FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 37, No 1 (2022), 137–158
https://doi.org/10.22190/FUMI210401012A
Original Scientific Paper

STABILITY AND ERROR OF THE NEW NUMERICAL SOLUTION OF FRACTIONAL RIESZ SPACE TELEGRAPH EQUATION WITH TIME DELAY

Malek A. Asl, Farhad D. Saei, Mohammad Javidi and Yaghoub Mahmoudi

Faculty of Science, Department of Mathematics
Tabriz Branch, Islamic Azad University, Tabriz, Iran

Abstract. In this paper, we propose a numerical method for Riesz space fractional telegraph equation with time delay. The Riesz fractional telegraph equation is approximated with the interpolating polynomial P2. First a system of fractional differential equations are obtained from the telegraph equation with respect to the time variable. Then our numerical algorithm is proposed. The convergence order and stability of the fractional order algorithms are proved. Finally, some numerical examples are constructed to describe the usefulness and profitability of the numerical method. Numerical results show that the accuracy of order $O(\Delta t^3)$.
Key words: fractional telegraph equation, delay equation, polynomial approximation, Riesz fractional equation, stability and convergence.

1. Introduction

In primitive definitions, the order of derivatives and integrals in calculus is called integers. Recently fractional calculus includes part of the applied mathematics research. Its origin comes from the hospital and Leibnitz's inquisition about considering the result, if m was taken as half in the m^{th} derivative of a function [1].

During the last three decades, fractional calculus has been recently applied to physics, biology, engineering, and other sciences [19]. Fractional calculus is an

---

Received April 1, 2021, accepted: December 20, 2021
Communicated by Hari Mohan Srivastava
Corresponding Author: Mohammad Javidi, Faculty of Science, Department of Mathematics, Tabriz Branch, Islamic Azad University, Tabriz, Iran | E-mail: Mo_javidi@tabrizu.ac.ir
2010 *Mathematics Subject Classification.* Primary 34A08; Secondary 33F05, 26A33

important part of the field of science and technology as it is the generality of ordinary differentiation and integration with arbitrary order [12, 22, 23, 24, 25].

Telegraph equations are a pair of linear differential equations due to their extensive applications in high frequency transmission lines, propagation of electrical signals [1]. The author of [7], implement the meshless method for solving the Riesz fractional equation.

By reference to classical books [5, 6] a delay fractional telegraph equation is a differential equation in which the derivative of the function at any time depends on the solution at previous time. It seems that, so far, an analytical solution is not presented for the Riesz space fractional telegraph equation with time delay. However, there are fewer works dealing with numerical methods for delay fractional differential equations [19]. Some authors investigated fractional order partial differential equations [2, 15, 17] and delay fractional partial differential equations [8]. Numerical approximations and solution techniques for the space-time fractional telegraph equations were studied in [14]. The development of efficient numerical methods to solve the Riesz space fractional telegraph equation with time delay is still an important issue. For more examples and details we refer [16, 20] and therein.

Since delay problem has important applications in many fields for example physics, electrical engineering, and telecommunications [20], then analytical solution will be a comprehensive answer to this phenomenon.

In this paper, we propose a new numerical method to approximate the solution of the Riesz space fractional telegraph equation with time delay based on the polynomial interpolation of degree three. The interpolation for the time variable and mesh schemes for the space variable is presented with error analysis and stability.

After introducing the numerical method to approximate the Riesz space fractional telegraph equation, numerical results show that the accuracy of the present scheme is of order $O(\Delta t^3)$.

The rest of this paper is as follows. In Section 2, we present some necessary definitions. In Section 3, we present our idea to approximate of Riesz space fractional telegraph equation with time delay and discretize them. In Section 4, a numerical method for solving Riesz space fractional telegraph equation with time delay and error analysis are outlined. In Section 5, the error and stability analysis are discussed based on the error estimate of the compound trapezoidal formula. Some example and their figures and tables in Section 6, shows the accuracy of the present scheme. Finally, the conclusions are included in the last section.

2. Preliminaries

In this section, we present some necessary definitions, preliminary facts and presentation that will be used further in this study. We focus on these definitions of fractional calculus.

Definition 2.1. [12] The Liouville-Caputo fractional derivative of order $\alpha > 0$ of the function $f \in C^m(I, \mathbb{R})$ is defined as:

$$ {}_d^C D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_d^t \frac{f^{(m)}(s)}{(t-s)^{\alpha-m+1}} ds, & m-1 < \alpha < m, \quad m \in N, \\ f^{(m)}(t), & \alpha = m, \end{cases} $$

where $C(I, \mathbb{R})$ denotes the Banach space of all continuous functions from $I = [0, t]$ into $\mathbb{R}$ and the norm

$$ \|f\|_\infty = \sup\{|f(t)| : t \in I\}, \quad t > 0, $$

$C^m(I, \mathbb{R})$ denotes the class of all real valued functions defined on $I = [0, t], \quad t > 0$ which have continuous mth order derivatives.

Definition 2.2. [18] The left and right Riemann-Liouville derivatives with order $\alpha > 0$ of the given function $f(t), t \in (d, e)$ are defined as:

$$ {}_d^{RL} D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_d^t (t-s)^{m-\alpha-1} f(s) ds, $$

$$ {}_t^{RL} D_e^\alpha f(t) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_t^e (s-t)^{m-\alpha-1} f(s) ds, $$

respectively, where m is a positive integer satisfying $m - 1 < \alpha < m$.

Definition 2.3. [18] The Riesz derivative with order $\beta > 0$ of the given function $f(x), \quad x \in (d, e)$ are defined by:

$$ {}_{RZ} D_x^\beta f(x) = C_\beta ({}_d^R D_x^\beta f(x) + {}_t^R D_e^\beta f(x)), $$

where $C_\beta = \frac{-1}{2\cos(\frac{\beta\pi}{2})}, \quad \beta \neq 2k+1, \quad k = 0, 1, 2, ...$

${}_{RZ} D_t^\beta f(t)$ is sometimes expressed as $\frac{\partial^\beta f(x)}{\partial |x|^\beta}$.

Definition 2.4. [12] The Mittag-Leffler function is defined by series when the real part of α is strictly positive

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, $$

where Γ is the Gamma function as the following form:

Definition 2.5. [18] The Grunwald-Letnikov fractional derivative for all $\alpha \in \mathbb{R}^+$ is defined as:

$$ {}_a^{GL} D_t^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{[\frac{t-a}{h}]} (-1)^k \binom{\alpha}{k} f(t - kh), $$

where $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}$.

3. The Model

If n,m are positive integers and $[d, e]$, $[0, T]$ is given, let $h = \frac{e-d}{n}$, $\Delta t = \frac{T}{m}$. The solution domain $[d, e] \times [0, T]$ is covered by a uniform grid of mesh points (x, t). Note that h and Δt are the uniform spatial step size and temporal step size.

For every $\beta \quad (1 < \beta \leqslant 2)$ the left and right Riemann-Liouville derivatives exist and match with the left and right Grunwald-Letnikov derivatives under suitable conditions. Then the Riesz derivative with order $\beta \quad (1 < \beta \leqslant 2)$ can be discretized By the standard, shifted Grunwald-Letnikov formulas, or fractional centered difference method [3].

Recently second-order and fourth-order methods are used for the Riesz space and time fractional diffusion equations. It points that these methods and techniques are useful for solving some other fractional differential equations with Riesz fractional derivatives.

