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CONVOLUTION CONDITIONS FOR CERTAIN SUBCLASSES OF MEROMORPHIC p-VALENT FUNCTIONS

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Abstract. In the present paper, two subclasses $\mathcal{MS}^p_{q,\lambda}(b; A, B)$ and $\mathcal{MK}^p_{q,\lambda}(b; A, B)$ of meromorphic multivalent functions by using q-derivative operator are defined in the punctured unit disc. Also, several properties including convolution properties, the necessary and sufficient condition and coefficient estimates for these subclasses are derived.
Key words: Meromorphic p-Valent functions, Hadamard product (or convolution), Subordination between analytic functions, q-derivative operator.

1. Introduction and definitions

Let Σ_p denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \qquad (p \in \mathbb{N}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $\mathbb{U}^* = \mathbb{U}\backslash\{0\}$, where $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. Let g and f be two analytic functions in $\mathbb{U}$, then function g is said to be subordinate to f if there exists an analytic function w in the unit disk $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $g(z) = f(w(z))$ $(z \in \mathbb{U})$. We denote this subordination by $g \prec f$. In particular, if the function f is univalent in $\mathbb{U}$ the above subordination is equivalent to $g(0) = f(0)$ *and* $f(\mathbb{U}) \subset g(\mathbb{U})$.

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Quantum calculus, or q-calculus is the subject of an extend investigation based on the different applications recognized for it over notable mathematical fields and in expansion to centrality to hypothetical physics. Recently, numerous authors have presented modern classes of analytic functions utilizing q-calculus. The q-calculus is an ordinary calculus without notion of limit point. The application of q-calculus was initiated by Jackson [6, 7, 8] to begin with investigated q-calculus applications, efficiently creating q-derivative and q-integral. By making use of q-calculus various functions classes in Geometric Function Theory are introduced and investigated from different view points and perspectives (see [1], [12], [17], [18], [22], [26] and references therein). Purpose of this paper is to introduce and study two subclasses of p-valent meromorphic functions by applying q-derivative operators in conjunction with the principle of subordinations.

For $0 < q < 1$, the q-derivative of a function f is defined by (see [5, 6, 7, 8])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \qquad (z \in \mathbb{U}), \tag{1.2}$$

provided that $f^{'}(0)$ exists.

From (1.2), it can be easily obtained that

$$D_q f(z) = \frac{-[p]_q}{q^p z^{p+1}} + \sum_{k=1}^{\infty} [k-p]_q a_k z^{k-p-1},$$

where

$$[k]_q = \frac{1-q^k}{1-q}.$$

As q→ 1^-, $[k]_q \to k$ and $\lim_{q \to 1^-} D_q f(z) = f^{'}(z)$. Also, we have

$$[k+p]_q = [k]_q + q^k [p]_q = q^p [k]_q + [p]_q,$$
$$[k-p]_q = q^{-p}[k]_q - q^{-p}[p]_q,$$
$$[0]_q = 0, [1]_q = 1.$$

For $f \in \Sigma_p$ given by (1.1) and $g \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \qquad (p \in \mathbb{N}),$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z).$$

Motivated essentially due to the work of Aouf [15], Seoudy et *al.* [16], Srivastava et *al.* [25], the following two subclasses of Σ_p by using the q-derivative operator D_q and the principle of subordination between analytic functions are define:

Definition 1.1. Let $0 < q < 1$, $-1 \leq B < A \leq 1$, $0 \leq \lambda < 1$ and $b \in \mathbb{C}\backslash\{0\}$. A function f belonging to Σ_p is said to be in the class $\mathcal{MS}^p_{q,\lambda}(b; A, B)$ if it satisfies

$$1 - \frac{1}{b}\left[\frac{zD_q f(z)}{(1-\frac{\lambda}{q^p})f(z) - \frac{\lambda}{[p]_q} zD_q f(z)} + \frac{[p]_q}{q^p}\right] \prec \frac{1+Az}{1+Bz}. \tag{1.3}$$

Definition 1.2. Let $0 < q < 1$, $-1 \leq B < A \leq 1$, $0 \leq \lambda < 1$ and $b \in \mathbb{C}\backslash\{0\}$. A function f belonging to Σ_p is said to be in the class $\mathcal{MK}^p_{q,\lambda}(b; A, B)$ if it satisfies

$$1 - \frac{1}{b}\left[\frac{zD_q\big(zD_q f(z)\big)}{(1-\frac{\lambda}{q^p})\big(zD_q f(z)\big) - \frac{\lambda}{[p]_q} zD_q\big(zD_q f(z)\big)} + \frac{[p]_q}{q^p}\right] \prec \frac{1+Az}{1+Bz}. \tag{1.4}$$

