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GEOMETRIC INEQUALITIES FOR DOUBLY WARPED PRODUCTS POINTWISE BI-SLANT SUBMANIFOLDS IN CONFORMAL SASAKIAN SPACE FORM

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Abstract. In this paper, we have established some geometric inequalities for the squared mean curvature in terms of warping functions of a doubly warped product pointwise bi-slant submanifold of a conformal Sasakian space form with a quarter symmetric metric connection. The equality cases havve also been considered. Moreover, some applications of obtained results are derived.
Keywords: doubly warped products, bi-slant submanifolds, quarter symmteric metric connection,conformal Sasakian space form.

1. Introduction

In 2000, B. Unal [17] introduced the notion of doubly warped products as a generalization of warped products and it states that: let N_1 and N_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively. Further, let us suppose that f_1 & f_2 are positive differentiable functions on N_1 and N_2 respectively. Then, the doubly warped product $N = {}_{f_2}N_1 \times_{f_1} N_2$ is defined as the product manifold $N_1 \times N_2$ equipped with the warped metric $g = f_2^2 g_1 + f_1^2 g_2$. In a meticulous manner, if $t_1 : N_1 \times N_2 \to N_1$ and $t_2 : N_1 \times N_2 \to N_2$ are natural projections, then the metric g is given by [17]

$$g(X,Y) = (f_2 \circ t_2)^2 g_1(\iota_1^* X, \iota_1^* Y) + (f_1 \circ t_1)^2 g_2(t_2^* X, t_2^* Y), \tag{1.1}$$

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for any vector fields X, Y on N, where $*$ denotes the symbol for tangent maps.

It is important to note that on a doubly warped product manifold $N = {}_{f_2}N_1 \times_{f_1} N_2$ if either f_1 or f_2 is constant on N but not both, then N is a single warped product. Furthermore, if both f_1 and f_2 are constant function on N, then N is locally a Riemannian product. A doubly warped product manifold is said to be proper if both f_1 and f_2 are non-constant functions on N.

On the other hand, the immersibility/non-immersibility of a Riemannian manifold in a space form is one of the most fundamental problems in the theory of submanifold which started with the most celebrated Nash embedding theorem [11]. In this theorem, actually Nash was aiming to take extrinsic help. However, due to the lack of control of the extrinsic properties of the submanifolds by the known intrinsic invariant, the aim cannot be reached. Motivated by this and to overcome the difficulties, Chen introduced new types of Riemannian invariants and established general optimal relationship between extrinsic invariants and intrinsic invariants on the submanifold. Motivated by Chen's result, several inequalities have been obtained by many geometers for warped products and doubly warped products in different setting of the ambient manifolds [4, 5, 8, 9, 10, 12, 13, 15, 16, 19, 20]. In this paper, we have studied doubly warped product pointwise bi-slant submanifolds isometrically immersed into a conformal Sasakian space form with a quarter symmetric metric connection. The inequalities which we shall obtain in this paper are very fascinating because we derive upper bound and lower bound for warping functions in terms of mean curvature, scalar curvature and pointwise constant φ-sectional curvature c. The obtained results generalize some other inequalities as special cases.

2. Preliminaries

Let $\tilde{N}$ be a Riemannian manifold with Riemannian metric g. A linear connection $\bar{\nabla}$ on $\tilde{N}$ is called a quarter-symmetric connection if its torsion tensor T given by

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y]$$

and satisfies

$$T(X, Y) = \pi(Y)\varphi X - \pi(X)\varphi Y,$$

where π is a 1-form and $\mathcal{V}$ is a vector field such that $\pi(X) = g(X, \mathcal{V})$ and φ is a (1,1) tensor field. If $\bar{\nabla} g = 0$, then $\bar{\nabla}$ is known as quarter-symmetric metric connection and $\bar{\nabla} g \neq 0$, then $\bar{\nabla}$ is known as quarter symmetric non-metric connection. In this setting, it is shown in [14], one can easily obtain a special quarter-symmetric connection defined as

$$\bar{\nabla}_X Y = \bar{\tilde{\nabla}}_X Y + \lambda_1 \pi(Y) X - \lambda_2 g(X, Y)\mathcal{V}. \tag{2.1}$$

This is a general class of connection in the sense of (2.1) can be obtained as:

1. when $\lambda_1 = \lambda_2 = 1$, then the above connection reduces to semi-symmetric metric connection.

2. when $\lambda_1 = 1$ and $\lambda_2 = 0$, then the above connection reduces to semi-symmetric non metric connection.

The curvature tensor with respect to $\bar{\nabla}$ is given by

$$\bar{\mathcal{R}}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{2.2}$$

Similarly, we can define the curvature tensor with respect to $\tilde{\bar{\nabla}}$. Now, using (2.1), the curvature tensor takes the following form [18]

$$\begin{aligned} \bar{\mathcal{R}}(X,Y,Z,W) &= \tilde{\bar{\mathcal{R}}}(X,Y,Z,W) + \lambda_1 \alpha(X,Z) g(Y,W) - \lambda_1 \alpha(Y,Z) g(X,W) \\ &\quad + \lambda_2 \alpha(Y,W) g(X,Z) - \lambda_2 \alpha(X,W) g(Y,Z) \\ &\quad + \lambda_2(\lambda_1 - \lambda_2) g(X,Z)\beta(Y,W) - \lambda_2(\lambda_1 - \lambda_2) g(Y,Z)\beta(X,W). \end{aligned} \tag{2.3}$$

where

$$\alpha(X,Y) = (\tilde{\bar{\nabla}}_X \pi)(Y) - \lambda_1 \pi(X)\pi(Y) + \frac{\lambda_2}{2} g(X,Y)\pi(\mathcal{V})$$

and

$$\beta(X,Y) = \frac{\pi(\mathcal{V})}{2} g(X,Y) + \pi(X)\pi(Y)$$

are $(0,2)$ tensors.

For simplicity, we denote by $tr(\alpha) = a$ and $tr(\beta) = b$.

Let N be an m-dimensional submanifold of a Riemannian manifold $\tilde{N}$ and ∇, $\tilde{\nabla}$ be the induced quarter symmetric-metric connection and Levi-Civita connection of N, respectively. Then the corresponding Gauss formulas are given by:

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \quad X,Y \in \Gamma(TN), \tag{2.4}$$

$$\tilde{\bar{\nabla}}_X Y = \tilde{\nabla}_X Y + \tilde{\sigma}(X,Y), \quad X,Y \in \Gamma(TN), \tag{2.5}$$

where $\tilde{\sigma}$ is the second fundamental form given by $\sigma(X,Y) = \tilde{\sigma}(X,Y) - \lambda_2 g(X,Y)\mathcal{V}^{\perp}$.