Now consider the following space-fractional telegraph delay equation with Riesz operation and fractional derivatives in time over a finite one-dimensional domain

$$\frac{\partial^\alpha}{\partial t^\alpha}(\frac{\partial^\alpha}{\partial t^\alpha}T(x,t)) + 2a\frac{\partial^\alpha}{\partial t^\alpha}T(x,t) + b^2T(x,t)$$

$$+W(x, t-\tau) = c\frac{\partial^\beta}{\partial |x|^\beta}T(x,t) + f(x,t), \tag{3.1}$$

subjected to the initial conditions

$$T(x,t) = u(x,t), \qquad d \leqslant x \leqslant e,$$

$$\frac{\partial^\alpha}{\partial t^\alpha}T(x,t) = S(x,t), \qquad d \leqslant x \leqslant e,$$

and Dirichlet boundary conditions

$$T(a,t) = T(b,t) = 0, \qquad 0 \leqslant t \leqslant T,$$

where $0 \leqslant b < a$ and $c > 0$ are constants, and $1 < \beta \leqslant 2, 0.5 < \alpha < 1$.
Equations of the form (3.1), arise in the study of electrical signals in a cable of transmission line and wave phenomena. In fact the telegraph equation is more suitable than ordinary diffusion equation in modeling reaction-diffusion [4]. Furthermore, we should mention that with the appropriate coefficient and forcing terms, the one-dimensional telegraph equation describes a diverse array of physical systems; for example, voltage and current signals in coaxial transmissions lines of negligible leakage conductance and/or resistance [11]. The Riesz space fractional operator $\frac{\partial^\beta}{\partial |x|^\beta}$ over $[d, e]$ is defined by right and left Riemann-Liouville fractional derivation [18] into following

$$\frac{\partial^\beta}{\partial |x|^\beta}T(x,t) = -\frac{1}{2\cos\frac{\beta\pi}{2}} \times \frac{1}{\Gamma(2-\beta)} \times \frac{\partial^2}{\partial x^2}\int_d^e \frac{T(s,t)}{|x-s|^{\beta-1}}ds, \tag{3.2}$$

The authors of [9], proposed the Chebyshev spectral collocation for one-dimensional linear hyperbolic telegraph equation. This method is very useful in providing highly accurate solutions to fractional partial differential equations. Another benefit of this method is using of spectral differentiation matrices. Ding et al. [3], used polynomial interpolation to design a novel high-order algorithm for the numerical estimation of fractional differential equations. They utilized Hadamard finite-part integral and the piecewise cubic interpolation polynomial to approximate the integral.

In this section, we present our idea to approximate of Riesz space fractional telegraph equation with time delay.

We discretize the space-fractional derivative operator through the following fractional central difference [16]:

$$\begin{aligned}\frac{\partial^\beta}{\partial|x|^\beta}T(x,t) &= -\frac{1}{h^\beta}\sum_{i=-\frac{e-x}{h}}^{\frac{x-d}{h}} \frac{(-1)^i\Gamma(\beta+1)}{\Gamma(\frac{\beta}{2}-i+1)\Gamma(\frac{\beta}{2}+i+1)}T(x-ih,t)+O(h^2)\\ &= -\frac{1}{h^\beta}\sum_{i=-\frac{e-x}{h}}^{\frac{x-d}{h}} v_iT(x-ih,t)+O(h^2),\end{aligned}$$

where $h\to 0$ and $1<\beta\leqslant 2$.

We introduce a new variable $S(x,t)=\frac{\partial^\alpha}{\partial t^\alpha}T(x,t)$ to transform (3.1) to the following equivalent system

$$\text{(3.3)}\qquad \begin{cases} \frac{d^\alpha}{dt^\alpha}T_i(t)=S_i(t), \qquad i=1,2,3,...,n-1\\ \frac{d^\alpha}{dt^\alpha}S_i(t)+2aS_i(t)+b^2T_i(t)=-c\frac{1}{h^\beta}\sum_{j=-\frac{e-x_i}{h}}^{j=\frac{x_i-d}{h}} v_jT_{i-j}(t)+F_i(t),\end{cases}$$

where $F_i(t)=f_i(t)-W(x_i,t-\tau)$.

Now, we define $T(x_i,t)=T_i(t), S(x_i,t)=S_i(t)$. By approximating [10] $\frac{\partial^\beta}{\partial|x|^\beta}T(x_i,t)$ by $\frac{1}{h^\beta}\sum_{j=-\frac{e-x_i}{h}}^{j=\frac{x_i-d}{h}} v_jT_{i-j}(t)$ where $v_j=\frac{(-1)^j\Gamma(\beta+1)}{\Gamma(\frac{\beta}{2}+j+1)\Gamma(\frac{\beta}{2}-j+1)}$.

Also from boundary conditions one can see $T_0(t)=T_n(t)=0$. By setting

$$T(t)=[T_1(t),T_2(t),...,T_{n-1}(t)]^t,$$

$$S(t)=[S_1(t),S_2(t),...,S_{n-1}(t)]^t,$$

we can rewrite (3.3), as the following matrix form

$$\text{(3.4)}\qquad \begin{cases}\frac{d^\alpha}{dt^\alpha}T(t)=S(t),\\ \frac{d^\alpha}{dt^\alpha}S(t)=-CT(t)-2aS(t)+F(t),\end{cases}$$

where

$$C=cB+b^2I_{n-1}.$$

The matrix I_{n-1} is the identity matrix of order n-1, and

$$B=\frac{1}{h^{\beta}}\begin{bmatrix} v_0 & v_{-1} & . & . & . & v_{-n+2} \\ v_1 & v_0 & . & . & . & v_{-n+3} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ v_{n-2} & v_{n-3} & . & . & . & v_0 \end{bmatrix}_{(n-1)(n-1)}.$$

Let

$$D=\begin{bmatrix} 0 & -I_{n-1} \\ C & 2aI_{n-1} \end{bmatrix}_{(2n-2)(2n-2)}.$$

If we set

$$Y(t)=[T_1(t),T_2(t),...,T_{n-1}(t),S_1(t),S_2(t),...,S_{n-1}(t)]^t,$$

Then, from (3.3), we obtain

$$\left\{\begin{array}{l} \frac{d^{\alpha}}{dt^{\alpha}}Y(t)=-DY(t)+G(t) \\ Y(0)=Y_0 \end{array}\right. , \tag{3.5}$$

where

$$G(t)=\begin{bmatrix} 0 \\ F(t) \end{bmatrix}_{(2n-2)\times 1}.$$

In order to obtain an error, we need the following theorems:

Theorem 3.1. *Assume that* $v_j=\frac{(-1)^j\Gamma(\beta+1)}{\Gamma(\frac{\beta}{2}+j+1)\Gamma(\frac{\beta}{2}-j+1)}$, $j=0,\pm1,\pm2,...$ *are the coefficients in the fractional central difference (3.3) for* $1<\beta\leqslant 2$. *Then*

1. $\sum\limits_{j=-\infty}^{\infty} v_j=0,$

2. $\forall m,n\in\mathbb{N}:\sum\limits_{\substack{j=-m\\ j\neq 0}}^{n}|v_j|=v_0$

Proof. (see [10], [12]). □

Theorem 3.2. *For the matrix D, we have*

$$\|D\|_{\infty}=Max\left\{1,\frac{2c}{h^{\beta}}(v_0+a)+b^2\right\}.$$

Proof. (see [10]). □

4. Numerical method

It is well known that the initial value problem (3.5), is equivalent to the following Volterra integral equation

$$Y(t) = Y(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1}(-DY(s) + G(s))ds, \tag{4.1}$$

Let $P(s) = -DY(s) + G(s)$. We consider (4.1) at $t = t_k$ $(k = 1, 2, ..., m-1)$ and rewrite it as the following form

$$Y(t_k) = Y(t_0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k - s)^{\alpha-1}(-DY(s) + G(s))ds, \tag{4.2}$$

Now we approximate $P(s)$ by its piecewise linear interpolation
$P(s) = -DY(s) + G(s)$ at the nodes t_0 and t_1 as the following form

$$\breve{P}_j(s) \simeq \frac{s - t_j}{t_{j-1} - t_j} P(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}} P(t_j). \tag{4.3}$$