We also conclude from the above definitions that

$$f \in \mathcal{MK}^p_{q,\lambda}(b; A, B) \Leftrightarrow -\frac{q^p}{[p]_q} zD_q f \in \mathcal{MS}^p_{q,\lambda}(b; A, B). \tag{1.5}$$

It may be pointed out here that these classes generalizes several previously studied function classes. We deem it proper to demonstrate briefly the relevant connections with some of the well-known classes. Indeed, we have

(i) $\mathcal{MS}^p_{q,0}(b; A, B) = \mathcal{MS}^*_{p,q}(b; A, B)$ and
$\mathcal{MK}^p_{q,0}(b; A, B) = \mathcal{MK}_{p,q}(b; A, B)$ (see [9]);

(ii) $\lim_{q\to 1^-} \mathcal{MS}^1_{q,0}(b; 1, -1) = \Sigma\mathcal{S}(b)$ and
$\lim_{q\to 1^-} \mathcal{MK}^1_{q,0}(b; 1, -1) = \Sigma\mathcal{K}(b)$ (see [2]);

(iii) $\lim_{q\to 1^-} \mathcal{MS}^1_{q,0}(b; A, B) = \Sigma\mathcal{S}^*_0(b; A, B)$ and
$\lim_{q\to 1^-} \mathcal{MK}^1_{q,0}(b; A, B) = \Sigma\mathcal{K}_0(b; A, B)$ (see [3]);

(iv) $\lim_{q\to 1^-} \mathcal{MS}^1_{q,0}(b; A, B) = \Sigma\mathcal{S}^*(b; A, B)$ and
$\lim_{q\to 1^-} \mathcal{MK}^1_{q,0}(b; A, B) = \Sigma\mathcal{K}(b; A, B)$ (see [4]);

(v) $\lim_{q\to 1^-} \mathcal{MS}^1_{q,0}[(1-\alpha)e^{-\iota\mu} cos\mu; 1, -1] = \Sigma\mathcal{S}^\mu_1(\alpha)$ and
$\lim_{q\to 1^-} \mathcal{MK}^1_{q,0}[(1-\alpha)e^{-\iota\mu} cos\mu; 1, -1] = \Sigma\mathcal{K}^\mu_1(\alpha)(\mu \in \mathbb{R},\ |\mu| < \frac{\pi}{2},\ 0 \leq \alpha < 1)$ (see [14]).

In the present investigations, we derive several properties including convolution properties, the necessary and sufficient condition and coefficient estimates for functions belonging to the subclasses $\mathcal{MS}^p_{q,\lambda}(b; A, B)$ and $\mathcal{MK}^p_{q,\lambda}(b; A, B)$. The inspiration of this paper is to renovate and generalize already known results.

2. Main Results

Unless otherwise mentioned, we assume throughout this section that $0 < q < 1$, $-1 \leq B < A \leq 1$, $0 \leq \lambda < 1$, $b \in \mathbb{C}\backslash\{0\}$ and $\theta \in [0, 2\pi)$.

Theorem 2.1. *If $f \in \Sigma_p$, then $f \in \mathcal{MS}^p_{q,\lambda}(b; A, B)$ if and only if*

$$z^p\left[f(z) * \frac{1 + \left\{\left(1 - \frac{\lambda}{q^p}\right)M(\theta) - \left(q + \frac{\lambda}{q^p[p]_q}\right)\right\}z}{z^p(1-z)(1-qz)}\right] \neq 0 \qquad (z \in \mathbb{U}^*), \tag{2.1}$$

where

$$M(\theta) = \frac{e^{-\iota\theta} + B}{(A - B)bq^p}. \tag{2.2}$$

Proof. It is easy to verify that for any function $f \in \Sigma_p$

$$f(z) * \frac{1}{z^p(1-z)} = f(z) \tag{2.3}$$

and

$$f(z) * \frac{1 - \left(q + \frac{1}{[p]_q}\right)z}{z^p(1-z)(1-qz)} = -\frac{q^p}{[p]_q} z D_q f(z). \tag{2.4}$$