Furthermore, the equation of Gauss is given by [18]:

$$\begin{aligned} \bar{\mathcal{R}}(X,Y,Z,W) &= \mathcal{R}(X,Y,Z,W) - g(\sigma(X,W),\sigma(Y,Z)) + g(\sigma(Y,W),\sigma(X,Z)) \\ &\quad + (\lambda_1 - \lambda_2) g(\sigma(Y,Z),\mathcal{V}^{\perp}) g(X,W) \\ &\quad + (\lambda_2 - \lambda_1) g(\sigma(X,Z),\mathcal{V}^{\perp}) g(Y,W). \end{aligned} \tag{2.6}$$

Now, let $\tilde{N}$ be a (2n+1) odd-dimensional Riemannian manifold. Then $\tilde{N}$ is said to be an almost contact metric manifold with structure (φ, ξ, η, g) if there exist a tensor φ of type $(1,1)$, a vector field ξ (structure vector field) and a 1-form η satisfying [3]

$$\varphi^2 X = -X + \eta(X)\xi, \;\; g(X,\xi) = \eta(X),$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.7}$$

for any X,Y on $\tilde{N}$. The 2-form Φ is called the fundamental 2-form in $\tilde{N}$ and the manifold is said to be a contact metric manifold if $\Phi = d\eta$. A Sasakian manifold is a normal contact metric manifold. In fact, an almost contact metric manifold is a Sasakian manifold if and only if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

A $(2n+1)$-dimensional Riemannian manifold $\tilde{N}$ endowed with the almost contact metric structure (φ, η, ξ, g) is called a conformal Sasakian manifold if for a C^∞ function $f : \tilde{N} \to \mathbb{R}$, there are [1]

$$\tilde{g} = exp(f)g, \tilde{\varphi} = \varphi, \tilde{\eta} = (exp(f))^{\frac{1}{2}}\eta, \tilde{\xi} = (exp(-f))^{\frac{1}{2}}\xi \tag{2.8}$$

is a Sasakian structure on $\tilde{N}$. Using Koszul formula, we derive the following relation between the connections $\tilde{\nabla}$ and ∇

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X,Y)\omega^{\#}\}, \tag{2.9}$$

for all vector fields X, Y on $\tilde{N}$, where $\omega(X) = X(f)$ and $g(\omega^{\#}, X) = \omega(X)$.

An almost contact metric manifold $(\tilde{N}, \varphi, \xi, \eta, g)$ whose curvature tensor satisfies

$$\begin{aligned} g(\tilde{\tilde{\mathcal{R}}}(X,Y)Z,W) \;\; = \;\; & exp(f)\Big\{\frac{c+3}{4}\big(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big) \\ & + \frac{c-1}{4}\big(\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\ & + g(X,Z)g(\xi,W)\eta(Y) - g(Y,Z)g(\xi,W)\eta(X) \\ & - g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) \\ & - 2g(\varphi X,Y)g(\varphi Z,W)\Big\} - \frac{1}{2}\big(B(X,Z)g(Y,W) \\ & - B(Y,Z)g(X,W) + B(Y,W)g(Y,Z) - B(X,W)g(Y,Z)\big) \\ & - \frac{1}{4}||\omega^{\#}||^2\big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\big), \end{aligned} \tag{2.10}$$

for any vector fields X, Y, Z, W tangent to $\tilde{N}$, where $B = \nabla\omega - \frac{1}{2}\omega \otimes \omega$, is said to be a conformal Sasakian space form [1].

From (2.1) and (2.10), we get

$$\begin{aligned} g(\bar{\mathcal{R}}(X,Y)Z,W) &= exp(f)\Big\{\frac{c+3}{4}\big(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big) \\ &+\frac{c-1}{4}\big(\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\ &+g(X,Z)g(\xi,W)\eta(Y) - g(Y,Z)g(\xi,W)\eta(X) \\ &-g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) \\ &-2g(\varphi X,Y)g(\varphi Z,W)\Big\} - \frac{1}{2}\big(B(X,Z)g(Y,W) \\ &-B(Y,Z)g(X,W) + B(Y,W)g(Y,Z) - B(X,W)g(Y,Z)\big) \\ &-\frac{1}{4}||\omega^{\#}||^2\big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\big) \\ &+\lambda_1\alpha(X,Z)g(Y,W) - \lambda_1\alpha(Y,Z)g(X,W) \\ &+\lambda_2 g(X,Z)\alpha(Y,W) - \lambda_2 g(Y,Z)\alpha(X,W) \\ &+\lambda_2(\lambda_1-\lambda_2)g(X,Z)\beta(Y,W) - \lambda_2(\lambda_1-\lambda_2)g(Y,Z)\beta(X,W). \end{aligned} \tag{2.11}$$

The squared norm of T at $p \in N$ is given by

$$||T||^2 = \sum_{i,j=1}^{m} g^2(Je_i, e_j), \tag{2.12}$$

where $\{e_1, \cdots, e_m\}$ is any orthonormal basis of the tangent space TN of N.

It was proved in [6] that a submanifold N of an almost Hermitian manifold $(\tilde{N}, J, g)$ is pointwise slant if and only if

$$T^2 = -\cos^2\theta(p)I, \quad \forall\, p \in N, \tag{2.13}$$

for some real-valued function $\theta(p)$ on N. A pointwise slant submanifold is *proper* if it contains neither totally real points nor complex points.

Clearly, it is easy to check that

$$g(TX, TY) = \cos^2\theta(p)g(X,Y), \tag{2.14}$$

$$g(FX, FY) = \sin^2\theta(p)g(X,Y), \tag{2.15}$$

for any $X, Y \in \Gamma(TN)$.

The following definition is given by Chen and Uddin in [8]:
A submanifold N of dimension m of an almost Hermitian manifold $\tilde{N}^{4n}$ is said to be a pointwise bi-slant submanifold if there exists a pair of orthogonal distributions $\mathfrak{D}_1$ and $\mathfrak{D}_2$, such that

(i) $TN = \mathfrak{D}_1 \oplus \mathfrak{D}_2$,

(ii) $J\mathfrak{D}_1 \perp \mathfrak{D}_2$ and $J\mathfrak{D}_2 \perp \mathfrak{D}_1$,

(iii) Each distribution $\mathfrak{D}_i$ is pointwise slant with slant function $\theta_i : TN - \{0\} \to \mathbb{R}$ for $i = 1, 2$.

In fact, pointwise bi-slant submanifold are more general class of submanifolds and bi-slant, pointwise semi-slant, semi-slant and CR-submanifolds are the particular cases of these submanifolds.

Since N is a pointwise bi-slant submanifold, we defined an adapted orthonormal frame as $m = 2d_1 + 2d_2$ follows

$$\{e_1, e_2 = \sec\theta_1 Te_1, \cdots, e_{2d_1-1}, e_{2d_1} = \sec\theta_1 Te_{2d_1-1},$$
$$\cdots, e_{2d_1+1}, e_{2d_1+2} = \sec\theta_2 Te_{2d_1+1}, \cdots, e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = \sec\theta_2 Te_{2d_1+2d_2-1}\}.$$

Thus, we defined it such that $g(e_1, Je_2) = -g(Je_1, e_2) = -g(Je_1, \sec\theta_1 Te_1)$, which implies that $g(e_1, Je_2) = -\sec\theta_1 g(Te_1, Te_2)$.