Let $\bar{Y}(t_j)$ be the approximate solution of $Y(t_j)$, $j = 0, 1$, which has been determined. By using relations (4.1) and (4.3). Also for $t_{j-2} \leqslant s \leqslant t_j$, by using

$$\begin{aligned} \widehat{P}_j(s) \quad &\simeq \quad \frac{(s - t_{j-1})(s - t_j)}{(t_{j-2} - t_{j-1})(t_{j-2} - t_j)} P(t_{j-2}) + \frac{(s - t_{j-2})(s - t_j)}{(t_{j-1} - t_{j-2})(t_{j-1} - t_j)} P(t_{j-1}) \\ &+ \frac{(s - t_{j-1})(s - t_{j-2})}{(t_j - t_{j-1})(t_j - t_{j-2})} P(t_j), \qquad j = 2, 3, ..., k. \end{aligned} \tag{4.4}$$

We can obtain the following formula

$$\begin{aligned} \int_{t_0}^{t_k} (t_k - s)^{\alpha-1} P(s) ds \quad &\simeq \quad \int_{t_0}^{t_1} (t_k - s)^{\alpha-1} \breve{P}_1(s) ds \\ &+ \sum_{j=2}^{k} \int_{t_{j-1}}^{t_j} (t_k - s)^{\alpha-1} \widehat{P}_j(s) ds. \end{aligned} \tag{4.5}$$

According to article [10]

$$\begin{aligned} \int_{t_0}^{t_1} (t_k - s)^{\alpha-1} \breve{P}_1(s) ds \quad &\simeq \quad \int_{t_0}^{t_1} (t_k - s)^{\alpha-1} \frac{s - t_1}{t_0 - t_1} P(t_0) ds \\ &+ \int_{t_0}^{t_1} (t_k - s)^{\alpha-1} \frac{s - t_0}{t_1 - t_0} P(t_1) ds \\ &= \quad \frac{t^{\alpha}}{\alpha(\alpha+1)} \Big\{ (k^{\alpha}(\alpha + 1 - k) + (k-1)^{\alpha+1}) P(t_0) \\ &+ (k^{\alpha+1} - (k-1)^{\alpha}(\alpha + k)) P(t_1) \Big\}. \end{aligned} \tag{4.6}$$

We can prove that

$$\begin{aligned}
\int_{t_{j-1}}^{t_j}(t_k-s)^{\alpha-1}ds &= \frac{t^\alpha}{\alpha}((k-j+1)^\alpha-(k-j)^\alpha),\\
\int_{t_{j-1}}^{t_j}s(t_k-s)^{\alpha-1}ds &= \frac{t^{\alpha+1}}{\alpha}((k-j+1)^\alpha(j-1+\frac{(k-j+1)}{(\alpha+1)})\\
&\quad -(k-j)^\alpha(j+\frac{(k-j)}{(\alpha+1)})),\\
\int_{t_{j-1}}^{t_j}s(t_k-s)^{\alpha-1}ds &= \frac{t^{\alpha+1}}{\alpha}((k-j+1)^\alpha(j-1+\frac{(k-j+1)}{(\alpha+1)})
\end{aligned}$$

$$-(k-j)^\alpha(j+\frac{(k-j)}{(\alpha+1)}))-\frac{(k-j)^\alpha t^{\alpha+2}}{\alpha}(j^2+\frac{2j(k-j)}{(\alpha+1)})+\frac{2(k-j)^2}{(\alpha+1)(\alpha+2)})). \tag{4.7}$$

By using relationship (4.4) for all $j=2,3,...,k$ we get

$$\begin{aligned}
\int_{t_{j-1}}^{t_j}(t_k-s)^{\alpha-1}\widehat{P}_j(s)ds &= \frac{P(t_{j-2})}{2t^2}\int_{t_{j-1}}^{t_j}(t_k-s)^{\alpha-1}(s^2-(t_j+t_{j-1})s+t_jt_{j-1})ds\\
&\quad -\frac{P(t_{j-1})}{t^2}\int_{t_{j-1}}^{t_j}(t_k-s)^{\alpha-1}(s^2-(t_j+t_{j-2})s+t_jt_{j-2})ds\\
&\quad +\frac{P(t_j)}{2t^2}\int_{t_{j-1}}^{t_j}(t_k-s)^{\alpha-1}(s^2-(t_{j-1}+t_{j-2})s+t_{j-1}t_{j-2})ds.
\end{aligned}$$

Using (4.7) shows that

$$\begin{aligned}
&\int_{t_{j-1}}^{t_j}(t_k-s)^{\alpha-1}\widehat{P}_j(s)ds\\
&= \frac{P(t_{j-2})t^\alpha}{2\alpha}\Bigg\{(k-j+1)^\alpha(\frac{2(k-j+1)^2}{(\alpha+1)(\alpha+2)}\\
&\quad -\frac{(k-j+1)}{(\alpha+1)})-(k-j)^\alpha(\frac{2(k-j)^2}{(\alpha+1)(\alpha+2)}+\frac{(k-j)}{(\alpha+1)})\Bigg\}\\
&\quad -\frac{P(t_{j-1})t^\alpha}{\alpha}\Bigg\{(k-j+1)^\alpha(\frac{2(k-j+1)^2}{(\alpha+1)(\alpha+2)}-1)\\
&\quad -(k-j)^\alpha(\frac{2(k-j)^2}{(\alpha+1)(\alpha+2)}+\frac{2(k-j)}{(\alpha+1)})\Bigg\}\\
&\quad -\frac{P(t_j)t^\alpha}{2\alpha}\Bigg\{(k-j+1)^\alpha(\frac{2(k-j+1)^2}{(\alpha+1)(\alpha+2)}\\
&\quad +\frac{(k-j+1)}{(\alpha+1)})-(k-j)^\alpha(2+\frac{2(k-j)^2}{(\alpha+1)(\alpha+2)}+\frac{3(k-j)}{(\alpha+1)})\Bigg\}.
\end{aligned}$$

Therefore

$$\sum_{j=2}^{k-2}\int_{t_{j-1}}^{t_j}(t_k-s)^{\alpha-1}\widehat{P}_j(s)ds=\frac{t^\alpha}{2}\left\{\sum_{j=2}^{k-2}(L_j+M_j+N_j)P(t_j)\right.$$

$$\left.+L_0P_0+(L_1+M_1)P_1+(M_{k-1}+N_{k-1})P_{k-1}+N_kP_k\right\}, \tag{4.8}$$

where

$$\begin{aligned}
L_j &= \frac{1}{\alpha(\alpha+1)(\alpha+2)}\sum_{j=0}^{k-2}\Big\{(k-j-1)^{\alpha+1}(2k-2j-\alpha-4)\\
&\qquad -(k-j-2)^{\alpha+1}(2k-2j+\alpha-2)\Big\},\\
M_j &= \frac{4}{\alpha(\alpha+1)(\alpha+2)}\sum_{j=1}^{k-1}\Big\{(k-j-1)^{\alpha+1}(k-j+\alpha+1)\\
&\qquad -(k-j)^{\alpha}((k-j)^2-\frac{(\alpha+1)(\alpha+2)}{2})\Big\},\\
N_j &= \frac{1}{\alpha(\alpha+1)(\alpha+2)}\sum_{j=2}^{k}\Big\{(k-j+1)^{\alpha+1}(2k-2j+\alpha+4)\\
&\qquad -(k-j)^{\alpha}(2(k-j)^2+3(k-j)(\alpha+2)+2(\alpha+1)(\alpha+2))\Big\}.
\end{aligned}$$

