differential equation with variable coefficients as follows,

$$\tau \cosh\theta \frac{\partial y(s,t_0)}{\partial t} = \sinh\theta \frac{\partial^2 y(s,t_0)}{\partial s \partial t}, \tag{3.16}$$

where since $\alpha(s)$ is an asymptotic on the surface pencil $\varphi(s,t)$, we have $\tau_g \neq 0$.The desired result is obtained from the solution of Eqn. (3.17). □

Now, suppose that $\varphi(s,t)$ is a timelike surface with a common timelike asymptotic curve $\alpha(s)$. Hence, the offset surface $\overline{\varphi}(s,t)$ of $\varphi(s,t)$ is also a timelike surface.

By a similar investigation we obtain the following theorem:

Theorem 3.5. *Let $\varphi(s,t)$ be a timelike surface pencil with a common timelike parametric and asymptotic curve $\alpha(s)$ or spacelike parametric and asymptotic curve $\alpha(s)$ with spacelike binormal. The image curve $\beta(s)$ of $\alpha(s)$ is a common asymptotic curve on the timelike offset surface pencil $\overline{\varphi}(s,t)$, if*

$$\begin{cases} x(s,t_0) = y(s,t_0) = z(s,t_0) \equiv 0. \\ y(s,t) = e^{\int \tau(s)\cot\theta(s)ds} \int \psi(t)\,dt + \xi(s), \end{cases} \tag{3.17}$$

where $A_1 \leq s \leq A_2$, $B_1 \leq t \leq B_2$, $\psi \in C^2$, $\xi \in C^1$.

4. Examples

4.1. Example 1

Unit speed timelike curve $\alpha(s) = \left(\frac{5}{3}s, \frac{4}{9}\cos(3s), \frac{4}{9}\sin(3s)\right)$ has Frenet vector fields as

$$\begin{cases} T(s) = \left(\frac{5}{3}, -\frac{4}{3}\sin(3s), \frac{4}{3}\cos(3s)\right), \\ N(s) = (0, -\cos(3s), -\sin(3s)), \\ B(s) = \left(-\frac{4}{3}, \frac{5}{3}\sin(3s), -\frac{5}{3}\cos(3s)\right), \end{cases}$$

and torsion $\tau(s) \equiv 5$. Choosing $\xi(s) \equiv 0$, $\psi(t) \equiv 1$, $t_0 = 0$ and $\theta(s) = \frac{\pi}{4}$ yields $y(s,t) = (t + c_1)e^{5s+c_2}$ and for $c_1 = c_2 = 0$, $y(s,t) = te^{5s}$. Letting $x(s,t) = z(s,t) \equiv 0$ Theorems 3.1 and 3.5 are satisfied. Thus, we obtain the timelike surface

$$\varphi(s,t) = \left(\frac{5}{3}s, \left(\frac{4}{9} - te^{5s}\right)\cos(3s), \left(\frac{4}{9} - te^{5s}\right)\sin(3s)\right),$$

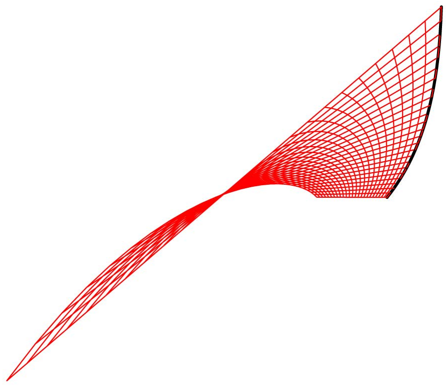

FIG. 4.1: Timelike surface $\varphi(s,t)$ and its asymptotic curve $\alpha(s)$.

$0 \leq s \leq 0.3,\ 0 \leq t \leq 0.2$, accepting $\alpha(s)$ as an asymptotic curve (Figure 4.1).

To obtain the offset surface of $\varphi(s,t)$, first we calculate

$$\widehat{n}(s,t) = \frac{1}{A}\left(4 - 9te^{5s}, 5\sin(3s), -5\cos(3s)\right),$$

where $A = \left|25 - \left(9te^{5s} - 4\right)^2\right|^{\frac{1}{2}}$. Now for $r = 3$, the image curve of $\alpha(s)$ is

$$\begin{aligned}\beta(s) &= \alpha(s) + 3\widehat{n}(s,0) \\ &= \left(\frac{5}{3}s + 4, \frac{4}{9}\cos(3s) + 5\sin(3s), \frac{4}{9}\sin(3s) - 5\cos(3s)\right).\end{aligned}$$

Using Eqn. (3.6), we get the offset timelike surface

$$\overline{\varphi}(s,t) = \left(\frac{5}{3}s - \frac{3\left(9te^{5s} - 4\right)}{A}, \left(\frac{4}{9} - te^{5s}\right)\cos(3s) + \frac{15\sin(3s)}{A},\right.$$
$$\left.\left(\frac{4}{9} - te^{5s}\right)\sin(3s) - \frac{15\cos(3s)}{A}\right),$$

$0 \leq s \leq 0.3,\ 0 \leq t \leq 0.2$, accepting $\beta(s)$ as an asymptotic curve (Figure 4.2).