Following (2.14), we get $g(e_1, Je_2) = cos\theta_1 g(e_1, e_2)$. Therefore, we easily obtained the following relation

$$||T||^2 = \sum_{i,j=1}^{m} g^2(e_i, Je_j) = (m_1\cos^2\theta_1 + m_2\cos^2\theta_2), \tag{2.16}$$

where $m_1 = \dim\mathfrak{D}_1$ and $m_2 = \dim\mathfrak{D}_2$.

Let $\varphi : N = {}_{f_2}N_1 \times_{f_1} N_2 \to \tilde{N}$ be isometric immersion of a doubly warped product ${}_{f_2}N_1 \times_{f_1} N_2$ into a Riemannian manifold of $\tilde{N}$ of constant sectional curvature c. Suppose that m_1, m_2 and m be the dimensions of N_1, N_2 and $N_1 \times_f N_2$, respectively. Then for unit vector fields X and Z tangent to N_1 and N_2 respectively, we have

$$\begin{aligned} \kappa(X \wedge Z) &= g\left(\nabla_Z\nabla_X X - \nabla_X\nabla_Z X, Z\right) \\ &= \frac{1}{f_1}\left\{\left(\nabla^1_X X\right) f_1 - X^2 f_1\right\} + \frac{1}{f_2}\left\{\left(\nabla^2_Z Z\right) f_2 - Z^2 f_2\right\}. \end{aligned} \tag{2.17}$$

If we consider the local orthonormal frame $\{e_1, e_2, \cdots, e_m\}$ such that $\{e_1, e_2, \cdots, e_{m_1}\}$ tangent to N_1 and $\{e_{m_1+1}, \cdots, e_m\}$ are tangent to N_2, then the sectional curvatre in terms of doubly warped product is defined by

$$\sum_{1\le i\le m_1}\sum_{m_1+1\le j\le m} \kappa\left(e_i \wedge e_j\right) = \frac{m_2\Delta_1 f_1}{f_1} + \frac{m_1\Delta_2 f_2}{f_2}, \tag{2.18}$$

for each $j = m_1 + 1, \cdots, m$.

In this context, we shall define another important Riemannian intrinsic invariant called the scalar curvature of $\tilde{N}^n$ and denoted by $\tilde{\tau}(T_p\tilde{N}^n)$, which at some p in $\tilde{N}^n$ is given as :

$$\tilde{\tau}(T_p\tilde{N}^n) = \sum_{1\le i<j\le n} \tilde{\kappa}_{ij}, \tag{2.19}$$

where $\tilde{\kappa}_{ij} = \tilde{\kappa}(e_i \wedge e_j)$. It is clear that first equality (2.19) is congruent to the following equation, which will be frequently used in the subsequent proof:

$$2\tilde{\tau}(T_p\tilde{N}^n) = \sum_{1\leq i\neq j\leq n} \tilde{\kappa}_{ij}. \tag{2.20}$$

Similarly, scalar curvature $\tilde{\tau}(L_p)$ of L-plane is given by

$$\tilde{\tau}(L_p) = \sum_{1\leq i<j\leq n} \tilde{\kappa}_{ij}. \tag{2.21}$$

An orthonormal basis of the tangent space T_pN is $\{e_1, \cdots, e_m\}$ such that $e_r = \{e_{m+1}, \cdots, e_{2n+1}\}$ belongs to the normal space $T^{\perp}N$. Then, we have

$$\sigma^r_{ij} = g(\sigma(e_i, e_j), e_r), \quad ||\sigma||^2 = \textstyle\sum_{i,j=1}^m g(\sigma(e_i, e_j), \sigma(e_i, e_j)) = \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^m (\sigma^r_{ij})^2,$$
$$||H||^2 = \tfrac{1}{m^2}\textstyle\sum_{i=1}^m g(\sigma(e_i, e_i), \sigma(e_i, e_i)),$$

where $\|H\|^2$ is the squared norm of the mean curvature vector H of N.

Let κ_{ij} and $\tilde{\kappa}_{ij}$ be the sectional curvature of the plane section spanned by e_i and e_j at p in a submanifold N^m and a Riemannian manifold $\tilde{N}^n$ respectively. Thus, κ_{ij} and $\tilde{\kappa}_{ij}$ are the intrinsic and the extrinsic sectional curvatures of the span $\{e_i, e_j\}$ at p. Thus from the Gauss Equation, we have

$$\begin{aligned}
2\tau(T_pN^m) &= \kappa_{ij} = 2\tilde{\tau}(T_pN^m) - \sum_{i,j=1}^m \big\{(\lambda_1-\lambda_2)g(\sigma(e_j,e_j),\mathcal{Q}^{\perp})g(e_i,e_i) \\
&+ (\lambda_2-\lambda_1)g(\sigma(e_i,e_j),\mathcal{Q}^{\perp})g(e_j,e_i)\big\} + \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^m \left(\sigma^r_{ii}\sigma^r_{jj} - (\sigma^r_{ij})^2\right) \\
&= \tilde{\kappa}_{ij} - \sum_{i,j=1}^m \big\{(\lambda_1-\lambda_2)g(\sigma(e_j,e_j),\mathcal{Q}^{\perp})g(e_i,e_i) \\
&+ (\lambda_2-\lambda_1)g(\sigma(e_i,e_j),\mathcal{Q}^{\perp})g(e_j,e_i)\big\} + \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^m \left(\sigma^r_{ii}\sigma^r_{jj} - (\sigma^r_{ij})^2\right).
\end{aligned}$$

(2.22)

The following consequences come from Gauss equation and (2.22)

$$\begin{aligned}
\tau(T_pN_1{}^{m_1}) &= \tilde{\tau}(T_pN_1^{m_1}) - \sum_{1\leq j<k\leq m_1} \big\{(\lambda_1-\lambda_2)g(\sigma(e_j,e_j),\mathcal{Q}^{\perp})g(e_k,e_k) \\
&+ (\lambda_2-\lambda_1)g(\sigma(e_j,e_k),\mathcal{Q}^{\perp})g(e_k,e_j)\big\} + \sum_{r=m+1}^{2n+1}\sum_{1\leq j<k\leq m_1} \left(\sigma^r_{jj}\sigma^r_{kk} - (\sigma^r_{jk})^2\right), \\
\tau(T_pN_2{}^{m_2}) &= \tilde{\tau}(T_pN_2^{m_2}) - \sum_{m_1+1\leq s<t\leq m} \big\{(\lambda_1-\lambda_2)g(\sigma(e_t,e_t),\mathcal{Q}^{\perp})g(e_s,e_s)
\end{aligned}$$

$$+ \quad (\lambda_2 - \lambda_1) g(\sigma(e_s, e_t), \mathcal{Q}^{\perp}) g(e_t, e_s) \big\} + \sum_{r=m+1}^{2n+1} \sum_{m_1+1 \leq s < t \leq m} m\big(\sigma_{ss}^r \sigma_{tt}^r - (\sigma_{st}^r)^2\big).$$

(2.23)

3. Main Inequalities

First, we recall the following result of B.-Y. Chen for later use.

Lemma 3.1. *[7] Let $m \geq 2$ and $a_1, \cdots, a_m, b$ be (m+1) real numbers such that*

$$\left(\sum_{i=1}^{m} a_i\right)^2 = (m-1)\left(\sum_{i=1}^{m} a_i^2 + b\right).$$

Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_m$.