Recall that

$$P(s)=-DY(s)+G(s).$$

Finally, we get our approximation

$$Y(t_k)=\left\{I+\frac{N_kt^\alpha D}{2\Gamma(\alpha+3)}\right\}^{-1}\left\{Y(t_0)+\frac{t^\alpha}{\Gamma(\alpha+2)}((k^\alpha(\alpha+1-k)+(k-1)^{\alpha+1})P_0\right.$$

$$+(k^{\alpha+1}-(k-1)^\alpha(\alpha+k))P_1)+\frac{t^\alpha}{2\Gamma(\alpha+3)}\left\{\sum_{j=2}^{k-2}(L_j+M_j+N_j)P(t_j)\right.$$

$$\left.\left.+L_0P_0+(L_1+M_1)P_1+(M_{k-1}+N_{k-1})P_{k-1}+N_kG_k\right\}\right\}. \tag{4.9}$$

5. Error and Stability Analysis

In this section, the error analysis for the proposed scheme in the previous section is discussed based on the error estimate of the compound trapezoidal formula. From previous section, we have

$$Y(t_k)=Y(0)+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(t_k-s)^{\alpha-1}P(s)ds$$

and

$$\bar{Y}(t_k) = \bar{Y}(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k - s)^{\alpha-1} \bar{P}(s) ds.$$

We can easily get that

$$Y(t_k) - \bar{Y}(t_k) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k - s)^{\alpha-1} (-DY(s) + G(s) - \bar{P}(s)) ds \tag{5.1}$$

where on each interval $[t_j, t_{j+1}]$, $j = 0, 1, ..., n-1$, we have

$$\begin{aligned} Y(t_k) - \bar{Y}(t_k) &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_k - s)^{\alpha-1} (-DY(s) + G(s) - \bar{P}(s)) ds \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{j=2}^{k-1} \int_{t_{j-1}}^{t_j} (t_k - s)^{\alpha-1} (-DY(s) + G(s) - \bar{P}(s)) ds, \end{aligned}$$

The reader refer to [10] for the first integral. But for the second integral on the interval $[t_j, t_{j+1}]$, $j = 0, 1, ..., n-1$, we have

$$P(s) - \bar{P}(s) = (s - t_{j-1})(s - t_j)(s - t_{j+1}) \frac{P'''(\delta_j)}{3!}, \quad t_j < \delta_j < t_{j+1}.$$

In this case, we get

$$\begin{aligned} & \left\| Y(t_k) - \bar{Y}(t_k) \right\|_\infty \\ \leqslant\ & \frac{1}{6\Gamma(\alpha)} \sum_{j=2}^{k-1} \int_{t_{j-1}}^{t_j} \left\| (t_k - s)^{\alpha-1} (s - t_{j-1})(s - t_j)(s - t_{j+1}) \right\|_\infty \cdot \\ & \left\| -DY'''(\delta_j) + G'''(\delta_j) \right\|_\infty ds \\ \leqslant\ & \frac{1}{6\Gamma(\alpha)} \sum_{j=2}^{k-1} \int_{t_{j-1}}^{t_j} \left\| (t_k - s)^{\alpha-1} (s - t_{j-1})(s - t_j)(s - t_{j+1}) \right\|_\infty \cdot \\ & (\|D\|_\infty \bar{Y} + \bar{G}) ds. \end{aligned} \tag{5.2}$$

Assuming

$$\begin{aligned} \bar{Y}_j &= \underset{t_{j-1} \leqslant s \leqslant t_{j+1}}{Max} \|Y'''(s)\|_\infty, \quad \bar{Y} = \underset{2 \leqslant j \leqslant k-1}{Max} \bar{Y}_j \\ \bar{G}_j &= \underset{t_{j-1} \leqslant s \leqslant t_{j+1}}{Max} \|G'''(s)\|_\infty, \quad \bar{G} = \underset{2 \leqslant j \leqslant k-1}{Max} \bar{G}_j \\ R &= \|D\|_\infty \bar{Y} + \bar{G} \end{aligned}$$

And knowing that

$$g(s) = (s - t_{j-1})(s - t_j)(s - t_{j+1}).$$

From there one can see that

$$|g(s)| \leqslant \frac{2\sqrt{3}}{9}\Delta t^3.$$

So we can write for relationship (5.2)

$$\begin{aligned}
& \left\|Y(t_k) - \bar{Y}(t_k)\right\|_\infty \\
\leqslant\ & \frac{R}{6\Gamma(\alpha)} \sum_{j=2}^{k-1} \int_{t_{j-1}}^{t_j} \left\|(t_k - s)^{\alpha-1}\right\|_\infty \|(s-t_{j-1})(s-t_j)(s-t_{j+1})\|_\infty ds \\
\leqslant\ & \frac{R\left|-\frac{2\sqrt{3}}{9}t^3\right|}{6\Gamma(\alpha)} \sum_{j=2}^{k-1} \left| \left(\frac{(t_k-s)^\alpha}{\alpha} \right]_{t_{j-1}}^{t_j} \right| \\
=\ & \frac{2\sqrt{3}R(1-(k-1)^\alpha)}{54\Gamma(\alpha+1)}\Delta t^{3+\alpha}.
\end{aligned}$$

Therefore

$$\left\|Y(t_k) - \bar{Y}(t_k)\right\|_\infty \leqslant O(\Delta t^3).$$

In sequence, we give the theoretical stability analysis of our scheme. A numerical initial value problem solver is stable if small perturbations in the initial conditions do not cause the numerical approximation to diverge away from the true solution provided the true solution of the initial value problem is bounded [13].

Theorem 5.1. *Let $Y(t_k)$ and $\bar{Y}(t_k)$ be numerical solutions in (4.1), with the initial conditions $Y(t_0)$ and $\bar{Y}(t_0)$, respectively. Then*

$$\left\|Y(t_k) - \bar{Y}(t_k)\right\|_\infty \leqslant E\left\|Y(t_0) - \bar{Y}(t_0)\right\|_\infty, \tag{5.3}$$

for any k, i.e. the new scheme is numerically stable. Where

$$E = \frac{Q(\Gamma(\alpha+1) + \frac{5}{2}T^\alpha\|D\|_\infty)}{\Gamma(\alpha+1) - T^\alpha\|D\|_\infty}.$$

Proof. This proof will be used based on mathematical induction. In view of the given initial condition, suppose that (5.3) is true for (j=1,2,...,k-1). We must prove that this also holds for j=k. Assume that