4.2. Example 2

The Frenet vector fields of the spacelike curve $\alpha(s) = \left(\frac{1}{3}\sinh\left(\sqrt{3}s\right), \frac{2\sqrt{3}}{3}s, \frac{1}{3}\cosh\left(\sqrt{3}s\right)\right)$ with timelike binormal are

$$\begin{cases} T(s) = \left(\frac{\sqrt{3}}{3}\cosh\left(\sqrt{3}s\right), \frac{2\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\sinh\left(\sqrt{3}s\right)\right), \\ N(s) = \left(\sinh\left(\sqrt{3}s\right), 0, \cosh\left(\sqrt{3}s\right)\right), \\ B(s) = \left(\frac{2\sqrt{3}}{3}\cosh\left(\sqrt{3}s\right), \frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\sinh\left(\sqrt{3}s\right)\right), \end{cases}$$

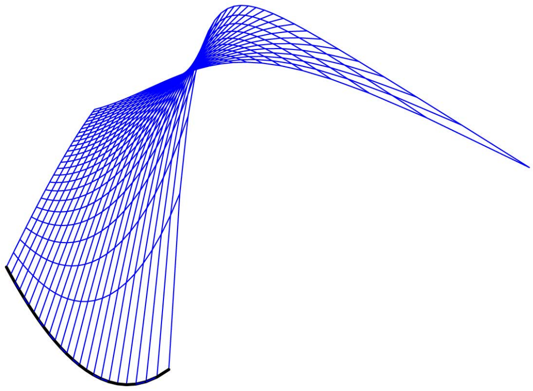

FIG. 4.2: Timelike offset surface $\overline{\varphi}(s,t)$ and its asymptotic curve $\beta(s)$.

and its torsion is $\tau(s) \equiv -2$. Choosing $\xi(s) \equiv 0$, $\psi(t) \equiv 1$, $t_0 = 0$ and $\theta(s) = \coth^{-1}\left(-\frac{1}{2}\right)$ yields $y(s,t) = (t + c_1) e^{s+c_2}$ and for $c_1 = c_2 = 0$, $y(s,t) = te^s$. Letting $x(s,t) = z(s,t) \equiv 0$, Theorems 3.1 and 3.4 are satisfied. Thus, we obtain the spacelike surface

$$\varphi(s,t) = \left((3 + te^s)\sinh\frac{s}{4}, \frac{5}{4}s, (3 + te^s)\cosh\frac{s}{4}\right),$$

$0 \leq s \leq 1$, $-1 \leq t \leq 1$, accepting $\alpha(s)$ as an asymptotic curve (Figure 4.3).

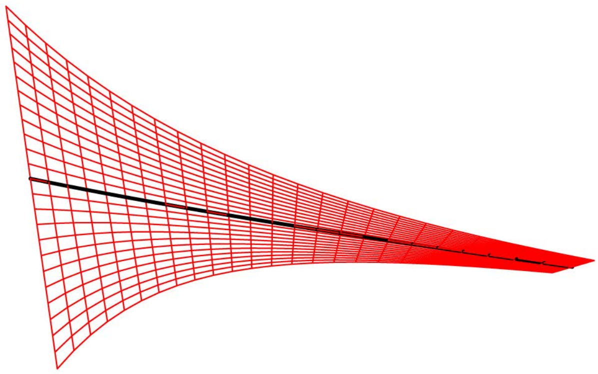

FIG. 4.3: Spacelike surface $\varphi(s,t)$ and its asymptotic curve $\alpha(s)$.

Using Eqn. (3.6), we get the offset spacelike surface

$$\overline{\varphi}(s,t) = \left((3 + te^s)\sinh\frac{s}{4} + \frac{20}{A}\cosh\frac{s}{4}, \frac{5}{4}s + \frac{4(te^s+3)}{A}, (3 + te^s)\cosh\frac{s}{4} + \frac{20}{A}\sinh\frac{s}{4}\right),$$

$0 \leq s \leq 5$, $0 \leq t \leq 5$, accepting $\beta(s)$ as an asymptotic curve (Figure 4.4).

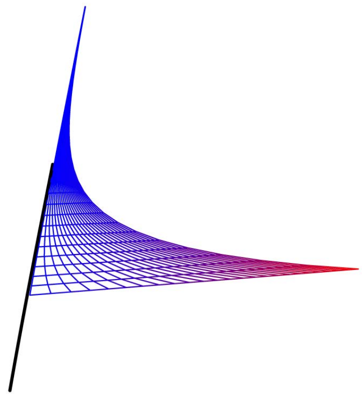

FIG. 4.4: Spacelike offset surface $\overline{\varphi}(s,t)$ and its asymptotic curve $\beta(s)$.

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