Now, we prove the following main result of this section.

Theorem 3.1. *Let $\tilde{N}(c)$ be a (2n+1)-dimensional conformal Sasakian space form and $\varphi :_{f_2} N_1 \times_{f_1} N_2 \to \tilde{N}(c)$ be an isometric immersion of an m-dimensional pointwise bi-slant doubly warped product into $\tilde{N}(c)$ equipped with quarter symmetric connection. Then*

(i) The relation between warping functions and the squared norm of mean curvature is given by

$$\begin{aligned} \frac{m_2 \Delta_1 f_1}{f_1} + \frac{m_1 \Delta_2 f_2}{f_2} \quad &\leq \quad \frac{m^2}{4}||H||^2 + exp(f)\Big\{\frac{(c+3)}{4} m_1 m_2 + \frac{1}{2} trB + \frac{1}{4} m_1 m_2 ||\omega^*||^2 \\ &- \quad \frac{(c-1)}{8}[2 + 3m_1 cos^2\theta_1 + 3m_2 cos^2\theta_2]\Big\} \\ &- \quad \frac{1}{2}\big\{(\lambda_1 + \lambda_2)a + \lambda_2(\lambda_1 - \lambda_2)b + 2m_1 m_2(\lambda_1 - \lambda_2)\pi(H)\big\}, \end{aligned} \tag{3.1}$$

where ∇ and Δ are the gradient and the laplacian operators, repectively and H is the mean curvature vector of N^m.

(ii) The equality case holds in (3.1) if and only if φ is a mixed totally geodesic isometric immersion and the following satisfies $m_1 H_1 = m_2 H_2$, where H_1 and H_2 are partial mean curvature vectors of H along $N_1^{m_1}$ and $N_2^{m_2}$, respectively and $\pi(H) = \frac{1}{m}\sum_{i=1}^{m} \pi(\sigma(e_i, e_j)) = g(\mathcal{Q}, H)$.

Proof. *let $\{e_1, ..., e_m\}$ and $\{e_{m+1}, ..., e_{2n+1}\}$ as orthonormal tangent frame and orthonormal normal frame on N, respectively. Putting $X = W = e_i$, $Y = Z = e_j$,*

$i \neq j$ in (2.21) and using(2.6), we obtain

$$\begin{aligned}
g(\bar{\mathcal{R}}(e_i, e_j, e_j, e_i) &= exp(f)\Big\{\frac{c+3}{4}\big(g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\big)\\
&+ \frac{c-1}{4}\big(\eta(e_i)\eta(e_j)g(e_j, e_i) - \eta(e_j)\eta(e_j)g(e_i, e_i)\\
&+ g(e_i, e_j)g(\xi, e_i)\eta(e_j) - g(e_j, e_j)g(\xi, e_i)\eta(e_i)\\
&- g(\varphi e_j, e_j)g(\varphi e_i, e_i) - g(\varphi e_i, e_j)g(\varphi e_j, e_i)\\
&- 2g(\varphi e_i, e_j)g(\varphi e_j, e_i)\Big\} - \frac{1}{2}\big(B(e_i, e_j)g(e_j, e_i)\\
&- B(e_j, e_j)g(e_i, e_i) + B(e_j, e_i)g(e_i, e_j) - B(e_i, e_i)g(e_j, e_j)\big)\\
&- \frac{1}{4}||\omega^{\#}||^2\big(g(e_i, e_j)g(e_j, e_i) - g(e_j, e_j)g(e_i, e_i)\big)\\
&+ \Lambda_1\alpha(e_i, e_j)g(e_j, e_i) - \Lambda_1\alpha(e_j, e_j)g(e_i, e_i)\\
&+ \lambda_2 g(e_i, e_j)\alpha(e_j, e_i) - \lambda_2 g(e_j, e_j)\alpha(e_i, e_i)\\
&+ \lambda_2(\lambda_1 - \lambda_2)g(e_i, e_j)\beta(e_j, e_i) - \lambda_2(\lambda_1 - \lambda_2)g(e_j, e_j)\beta(e_i, e_i),\\
&- (\lambda_1 - \lambda_2)g(h(e_j, e_j), \mathcal{P}^{\perp})g(e_i, e_i)\\
&- (\lambda_2 - \lambda_1)g(h(e_i, e_j), \mathcal{P}^{\perp})g(e_j, e_i)
\end{aligned} \tag{3.2}$$

By taking summation $1 \leq i, j \leq m$ and using Gauss equation with (3.2), we have

$$\begin{aligned}
2\tau &= exp(f)\Big\{\frac{(c+3)}{4}m(m-1) + \frac{(c-1)}{4}\big(2 - 2m + 3||P||^2\Big\} + (m-1)trB\\
&+ \frac{1}{4}m(m-1)||\omega^{\#}||^2 + (\lambda_1 + \lambda_2)(1-m)a + \lambda_2(\lambda_1 - \lambda_2)(1-m)b\\
&+ (\lambda_2 - \lambda_1)m(m-1)\pi(\mathcal{H}) + m^2||\mathcal{H}||^2 - ||\sigma||^2\\
&= exp(f)\Big\{\frac{(c+3)}{4}m(m-1) + \frac{(c-1)}{4}\big(2 - 2m + 3m_1cos^2\theta_1 + 3m_2cos^2\theta_2\big)\\
&+ (m-1)trB + \frac{1}{4}m(m-1)||\omega^{\#}||^2\Big\} + (\lambda_1 + \lambda_2)(1-m)a\\
&+ \lambda_2(\lambda_1 - \lambda_2)(1-m)b + (\lambda_2 - \lambda_1)m(m-1)\pi(\mathcal{H}) + m^2||\mathcal{H}||^2 - ||\sigma||^2,
\end{aligned} \tag{3.3}$$

where

$$||P||^2 = \sum_{i,j=1}^{m} g^2(\varphi e_i, e_j) \quad \text{and} \quad \pi(\mathcal{H}) = \frac{1}{m}\sum_{j=1}^{m}\pi(h(e_j, e_j)) = g(\mathcal{V}^{\perp}, \mathcal{H}).$$

Let us assume that

$$\begin{aligned}
\delta &= 2\tau - \Big\{exp(f)\big\{\frac{(c+3)}{8}m_1(m_1 - 1) + \frac{(c-1)}{8}(2 - 2m_1) + \frac{(c-1)}{4}3m_1cos^2\theta_1\\
&+ \frac{1}{2}(m_1 - 1)trB + \frac{1}{8}m_1(m_1 - 1)||\omega^{\#}||^2\big\}
\end{aligned}$$

$$+ \frac{(c+3)}{8}m_2(m_2-1) + \frac{(c-1)}{8}(2-2m_2) + \frac{(c-1)}{4}3m_2cos^2\theta_2$$
$$+ \frac{1}{2}(m_2-1)trB + \frac{1}{8}m_2(m_2-1)||\omega^{\#}||^2\Big\} + (\lambda_1+\lambda_2)(1-m)a$$
$$+ \lambda_2(\lambda_1-\lambda_2)(1-m)b + (\lambda_2-\lambda_1)m(m-1)\pi(\mathcal{H})\Big\} - \frac{n^2}{2}||H||^2. \tag{3.4}$$