$$
\begin{aligned}
Y(t_k) \;=\; & Y(t_0)+\frac{1}{\Gamma(\alpha)}\left\{\int_{t_0}^{t_1}(t_k-s)^{\alpha-1}\frac{s-t_1}{t_0-t_1}P(t_0)ds\right.\\
&+\int_{t_0}^{t_1}(t_k-s)^{\alpha-1}\frac{s-t_0}{t_1-t_0}P(t_1)ds\\
&+\sum_{j=2}^{k}\int_{t_{j-1}}^{t_j}(t_k-s)^{\alpha-1}\frac{(s-t_{j-1})(s-t_j)}{(t_{j-2}-t_{j-1})(t_{j-2}-t_j)}P(t_{j-2})ds\\
&+\sum_{j=2}^{k}\int_{t_{j-1}}^{t_j}(t_k-s)^{\alpha-1}\frac{(s-t_{j-2})(s-t_j)}{(t_{j-1}-t_{j-2})(t_{j-1}-t_j)}P(t_{j-1})ds\\
&\left.+\sum_{j=2}^{k}\int_{t_{j-1}}^{t_j}(t_k-s)^{\alpha-1}\frac{(s-t_{j-1})(s-t_{j-2})}{(t_j-t_{j-1})(t_j-t_{j-2})}P(t_j)ds\right\}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
& \left\|Y(t_k)-\bar{Y}(t_k)\right\|_\infty\\
\leqslant\; & \left\|Y(t_0)-\bar{Y}(t_0)\right\|_\infty\\
&+\frac{\|D\|_\infty}{\Gamma(\alpha)}\left\{\left|-\frac{1}{t}\int_{t_0}^{t_1}(t_k-s)^{\alpha-1}(s-t_1)ds\right|\left\|Y(t_0)-\bar{Y}(t_0)\right\|_\infty\right.\\
&+\left|\frac{1}{t}\int_{t_0}^{t_1}(t_k-s)^{\alpha-1}(s-t_0)ds\right|\left\|Y(t_1)-\bar{Y}(t_1)\right\|_\infty\\
&+\sum_{j=0}^{k-2}\left|\frac{1}{2t^2}\int_{t_{j+1}}^{t_{j+2}}(t_k-s)^{\alpha-1}(s-t_{j+1})(s-t_{j+2})ds\right|\left\|Y(t_j)-\bar{Y}(t_j)\right\|_\infty\\
&+\sum_{j=1}^{k-1}\left|\frac{-1}{t^2}\int_{t_j}^{t_{j+1}}(t_k-s)^{\alpha-1}(s-t_{j-1})(s-t_{j+1})ds\right|\left\|Y(t_j)-\bar{Y}(t_j)\right\|_\infty\\
&+\sum_{j=2}^{k-1}\left|\frac{1}{2t^2}\int_{t_{j-1}}^{t_j}(t_k-s)^{\alpha-1}(s-t_{j-1})(s-t_{j-2})ds\right|\left\|Y(t_j)-\bar{Y}(t_j)\right\|_\infty\\
&\left.+\left|\frac{1}{2t^2}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}(s-t_{k-1})(s-t_{k-2})ds\right|\left\|Y(t_k)-\bar{Y}(t_k)\right\|_\infty\right\}.
\end{aligned}
$$

Since

$$
\left|-\frac{1}{t}\int_{t_0}^{t_1}(t_k-s)^{\alpha-1}(s-t_1)ds\right|\leqslant\left|\frac{\tilde{s}_1-t_1}{-t}\right|_{t_0\leqslant\tilde{s}_1\leqslant t_1}\int_{t_0}^{t_1}(t_k-s)^{\alpha-1}ds\leqslant\frac{1}{\alpha}(t_k^\alpha-t_{k-1}^\alpha),
$$

$$
\left|\frac{1}{t}\int_{t_0}^{t_1}(t_k-s)^{\alpha-1}(s-t_0)ds\right|\leqslant\left|\frac{\tilde{s}_1-t_0}{t}\right|_{t_0\leqslant\tilde{s}_1\leqslant t_1}\int_{t_0}^{t_1}(t_k-s)^{\alpha-1}ds\leqslant\frac{1}{\alpha}(t_k^\alpha-t_{k-1}^\alpha),
$$

Combining above results, we can derive the following inequalities

$$\begin{aligned}
&\sum_{j=0}^{k-2}\left|\frac{1}{2t^2}\int_{t_{j+1}}^{t_{j+2}}(t_k-s)^{\alpha-1}(s-t_{j+1})(s-t_{j+2})ds\right| \\
\leqslant\ &\sum_{j=0}^{k-2}\left|\underset{t_{j+1}\leqslant\tilde{s}_1\leqslant t_{J+2}}{\frac{(\tilde{s}_1-t_{j+1})(\tilde{s}_1-t_{j+2})}{2t^2}}\right|\int_{t_{j+1}}^{t_{j+2}}(t_k-s)^{\alpha-1}ds\leqslant\frac{1}{2\alpha}t_{k-1}^{\alpha}, \\
&\sum_{j=1}^{k-1}\left|\frac{-1}{t^2}\int_{t_j}^{t_{j+1}}(t_k-s)^{\alpha-1}(s-t_{j-1})(s-t_{j+1})ds\right| \\
\leqslant\ &\sum_{j=1}^{k-1}\left|\underset{t_j\leqslant\tilde{s}_2\leqslant t_{J+1}}{\frac{(\tilde{s}_2-t_{j-1})(\tilde{s}_2-t_{j+1})}{-t^2}}\right|\int_{t_j}^{t_{j+1}}(t_k-s)^{\alpha-1}ds\leqslant\frac{2}{\alpha}t_{k-1}^{\alpha}, \\
&\sum_{j=2}^{k-1}\left|\frac{1}{2t^2}\int_{t_{j-1}}^{t_j}(t_k-s)^{\alpha-1}(s-t_{j-1})(s-t_{j-2})ds\right| \\
\leqslant\ &\sum_{j=2}^{k-1}\left|\underset{t_{j-1}\leqslant\tilde{s}_3\leqslant t_J}{\frac{(\tilde{s}_3-t_{j-2})(\tilde{s}_3-t_{j-1})}{2t^2}}\right|\int_{t_{j-1}}^{t_j}(t_k-s)^{\alpha-1}ds\leqslant\frac{1}{\alpha}(t_{k-1}^{\alpha}-t_1^{\alpha}), \\
&\left|\frac{1}{2t^2}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}(s-t_{k-1})(s-t_{k-2})ds\right|\leqslant\frac{t^{\alpha}}{\alpha}.
\end{aligned}$$

Let us

$$\begin{gathered}
\forall j: 0\leqslant j\leqslant k-1 \\
\left\|Y(t_j)-\bar{Y}(t_j)\right\|_{\infty}\leqslant Q_j\left\|Y(t_0)-\bar{Y}(t_0)\right\|_{\infty}, \\
Q=\underset{0\leqslant j\leqslant k-1}{Max}\ Q_j
\end{gathered}$$

In this case

$$\begin{aligned}
&\left\|Y(t_k)-\bar{Y}(t_k)\right\|_{\infty}-\frac{\|D\|_{\infty}t^{\alpha}}{\alpha\Gamma(\alpha)}\left\|Y(t_k)-\bar{Y}(t_k)\right\|_{\infty} \\
\leqslant\ &Q\left\|Y(t_0)-\bar{Y}(t_0)\right\|_{\infty}+\frac{\|D\|_{\infty}}{\Gamma(\alpha)}\cdot \\
&\left\{\frac{Q}{\alpha}(t_k^{\alpha}-t_{k-1}^{\alpha})+\frac{Q}{\alpha}(t_k^{\alpha}-t_{k-1}^{\alpha})+\frac{Q}{2\alpha}t_{k-1}^{\alpha}+\frac{2Q}{\alpha}t_{k-1}^{\alpha}+\frac{Q}{\alpha}(t_{k-1}^{\alpha}-t_1^{\alpha})\right\}\cdot \\
&\left\|Y(t_0)-\bar{Y}(t_0)\right\|_{\infty}
\end{aligned}$$

$$\begin{aligned}
\left\|Y(t_k)-\bar{Y}(t_k)\right\|_{\infty} &\leqslant \frac{Q\Gamma(\alpha+1)}{\Gamma(\alpha+1)-\|D\|_{\infty}t^{\alpha}}\left\{1+\frac{5\|D\|_{\infty}t_k^{\alpha}}{2\Gamma(\alpha+1)}\right\}\left\|Y(t_0)-\bar{Y}(t_0)\right\|_{\infty} \\
&\leqslant \frac{Q(\Gamma(\alpha+1)+\frac{5}{2}T^{\alpha}\|D\|_{\infty})}{\Gamma(\alpha+1)-T^{\alpha}\|D\|_{\infty}}\left\|Y(t_0)-\bar{Y}(t_0)\right\|_{\infty},
\end{aligned}$$

where $T = Max(t_j)$; $\quad j = 0, 1, 2, ..., m$. Therefore

$$\left\|Y(t_k) - \bar{Y}(t_k)\right\|_\infty \leqslant E\left\|Y(t_0) - \bar{Y}(t_0)\right\|_\infty.$$

Now, applying the mathematical induction and choosing suitable E leads to the end of the proof. $\square$

6. Numerical examples

In this section, two examples for which the exact solutions are known are solved by the proposed method to illustrate the efficiency and effectiveness of the suggested numerical scheme. We estimate the maximum error and show its values graphically in different modes. Both examples and their figures and Tables show that the accuracy of the present scheme. The distinction between the measured value of the approximate solution and its absolute error, is given by

$$|Y(x,t) - T(x,t)|\,,$$

where $Y(x,t)$ and $T(x,t)$ are the exact and the numerical solution at the point (x,t), respectively.