Then, from (3.3) and (3.4), we have

$$m^2||H||^2 = 2(\delta + ||\sigma||^2). \tag{3.5}$$

Thus, the orthonormal frame $\{e_1, \cdots, e_m\}$ the proceeding equation takes the following form

$$\left(\sum_{i=1}^{m}\sigma_{ii}^{m+1}\right)^2 = 2\Big\{\delta + \sum_{i=1}^{m}(\sigma_{ii}^{m+1})^2 + \sum_{i\neq j}(\sigma_{ij}^{m+1})^2 + \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m}(\sigma_{ij}^{r})^2\Big\}. \tag{3.6}$$

By using the algebraic Lemma 3.1 and relation (3.6), we have

$$2\sigma_{11}^{m+1}\sigma_{22}^{m+1} \geq \sum_{i\neq j}(\sigma_{ij}^{m+1})^2 + \sum_{i,j=1}^{m}\sum_{r=m+2}^{2n+1}(\sigma_{ij}^{r})^2 + \delta. \tag{3.7}$$

If we substitute $a_1 = \sigma_{11}^{m+1}$, $a_2 = \sum_{i=2}^{m_1}\sigma_{ii}^{m+1}$ and $a_3 = \sum_{t=m_1+1}^{m}\sigma_{tt}^{m+1}$ in the above equation (3.6), we have

$$\begin{aligned}\left(\sum_{i=1}^{m}a_i\right)^2 &= 2\Big\{\delta + \sum_{i=1}^{m}a_i^2 + \sum_{i\neq j\leq m}(\sigma_{ij}^{m+1})^2 + \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m}(\sigma_{ij}^{r})^2 \\ &\quad - \sum_{2\leq j\neq k\leq m_1}\sigma_{jj}^{m+1}\sigma_{kk}^{m+1} - \sum_{m_1+1\leq s\neq t\leq m}\sigma_{ss}^{m+1}\sigma_{tt}^{m+1}\Big\}.\end{aligned} \tag{3.8}$$

Thus a_1, a_2, a_3 satisfy the Chen's Lemma (for $m = 3$), that is

$$\left(\sum_{i=1}^{3}a_i\right)^2 = 2\left(b + \sum_{i=1}^{3}a_i^2\right).$$

Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3$. In the case of under considering, this means that

$$\begin{aligned}\sum_{1\leq j<k\leq m_1}\sigma_{jj}^{m+1}\sigma_{kk}^{m+1} + \sum_{m_1+1\leq s<t\leq m}\sigma_{ss}^{m+1}\sigma_{tt}^{m+1} &\geq \tfrac{\delta}{2} + \textstyle\sum_{1\leq\alpha_3<\beta_3\leq m}(\sigma_{\alpha_3\beta_3}^{m+1})^2 \\ &\quad + \textstyle\sum_{r=m+1}^{2n+1}\sum_{\alpha_3\beta_3=1}^{m}(\sigma_{\alpha_3\beta_3}^{r})^2.\end{aligned} \tag{3.9}$$

Equality holds if and only if

$$\sum_{i=1}^{m_1}\sigma_{ii}^{m+1} = \sum_{t=m_1+1}^{m}\sigma_{tt}^{m+1}. \tag{3.10}$$

Again, using Gauss equation, we derive

$$m_2\frac{\Delta_1 f_1}{f_1}+m_1\frac{\Delta_2 f_2}{f_2} = \tau-\sum_{1\le j<k\le m_1}\kappa(e_j\wedge e_k)-\sum_{m_1+1\le s<t\le m}\kappa(e_s\wedge e_t). \tag{3.11}$$

Then, the scalar curvature for the conformal Sasakian space form with quarter-symmetric connection from (2.22), we get

$$\begin{aligned} m_2\frac{\Delta_1 f_1}{f_1}+m_1\frac{\Delta_2 f_2}{f_2} &= \tau-exp(f)\Big\{\frac{(c+3)}{8}m_1(m_1-1)+\frac{(c-1)}{8}(2-2m_1)\\ &+ \frac{(c-1)}{4}3m_1cos^2\theta_1+\frac{1}{2}(m_1-1)trB+\frac{1}{8}m_1(m_1-1)||\omega^{\#}||^2\Big\}\\ &- \frac{1}{2}\big\{(\lambda_1+\lambda_2)(1-m_1)a+\lambda_2(\lambda_1-\lambda_2)(1-m_1)b\\ &+ (\lambda_2-\lambda_1)m_1(m_1-1)\pi(H)\big\}-\sum_{r=m+1}^{2n+1}\sum_{m_1+1\le j<k\le m}\left(\sigma^r_{jj}\sigma^r_{kk}-(\sigma^r_{jk})^2\right)\\ &- exp(f)\Big\{\frac{(c+3)}{8}m_2(m_2-1)+\frac{(c-1)}{8}(2-2m_2)\\ &+ \frac{(c-1)}{4}3m_2cos^2\theta_2+\frac{1}{2}(m_2-1)trB+\frac{1}{8}m_2(m_2-1)||\omega^{\#}||^2\Big\}\\ &- \frac{1}{2}\big\{(\lambda_1+\lambda_2)(1-m_2)a+\lambda_2(\lambda_1-\lambda_2)(1-m_2)b\\ &+ (\lambda_2-\lambda_1)m_2(m_2-1)\pi(H)\big\}-\sum_{r=m+1}^{2n+1}\sum_{m_1+1\le s<t\le m}\left(\sigma^r_{ss}\sigma^r_{tt}-(\sigma^r_{st})^2\right). \end{aligned} \tag{3.12}$$

Now making use of (3.9) and (3.12), we have

$$\begin{aligned} m_2\frac{\Delta_1 f_1}{f_1}+m_1\frac{\Delta_2 f_2}{f_2} &\le \tau-exp(f)\Big\{\frac{(c+3)}{8}[m(m-1)-2m_1m_2]+\frac{(c-1)}{8}(4-2m)\\ &+ \frac{1}{2}(m-2)trB+\frac{1}{8}[m(m-1)-2m_1m_2]||\omega^{\#}||^2\\ &+ \frac{(c-1)}{4}[3m_1cos^2\theta_1+3m_2cos^2\theta_2]\Big\}\\ &+ \frac{1}{2}\big\{(\lambda_1+\lambda_2)(2-m)a+\lambda_2(\lambda_1-\lambda_2)(2-m)b\\ &+ (\lambda_2-\lambda_1)[m(m-1)-2m_1m_2]\pi(H)\big\}-\frac{\delta}{2}. \end{aligned} \tag{3.13}$$

Using (3.4) in the above equation, we obtain

$$\frac{m_2\Delta_1 f_1}{f_1}+\frac{m_1\Delta_2 f_2}{f_2} \le \frac{m^2}{4}||H||^2+exp(f)\Big\{\frac{(c+3)}{4}m_1m_2+\frac{1}{2}trB+\frac{1}{4}m_1m_2||\omega^*||^2$$

$$- \frac{(c-1)}{8}[2+3m_1cos^2\theta_1+3m_2cos^2\theta_2]\Big\}$$

$$- \frac{1}{2}\big\{(\lambda_1+\lambda_2)a+\lambda_2(\lambda_1-\lambda_2)b+2m_1m_2(\lambda_1-\lambda_2)\pi(H)\big\}, \tag{3.14}$$

which is inequality (3.1). The equality sign holds in (3.1) if and only if the leaving term in (3.9) and (3.10) imply that

$$\sum_{r=m+1}^{2n+1}\sum_{i=1}^{m_1}\sigma_{ii}^r=\sum_{r=m+1}^{2n+1}\sum_{t=m_1+1}^{m}\sigma_{tt}^r=0, \tag{3.15}$$

and $m_1H_1=m_2H_2$.