Example 6.1. Consider the following Riesz space fractional telegraph equation with time delay

$$\frac{\partial^\alpha}{\partial t^\alpha}(\frac{\partial^\alpha}{\partial t^\alpha}T(x,t)) + 2a\frac{\partial^\alpha}{\partial t^\alpha}T(x,t) + b^2T(x,t) + W(x,t-\tau) = c\frac{\partial^\beta}{\partial |x|^\beta}T(x,t) + f(x,t),$$

where the initial and boundary conditions are

$$T(0,t) = T(1,t) = 0, \qquad 0 \leqslant t \leqslant 1$$

$$T(x,0) = 0, \quad \frac{\partial^\alpha}{\partial t^\alpha}T(x,0) = 0,$$

and the inhomogeneous term is

$$\begin{aligned} F(x,t) \quad = \quad & x^2(1-x)^2\left\{t^{2-2\alpha}E_{2,3-2\alpha}(-t^2) + 2at^{2-\alpha}E_{2,3-\alpha}(-t^2) + 2b^2\sin^2\frac{t}{2}\right\} \\ & + \frac{2c\sin^2\frac{t}{2}}{2\cos\frac{\beta\pi}{2}}\left\{\frac{\Gamma(5)}{\Gamma(5-\beta)}(x^{4-\beta} + (1-x)^{4-\beta}) - 2\frac{\Gamma(4)}{\Gamma(4-\beta)}(x^{3-\beta} + (1-x)^{3-\beta})\right. \\ & \left. + \frac{\Gamma(3)}{\Gamma(3-\beta)}(x^{2-\beta} + (1-x)^{2-\beta})\right\} - x^6(1-x)^8\sin^2(\frac{1}{2}(t-\tau)). \end{aligned}$$

This problem has the exact solution $T(x,t) = 2x^2(1-x^2)\sin^2\frac{t}{2}$.
We use the method of (4.9) to solve this problem for $a = 5$, $\quad b = .25$, $\quad c = 1$. The numerical solution are shown in Table 6.1 with take $\tau = .0001$, $\alpha = .9$, $\quad \beta = 1.9$ we find that, the numerical results fit well with the theoretical analysis. Table 6.2 shows the maximum error in difference α, β.
Figures 6.1 and 6.2 show that the analytical and numerical solution. Figure 6.3 shows the comparison between the different β in $hx = \Delta t = 1/5$. Figure 6.4 shows that the comparison between the analytical and numerical solutions in terms of time and space at $\alpha = 0.9, hx = 1/100$.

Table 6.1: Maximum errors and temporal convergence order of example 1 for $\beta = 1.9$

hx=Δt	Maximum error	Temporal convergence order
$\frac{1}{9}$	2.131321e-4	-
$\frac{1}{18}$	4.463033e-5	2.255651
$\frac{1}{36}$	7.085532e-6	2.655076
$\frac{1}{72}$	8.858860e-7	2.999683

Table 6.2: Maximum errors in difference α, β of example 1

hx=Δt	τ	α	β	The maximum error
.01	.1	.6	1.3	1.898990e-6
.01	.1	.6	1.6	1.688523e-6
.01	.1	.6	1.9	2.438964e-6
.01	.1	.7	1.3	1.883214e-6
.01	.1	.7	1.6	1.694502e-6
.01	.1	.7	1.9	2.157194e-6
.01	.1	.8	1.3	1.934076e-6
.01	.1	.8	1.6	1.765036e-6
.01	.1	.8	1.9	1.521870e-6
.01	.1	.9	1.3	3.927010e-6
.01	.1	.9	1.6	2.651678e-6
.01	.1	.9	1.9	7.383265e-7

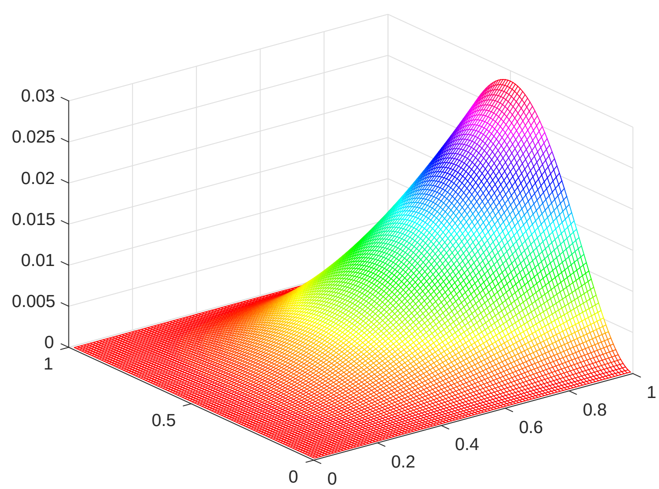

FIG. 6.1: The analytical solution $T(x,t) = 2x^2(1-x^2)\sin^2\frac{t}{2}$ for example 1

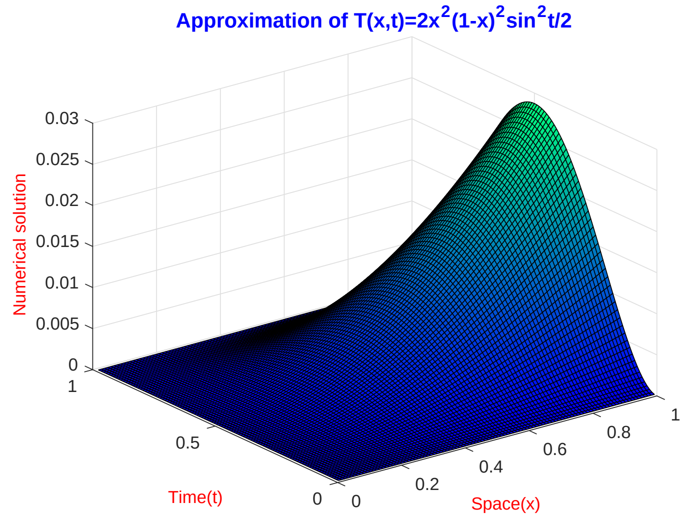

Fig. 6.2: The numerical solution at $\alpha = .9$ and $hx = 1/100$ for example 1

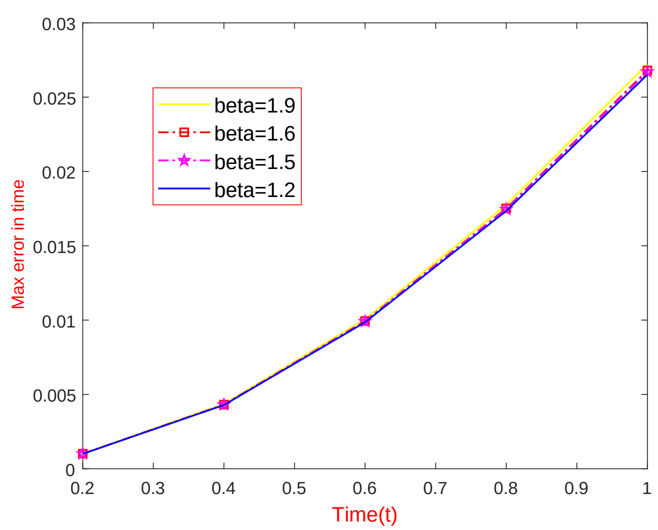

Fig. 6.3: Comparison between the numerical solution for $hx = 1/5$ at different β for example 1