Moreover from (3.10), we obtain

$$\sigma_{jt}=0,\ \forall\ 1\leq j\leq m_1, m+1\leq t\leq m, m+1\leq r\leq 2n+1. \tag{3.16}$$

This shows that φ *is a mixed, totally geodesic immersion. The converse part of (3.16) is true for pointwise bi-slant warped product immersion into conformal Sasakian space form. Hence, the proof is complete.* □

Following corollaries are easy consequence of the above theorem.

Corollary 3.1. *Let* $\tilde{N}(c)$ *be a (2n+1)-dimensional conformal Sasakian space form and* $\varphi :_{f_2} N_1\times_{f_1} N_2\to\tilde{N}(c)$ *be an isometric immersion of an m-dimensional pointwise semi-slant doubly warped product into* $\tilde{N}(c)$ *equipped with quarter symmetric connection. Then*

$$\frac{m_2\Delta_1 f_1}{f_1}+\frac{m_1\Delta_2 f_2}{f_2} \leq \frac{m^2}{4}||H||^2+exp(f)\Big\{\frac{(c+3)}{4}m_1m_2+\frac{1}{2}trB+\frac{1}{4}m_1m_2||\omega^*||^2$$

$$- \frac{(c-1)}{8}[2+3m_1+3m_2cos^2\theta_2]\Big\}$$

$$- \frac{1}{2}\big\{(\lambda_1+\lambda_2)a+\lambda_2(\lambda_1-\lambda_2)b+2m_1m_2(\lambda_1-\lambda_2)\pi(H)\big\}. \tag{3.17}$$

Similarly, if $\theta_1=\pi/2$ and $\theta_2=\theta$, in Theorem 3.1, then we have

Corollary 3.2. *Let* $\tilde{N}(c)$ *be a (2n+1)-dimensional conformal Sasakian space form and* $\varphi :_{f_2} N_1\times_{f_1} N_2\to\tilde{N}(c)$ *be an isometric immersion of an m-dimensional pointwise hemi-slant doubly warped product into* $\tilde{N}(c)$ *equipped with quarter symmetric connection. Then*

$$\frac{m_2\Delta_1 f_1}{f_1}+\frac{m_1\Delta_2 f_2}{f_2} \leq exp(f)\Big\{\frac{(c+3)}{4}m_1m_2+\frac{1}{2}trB+\frac{1}{4}m_1m_2||\omega^*||^2$$

$$- \frac{(c-1)}{8}[2+3m_2cos^2\theta]\Big\}$$

$$- \frac{1}{2}\big\{(\lambda_1+\lambda_2)a+\lambda_2(\lambda_1-\lambda_2)b+2m_1m_2(\lambda_1-\lambda_2)\pi(H)\big\}. \tag{3.18}$$

Also, if $\theta_1 = 0$ and $\theta_2 = \pi/2$, in Theorem 3.1, then we have

Corollary 3.3. *Let $\tilde{N}(c)$ be a (2n+1)-dimensional conformal Sasakian space form and $\varphi :_{f_2} N_1 \times_{f_1} N_2 \to \tilde{N}(c)$ be an isometric immersion of an m-dimensional from poitwise CR-doubly warped product into $\tilde{N}(c)$ equipped with quarter symmetric connection. Then*

$$\begin{aligned}\frac{m_2\Delta_1 f_1}{f_1} + \frac{m_1\Delta_2 f_2}{f_2} \quad &\leq \quad \frac{n^2}{4}||H||^2 + exp(f)\Big\{\frac{(c+3)}{4}m_1m_2 + \frac{1}{2}trB + \frac{1}{4}m_1m_2||\omega^*||^2 \\ &- \quad \frac{(c-1)}{8}[2+3m_1]\Big\} \\ &- \quad \frac{1}{2}\big\{(\lambda_1+\lambda_2)a + \lambda_2(\lambda_1-\lambda_2)b + 2m_1m_2(\lambda_1-\lambda_2)\pi(H)\big\}. \qquad (3.19)\end{aligned}$$

Furthermore, we have the following corollary of Theorem 3.1

Corollary 3.4. *Let $\tilde{N}(c)$ be a (2n+1)-dimensional conformal Sasakian space form and $\varphi :_{f_2} N_1 \times_{f_1} N_2 \to \tilde{N}(c)$ be an isometric minimal immersion of an m-dimensional pointwise bi-slant doubly warped product into $\tilde{N}(c)$ equipped with quarter symmetric connection. Then the following inequality holds:*

$$\begin{aligned}\frac{m_2\Delta_1 f_1}{f_1} + \frac{m_1\Delta_2 f_2}{f_2} \quad &\leq \quad exp(f)\Big\{\frac{(c+3)}{4}m_1m_2 + \frac{1}{2}trB + \frac{1}{4}m_1m_2||\omega^*||^2 \\ &- \quad \frac{(c-1)}{8}[2+3m_1cos^2\theta_1 + 3m_2cos^2\theta_2]\Big\} \\ &- \quad \frac{1}{2}\big\{(\lambda_1+\lambda_2)a + \lambda_2(\lambda_1-\lambda_2)b + 2m_1m_2(\lambda_1-\lambda_2)\pi(H)\big\}. \qquad (3.20)\end{aligned}$$

For the semi-symmetric metric connection $\lambda_1 = \lambda_2 = 1$, we have

Theorem 3.2. *Let $\tilde{N}(c)$ be a (2n+1)-dimensional conformal Sasakian space from and $\varphi :_{f_2} N_1 \times_{f_1} N_2 \to \tilde{N}(c)$ be an isometric immersion of an m-dimensional pointwise bi-slant doubly warped product into $\tilde{N}(c)$ equipped with semi-symmetric connection. Then the following inequality holds:*

$$\begin{aligned}\frac{m_2\Delta_1 f_1}{f_1} + \frac{m_1\Delta_2 f_2}{f_2} \quad &\leq \quad \frac{n^2}{4}||H||^2 + exp(f)\Big\{\frac{(c+3)}{4}m_1m_2 + \frac{1}{2}trB + \frac{1}{4}m_1m_2||\omega^*||^2 \\ &- \quad \frac{(c-1)}{8}[2+3m_1cos^2\theta_1 + 3m_2cos^2\theta_2]\Big\} - a. \qquad (3.21)\end{aligned}$$

For the semi-symmetric metric nonmetric connection, if we put $\lambda_1 = 1$ and $\lambda_2 = 0$ in Theorem 3.1, then we have

Theorem 3.3. *Let $\tilde{N}(c)$ be a (2n+1)-dimensional conformal Sasakian space form and $\varphi :_{f_2} N_1 \times_{f_1} N_2 \to \tilde{N}(c)$ be an isometric immersion of an m-dimensional pointwise bi-slant doubly warped product into $\tilde{N}(c)$ equipped with semi-symmetric metric non metric connection satisfies the following inequality*

$$\begin{aligned}\frac{m_2\Delta_1 f_1}{f_1} + \frac{m_1\Delta_2 f_2}{f_2} \leq\ & \frac{n^2}{4}||H||^2 + exp(f)\Big\{\frac{(c+3)}{4}m_1m_2 + \frac{1}{2}trB + \frac{1}{4}m_1m_2||\omega^*||^2 \\ -\ & \frac{(c-1)}{8}[2 + 3m_1cos^2\theta_1 + 3m_2cos^2\theta_2]\Big\} \\ -\ & \frac{1}{2}(a + 2m_1m_2\pi(H)). \qquad (3.22)\end{aligned}$$

Next, we have the following theorem

Theorem 3.4. *Let $\tilde{N}(c)$ be a (2n+1)-dimensional conformal Sasakian space form and $\varphi :_{f_2} N_1 \times_{f_1} N_2 \to \tilde{N}(c)$ be an isometric immersion of an m-dimensional pointwise bi-slant doubly warped product into $\tilde{N}(c)$ equipped with quarter symmetric connection. Then*

$$\begin{aligned}(i)\ \ (\frac{\Delta_1 f_1}{m_1 f_1}) + (\frac{\Delta_2 f_2}{m_2 f_2}) \geq\ & \tau - \frac{m^2(m-2)}{2(m-1)}||H||^2 \\ -\ & exp(f)\Big\{\frac{(c+3)}{8}(m+1)(m-2) + \frac{(c-1)}{8}\big(2 - 2m + 3m_1cos^2\theta_1 \\ +\ & 3m_2cos^2\theta_2\big) + \frac{1}{2}(m-1)trB + \frac{1}{8}m(m-1)||\omega^{\#}||^2\Big\} \\ -\ & \frac{1}{2}\big\{(\lambda_1+\lambda_2)(1-n)a + \lambda_2(\lambda_1-\lambda_2)(1-n)b \\ +\ & (\lambda_1-\lambda_2)n(n-1)\pi(H)\big\}, \qquad (3.23)\end{aligned}$$

where $m_i = dimN_i$, i=1,2 and Δ^i is the laplacian operator on N_i, i=1,2.

(ii) If the equality sign holds in (3.23), then the equality sign in (3.36) holds auotomatically.

(iii) If $m = 2$, then equality sign in (3.23) holds identically.

Proof. Let us consider that $_{f_2}N_1 \times_{f_1} N_2$ be an isometric immersion of an m-dimensional pointwise bi-slant doubly warped product $\tilde{N}(c)$ with pointwise φ-sectional curvature c endowed with quarter symmetric connection. Then from the equation of Gauss, we obtain

$$2\tau = exp(f)\Big\{\frac{(c+3)}{4}m(m-1) + \frac{(c-1)}{4}\big(2 - 2m + 3m_1cos^2\theta_1 + 3m_2cos^2\theta_2\big)$$

$$+ \quad (m-1)trB + \frac{1}{4}m(m-1)||\omega^{\#}||^2\Big\} + (\lambda_1+\lambda_2)(1-m)a$$

$$(3.24) + \quad \lambda_2(\lambda_1-\lambda_2)(1-m)b + (\lambda_2-\lambda_1)m(m-1)\pi(\mathcal{H}) + m^2||\mathcal{H}||^2 - ||\sigma||^2.$$

Now, we consider that

$$\delta = 2\tau - exp(f)\Big\{\frac{(c+3)}{4}(m+1)(m-2) + \frac{(c-1)}{4}\big(2-2m+3m_1cos^2\theta_1+3m_2cos^2\theta_2\big)$$
$$+ \quad (m-1)trB + \frac{1}{4}m(m-1)||\omega^{\#}||^2\Big\} - (\lambda_1+\lambda_2)(1-m)a$$
$$- \quad \lambda_2(\lambda_1-\lambda_2)(1-m)b - (\lambda_2-\lambda_1)m(m-1)\pi(\mathcal{H}) - \frac{m^2(m-2)}{m-1}||\mathcal{H}||^2.$$
$$(3.25)$$

Then from (3.24) and (3.25), it follows that

$$(3.26) \qquad m^2||H||^2 = (m-1)\big\{||\sigma||^2 + \delta - exp(f)\frac{(c+3)}{2}\big\}.$$

Let $\{e_1,\cdots,e_m\}$ *be an orthonormal frame, the equation takes the following form*

$$\left(\sum_{r=m+1}^{2n+1}\sum_{i=1}^{m}\sigma_{ii}^r\right)^2 = (m-1)\Big\{\delta + \sum_{r=m+1}^{2n+1}\sum_{i=1}^{m}(\sigma_{ii}^r)^2 + \sum_{r=m+1}^{2n+1}\sum_{i<j}(\sigma_{ij}^r)^2$$
$$(3.27) \qquad + \sum_{r=m+2}^{2n+1}\sum_{i,j=1}^{m}(\sigma_{ij}^r)^2 - exp(f)\frac{(c+3)}{2}\Big\},$$

which implies that

$$\left(\sigma_{11}^{m+1} + \sum_{i=2}^{m_1}\sigma_{ii}^{m+1} + \sum_{t=m_1+1}^{m}\sigma_{tt}^{m+1}\right)^2 = \delta + (\sigma_{11}^{m+1})^2 + \sum_{i=2}^{m_1}(\sigma_{ii}^{m+1})^2$$
$$+ \sum_{t=m_1+1}(\sigma_{tt}^{m+1})^2 + \sum_{2\le j\ne l\le m_1}\sigma_{jj}^{m+1}\sigma_{ll}^{m+1}$$
$$- \sum_{m_1+1\le t\ne s\le m_1}(\sigma_{jj}^{m+1})(\sigma_{ll}^{m+1}) + \sum_{i<j=1}^{m}(\sigma_{ij}^{m+1})^2$$
$$(3.28) \qquad + \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m}(\sigma_{ij}^r)^2 - exp(f)\frac{(c+3)}{2}.$$

Let us consider that $b_1 = \sigma_{11}^{m+1}$, $b_2 = \sum_{i=2}^{m_1}(\sigma_{ii}^{m+1})^2$ *and* $b_2 = \sum_{t=m_1}^{m}(\sigma_{tt}^{m+1})^2$. *Then from (3.1) and the equation (3.28), we have*

$$\frac{\delta}{2} - exp(f)\frac{(c+3)}{2} + \sum_{i<j=1}^{m}(\sigma_{ij}^{m+1})^2) + \frac{1}{2}\sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m}(\sigma_{ij}^r)^2 \le \sum_{2\le j\ne+l\le m_1}\sigma_{jj}^{m+1}\sigma_{ll}^{m+1}$$
$$(3.29) \qquad + \sum_{m_1+1\le t\ne s\le m}\sigma_{tt}^{m+1}\sigma_{ss}^{m+1}.$$