Example 6.2. Consider the following Riesz space fractional telegraph equation with time delay and constant coefficients

$$\frac{\partial^\alpha}{\partial t^\alpha}(\frac{\partial^\alpha}{\partial t^\alpha}T(x,t)) + 2a\frac{\partial^\alpha}{\partial t^\alpha}T(x,t) + b^2T(x,t) + W(x,t-\tau) = c\frac{\partial^\beta}{\partial |x|^\beta}T(x,t) + f(x,t),$$

With initial condition

$$T(0,t) = T(1,t) = 0, \qquad 0 \leqslant t \leqslant 1$$

and boundary conditions

$$T(x,0) = 0, \quad \frac{\partial^\alpha}{\partial t^\alpha}T(x,0) = 0,$$

since

$$\frac{\partial^\beta}{\partial |x|^\beta}T(x,t) = -\frac{1}{2\cos\frac{\beta\pi}{2}} \times \frac{1}{\Gamma(2-\beta)} \times \frac{\partial^2}{\partial x^2}\int_a^b \frac{T(s,t)}{|x-s|^{\lambda-1}}ds$$

and the forced term is

$$\begin{aligned} F(x,t) \quad = \quad & x^2(1-x)^2\left\{\frac{6t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{12at^{3-\alpha}}{\Gamma(4-\alpha)} + b^2t^3\right\} + \frac{ct^3}{2\cos\frac{\beta\pi}{2}} \\ & \times\left\{\frac{\Gamma(5)}{\Gamma(5-\beta)}(x^{4-\beta} + (1-x)^{4-\beta}) - 2\frac{\Gamma(4)}{\Gamma(4-\beta)}(x^{3-\beta} + (1-x)^{3-\beta}) \right. \\ & \left. + \frac{\Gamma(3)}{\Gamma(3-\beta)}(x^{2-\beta} + (1-x)^{2-\beta})\right\} - x^6(1-x)^6(t-\tau)^6. \end{aligned}$$

The above equation has the exact solution $T(x,t) = x^2(1-x^2)t^3$. We used our proposed method to solve this problem for $a = 3, \quad b = 1, \quad c = 1$. In this test, corresponding with the last example the computational results are tabulated in Table 6.3 and 6.4.
Figure 6.5 shows the analytical solution for $T(x,t) = x^2(1-x^2)t^3$. Figure 6.6 shows the numerical solution of $T(x,t)$ at $\alpha = .9$ and $hx = 1/100$. Figure 6.7 shows a comparison between the numerical solution for $hx = 1/5$ at different β. At least 6.8 the comparison between the analytical and numerical solutions in terms of time and space has been shown.

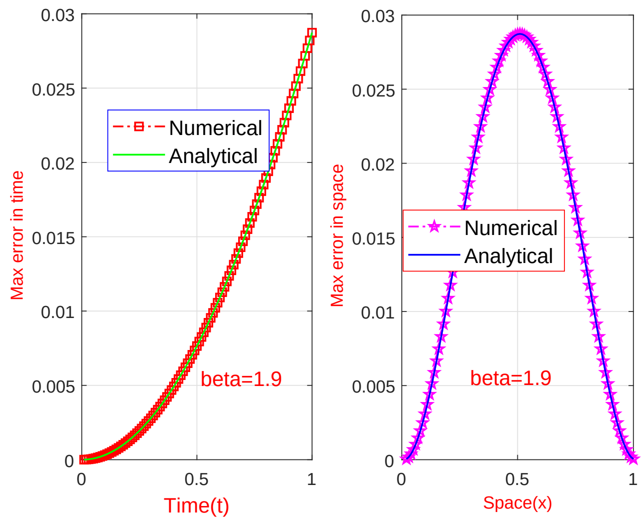

FIG. 6.4: Comparison between the analytical and numerical solutions in terms of time and space for example 1 at $\alpha = .9$ and $hx = 1/100$

Table 6.3: Maximum errors and temporal convergence order of example 2 for $\beta = 1.9$

hx=Δt	Maximum error	Temporal convergence order
$\frac{1}{8}$	6.758348e-4	-
$\frac{1}{16}$	1.677320e-4	2.010514
$\frac{1}{32}$	3.963873e-5	2.081175
$\frac{1}{64}$	7.869601e-6	2.332549
$\frac{1}{128}$	9.834845e-7	3.000316

Table 6.4: Maximum errors in difference α, β, τ of example 2

α	$hx = \Delta t$	β	τ	The maximum error
.6	$\frac{1}{6}$	1.2	.1	4.364050e-4
.6	$\frac{1}{12}$	1.8	.01	3.650731e-4
.6	$\frac{1}{24}$	1.9	.001	1.089647e-4
.8	$\frac{1}{6}$	1.2	.1	3.302924e-4
.8	$\frac{1}{12}$	1.8	.01	2.806825e-4
.8	$\frac{1}{24}$	1.9	.001	8.471009e-5
.9	$\frac{1}{6}$	1.2	.1	2.851935e-4
.9	$\frac{1}{12}$	1.8	.01	2.404663e-4
.9	$\frac{1}{24}$	1.9	.001	7.280021e-5

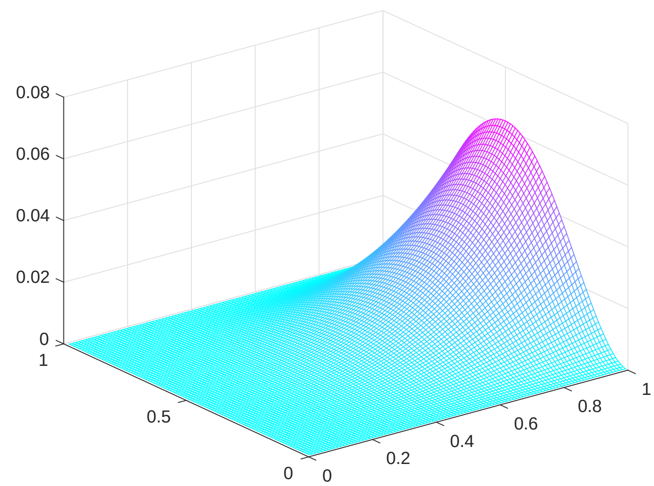

FIG. 6.5: The analytical solution $T(x,t) = x^2(1 - x^2)t^3$ for example 2

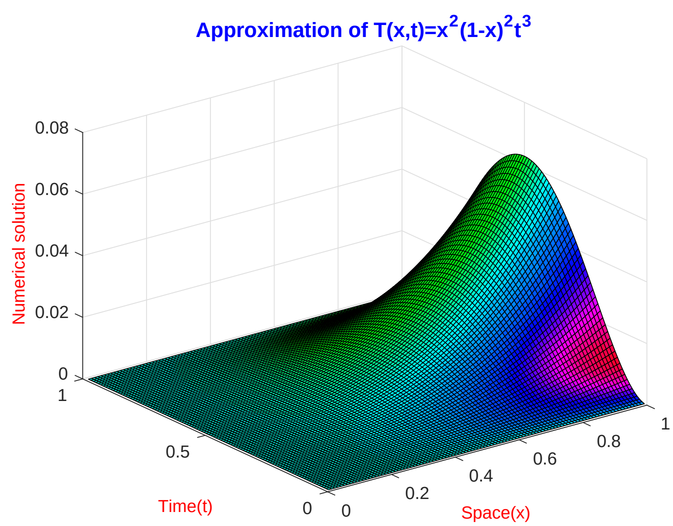

FIG. 6.6: The numerical solution at $\alpha = .9$ and $hx = 1/100$ for example 2

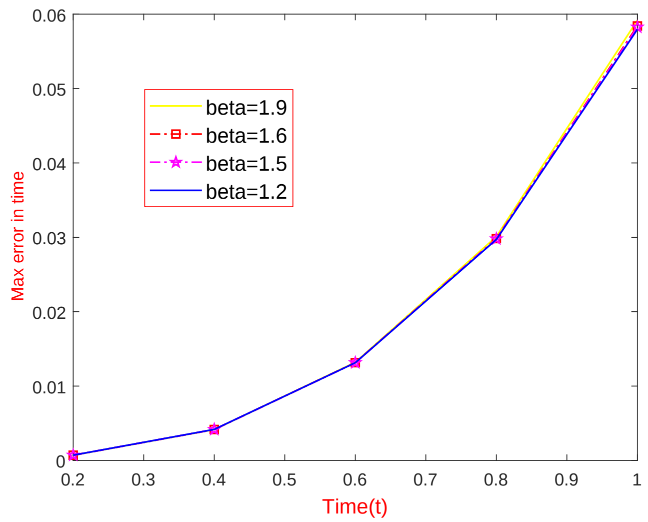

FIG. 6.7: Comparison between the numerical solution for $hx = 1/5$ at different β for example 2

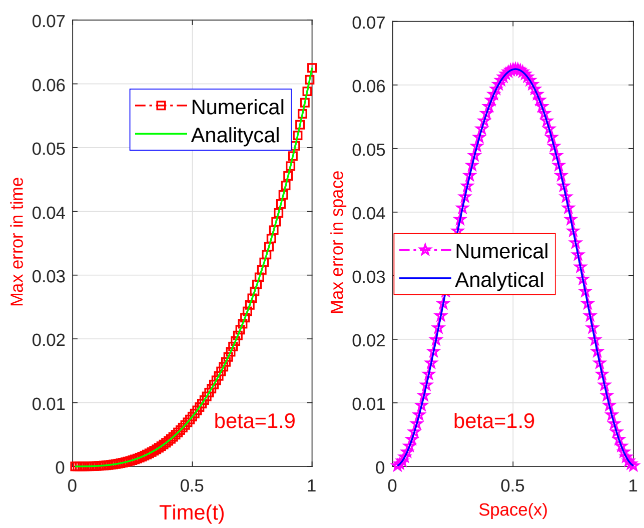

FIG. 6.8: Comparison between the analytical and numerical solutions in terms of time and space for example 2 at $\alpha = .9$ and $hx = 1/100$

7. Conclusion

Riesz derivative operators are used in some partial differential equations such as wave equation, diffusion equation, telegraph equations, and some other partial differential equations. This paper provides an iterative solution to the Riesz space telegraph equation with time delay. We present an algorithm to approximation based on the piecewise polynomial interpolation of degree 2 that is used for discretizing of Riesz space fractional telegraph equation. The approximate results approach in analytic form with order $O(\Delta t^3)$. The conclusions are verified and compared by two numerical examples. We believe that this approximation will be possible to have a better comprehension of the telegraph equation with time delay in electrical systems and transmission lines. The gained results show that, this method required less amount of similar numerical methods. This claim can be substantiated by other examples.

REFERENCES

1. M. Ahmad, A. A. Bhat, and J. A. I. N. Renu: *Space time fractional telegraph equation and its application by using Adomian decomposition method*. Journal of New Theory. **22** (2018), 73–81.
2. M. S. Asl, M. Javidi, and Y. Yan: *A novel high-order algorithm for the numerical estimation of fractional differential equations*. Elsevier J. Computational and Applied Mathematics. **16(2)** (2007), 180–201.
3. H. Ding, C. Li, and Y. Chen: *High-order algorithms for Riesz derivative and their applications II*. Journal of Computational Physics. **293** (2015), 218–237.
4. M. Dosti and A. Nazemi: *Quartic B-spline collocation method for solving one-dimensional hyperbolic telegraph equation*. Journal of Information and Computing Science. **7(2)** (2012), 83–90.
5. R. D. Driver: *Ordinary and delay differential equations*. Springer Science, Business Media, 2012.
6. J. K. Hale and S. M. V. Lunel: *Introduction to functional differential equations*. Springer Science, Business Media, 1993.
7. M. Ilkhan, E. E. Kara, and F. Usta: *Numerical Solution of Riesz Fractional Differential Equation via Meshless Method*. Conference Proceedings of Science and Technology. **2(1)** (2019), 94–96.
8. Z. Jackiewicz and B. Zubik-Kowal: *Spectral collocation and waveform relaxation methods for nonlinear delay partial differential equations*. Applied Numerical Mathematics. **56(3-4)** (2006), 433–443.
9. M. Javidi: *Chebyshev spectral collocation method for computing numerical solution of telegraph equation*. Computational Methods for Differential Equations. **1(1)** (2013), 16–29.
10. M. Javidi, M. A. Asl, F. D. Saei, and Y. Mahmoudi: *Numerical solution of fractional Riesz space telegraph equation*. Computational Methods for Differential Equations. (2020) Feb 13.

11. P. M. Jordan, M. R. Meyer, and A. Puri: *Causal implications of viscous damping in compressible fluid flows*. Physical review E. **26(2)** (2000), 7918.
12. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo: *Theory and applications of fractional differential equations*. Elsevier, 2006.
13. E. C. Li, A. Chen, and J. Ye: *Numerical approaches to fractional calculus and fractional ordinary differential equation*. Journal of Computational Physics. **230(9)** (2011), 3352–3368.
14. S. Mohammadian and et al.: *Solution of fractional telegraph equation with Riesz space-fractional derivative*. AIMS MATHEMATICS. **4(6)** (2019), 1664–+.
15. S. Momani: *Analytic and approximate solutions of the space-and time-fractional telegraph equations*. Applied Mathematics and Computation. **170(2)** (2005), 1126–1134.
16. M. L. Morgado, N. J. Ford, and P. M. Lima: *Analysis and numerical methods for fractional differential equations with delay*. International Journal of computational and applied mathematics. **252** (2013), 159–168.
17. Z. Odibat and S. Momani: *A generalized differential transform method for linear partial differential equations of fractional order*. Applied Mathematics Letters. **21(2)** (2008), 194–199.
18. I. Podlubny: *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Elsevier, 1998.
19. M. S. Heris, M. Javidi, and M. A. Bashir: *Stability and convergence of difference methods for two-dimensional Riesz space fractional advection-dispersion equations with delay*. Computational and Mathematical Methods. **2(3)** (2020), e–1084.
20. S. Saghali, M. Javidi, and F. D. Saei: *Analytical solution of a fractional differential equation in the theory of viscoelastic fluids*. International Journal of Applied and Computational Mathematics. **5(3)** (2019), 53.
21. Z. Wang, X. Huang, and G. Shi: *Analysis of nonlinear dynamics and chaos in a fractional order financial system with time delay*. Computers Mathematics with Applications. **62(3)** (2011), 1531–1539.
22. H. M. Srivastava *A Survey of Some Recent Developments on Higher Transcendental Functions of Analytic Number Theory and Applied Mathematics*. Symmetry, **13(12)** (2021), 2294.
23. H. M. Srivastava *An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions.* Journal of Advanced Engineering and Computation, **5(3)**(2021), 135-166.
24. H. M. Srivastava, *Fractional-order derivatives and integrals: Introductory overview and recent developments.* Kyungpook Mathematical Journal, **60(1)**(2020), 73-116.
25. S. K. Upadhyay, *Asymptotic series of general symbol and pseudo-differential operators.* Integral Transforms and Special Functions, **14(1)**(2003), 87-95.