Equality holds if and only if

$$\sum_{i=1}^{m_1} \sigma_{ii}^{m+1} = \sum_{t=m_1+1}^{m} \sigma_{tt}^{m+1}. \tag{3.30}$$

On the other hand from (3.29) and the definition of scalar curvature, we have

$$\begin{aligned}\kappa(e_1 \wedge e_{m_1+1}) &\geq \sum_{r=m+1}^{2n+1} \sum_{j \in P_{1m_1+1}} (\sigma_{1j}^r)^2 + \frac{1}{2} \sum_{r=m+1}^{2n+1} \sum_{j \in P_{1m_1+1}}^{i \neq j} (\sigma_{ij}^r)^2 \\ &+ \sum_{r=m+1}^{2n+1} \sum_{j \in P_{1m_1+1}} (\sigma_{m_1+1j}^r)^2 + \frac{1}{2} \sum_{r=m+1}^{2n+1} \sum_{i,j \in P_{1m_1+1}} (\sigma_{1j}^r)^2 \\ &+ \frac{1}{2} \sum_{r=m+1}^{2n+1} \sum_{i,j=1}^{m_1+1} (\sigma_{1j}^r)^2 + \frac{\delta}{2},\end{aligned}$$

where $P_{1m_1+1} = \{1, ..., m\} - \{1, m_1 + 1\}$. *Thus, it implies that*

$$\kappa(e_1 \wedge e_{m_1+1}) = \frac{\delta}{2}, \tag{3.31}$$

Since, $N =_{f_2} N_1 \times_{f_1} N_2$ *is a pointwise bi-slant doubly warped product submanifold, we have* $\nabla_X Z = \nabla_Z X = (X \ln f_1) Z + (Z \ln f_2) X$, *for any unit vector fields* X *and* Z *tangent to* N_1 *and* N_2, *respectively. Then from (2.18),(3.25) and* (3.31), *the scalar curvature derives as;*

$$\begin{aligned}\tau &\leq \frac{1}{f_1}\left\{(\nabla_{e_1} e_1) f_1 - e_1^2 f_1\right\} + \frac{1}{f_2}\left\{(\nabla_{e_2} e_2) f_2 - e_2^2 f_2\right\} \\ &+ exp(f)\left\{\frac{(c+3)}{8}(m+1)(m-2) + \frac{(c-1)}{8}\left(2 - 2m + 3m_1 cos^2\theta_1 + 3m_2 cos^2\theta_2\right)\right. \\ &+ \left.\frac{1}{2}(m-1)trB + \frac{1}{8}m(m-1)||\omega^{\#}||^2\right\} \\ &+ \frac{1}{2}\{(\lambda_1 + \lambda_2)(1-m)a + \lambda_2(\lambda_1 - \lambda_2)(1-m)b + (\lambda_2 - \lambda_1)m(m-1)\pi(H)\}.\end{aligned} \tag{3.32}$$

Let the equality holds in (3.32), *then all leaving terms in (3.29) and* (3.31), *we obtain the follwing conditions, i.e.*

$$\begin{aligned}&\sigma_{1j}^r = 0, \quad \sigma_{jm_1+1}^r = 0, \quad \sigma_{ij}^r = 0, \quad \text{where} \quad i \neq j, \quad \text{and} \quad r \in \{m+1, \cdots, 2n+1\} \\ &\sigma_{1j}^r = \sigma_{jm_1+1}^r = \sigma_{ij}^r = 0, \quad \text{and} \quad \sigma_{11}^r + \sigma_{m_1+1m_1+1}.\end{aligned} \tag{3.33}$$

Similarly, we extend the relation (3.32) as follows

$$\tau \leq \frac{1}{f_1}\left\{(\nabla_{e_a} e_\alpha) f_1 - e_\alpha^2 f_1\right\} + \frac{1}{f_2}\left\{(\nabla_{e_\beta} e_\beta) f_2 - e_\beta^2 f_2\right\}$$

$$+ \frac{m^2(m-2)}{2(m-1)}||H||^2 + exp(f)\Big\{\frac{(c+3)}{8}(m+1)(m-2) + \frac{(c-1)}{8}\big(2-2m+3m_1cos^2\theta_1$$
$$+ 3m_2cos^2\theta_2) + \frac{1}{2}(m-1)trB + \frac{1}{8}m(m-1)||\omega^{\#}||^2\Big\}$$
$$+ \frac{1}{2}\{(\lambda_1+\lambda_2)(1-m)a + \lambda_2(\lambda_1-\lambda_2)(1-m)b + (\lambda_2-\lambda_1)m(m-1)\pi(H)\}. \quad (3.34)$$

for any $\alpha = 1, \cdots, m_1$ *and* $\beta = m_1+1, \cdots m$. *Taking the summing up* α *from 1 to* m_1 *and summing up* β *from* m_1+1 *to* m_2 *repectively, we arrive at*

$$m_1m_2\tau \leq \frac{m_2\Delta_1 f_1}{f_1} + \frac{m_1\Delta_2 f_2}{f_2} + exp(f)\Big\{\frac{(c+3)}{8}(m+1)(m-2)$$
$$+ \frac{(c-1)}{8}\big(2-2m+3m_1cos^2\theta_1+3m_2cos^2\theta_2)$$
$$+ \frac{1}{2}(m-1)trB + \frac{1}{8}m(m-1)||\omega^{\#}||^2\Big\}$$
$$+ \frac{1}{2}\{(\lambda_1+\lambda_2)(1-m)a + \lambda_2(\lambda_1-\lambda_2)(1-m)b + (\lambda_2-\lambda_1)m(m-1)\pi(H)\}. \quad (3.35)$$

Similarly, the equality sign holds in (3.35) identically. Thus the equality sign in (3.32) holds for each $\alpha \in \{1, \cdots, n_1\}$ *and* $\beta \in \{n_1+1, \cdots, n\}$. *Then we get*

$$\sigma^r_{\alpha j} = 0, \quad \sigma^r_{ij} = 0, \quad \sigma^r_{ij} = 0, \quad \text{where} \quad i \neq j, \quad \text{and} \quad r \in \{n+1, \cdots, 2m+1\}$$
$$\sigma^r_{\alpha j} = \sigma^r_{ij} = \sigma^r_{ij} = 0, \quad \text{and} \quad \sigma^r_{\alpha\alpha} + \sigma^r_{\beta\beta} = 0, \quad i,j \in P_{1n_1+1}, r = n+2, \cdots, 2m+1. \quad (3.36)$$

Moreover, If $m = 2$. *Then* $m_1 = m_2 = 1$. *thus from* (2.18), *we get* $\tau = \Delta_1 f_1 + \Delta_2 f_2$. *Hence the equality in (3.23) holds, which proves the theorem completely.* □

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