First, if $f \in \mathcal{MS}^p_{q,\lambda}(b; A, B)$, in order to prove that (2.1) holds we will write (1.3) by using the definition of the subordination, that is

$$-\frac{q^p}{[p]_q}\frac{zD_q f(z)}{(1 - \frac{\lambda}{q^p})f(z) - \frac{\lambda}{[p]_q} zD_q f(z)} = \frac{1 + \left[B + (A-B)b\frac{q^p}{[p]_q}\right]w(z)}{1 + Bw(z)} \quad (z \in \mathbb{U}^*),$$

where w is a Schwarz function, hence

$$\begin{aligned} z^p\Big[&- q^p\left(1 + Be^{\iota\theta}\right)zD_q f(z) \\ &- \left([p]_q + \left(B[p]_q + (A-B)bq^p\right)e^{\iota\theta}\right)\left((1 - \frac{\lambda}{q^p})f(z) - \frac{\lambda}{[p]_q} zD_q f(z)\right)\Big] \neq 0. \end{aligned} \tag{2.5}$$

Now from (2.3) and (2.4), we may write (2.5) as

$$\begin{aligned} z^p\Big[&\left(1+Be^{\iota\theta}\right)\left(f(z) * \frac{\left\{1 - \left(q + \frac{1}{[p]_q}\right)z\right\}[p]_q}{z^p(1-z)(1-qz)}\right) - \{[p]_q + (B[p]_q \\ &+ (A-B)bq^p)e^{\iota\theta}\}\Big\{(1 - \frac{\lambda}{q^p})\left(f(z) * \frac{1}{z^p(1-z)}\right) + \frac{\lambda}{q^p}\left(f(z) * \frac{\left\{1 - \left(q + \frac{1}{[p]_q}\right)z\right\}}{z^p(1-z)(1-qz)}\right)\Big\}\Big] \neq 0, \end{aligned}$$

$$(z \in \mathbb{U}^*)$$

which is equivalent to

$$z^p\Big[f(z)*\frac{1+\Big\{(1-\frac{\lambda}{q^p})\frac{1+Be^{\iota\theta}}{(A-B)bq^pe^{\iota\theta}}-(q+\frac{\lambda}{q^p[p]_q})\Big\}z}{z^p(1-z)(1-qz)}\big[-(A-B)bq^pe^{\iota\theta}\big]\Big]\neq 0$$

or

$$z^p\left[f(z)*\frac{1+\Big\{(1-\frac{\lambda}{q^p})\frac{e^{-\iota\theta}+B}{(A-B)bq^p}-(q+\frac{\lambda}{q^p[p]_q})\Big\}z}{z^p(1-z)(1-qz)}\right]\neq 0 \qquad (z\in\mathbb{U}^*),$$

which leads to (2.1), which proves the necessary part of Theorem 2.1.
Reversely, suppose that $f\in\Sigma_p$ satisfies the condition (2.1). Since it was shown in the first part of the proof that assumption (2.1) is equivalent to (2.5), we obtain that

$$(2.6)\quad -\frac{q^p}{[p]_q}\frac{zD_qf(z)}{(1-\frac{\lambda}{q^p})f(z)-\frac{\lambda}{[p]_q}zD_qf(z)}\neq\frac{1+\Big[B+(A-B)b\frac{q^p}{[p]_q}\Big]e^{\iota\theta}}{1+Be^{\iota\theta}} \qquad (z\in\mathbb{U}^*),$$

and let us assume that

$$\varphi(z)=-\frac{q^p}{[p]_q}\frac{zD_qf(z)}{(1-\frac{\lambda}{q^p})f(z)-\frac{\lambda}{[p]_q}zD_qf(z)}\ \ \textit{and}\ \ \psi(z)=\frac{1+\Big[B+(A-B)b\frac{q^p}{[p]_q}\Big]z}{1+Bz}.$$

The relation (2.6) means that

$$\varphi(\mathbb{U}^*)\cap\psi(\partial\mathbb{U}^*)=\emptyset.$$

Thus, the simply connected domain is included in a connected component of $\mathbb{C}\backslash\psi(\partial\mathbb{U}^*)$. Therefore, using the fact that $\varphi(0)=\psi(0)$ and the univalence of the function ψ, it follows that $\varphi(z)\prec\psi(z)$, which implies that $f\in\mathcal{MS}^p_{q,\lambda}(b;A,B)$. Thus, the proof of Theorem 2.1 is completed. $\square$

Theorem 2.2. *If $f\in\Sigma_p$, then $f\in\mathcal{MK}^p_{q,\lambda}(b;A,B)$ if and only if*

$$(2.7)\quad z^p\Big[f(z)*\frac{1-[\{(q+\frac{\lambda}{q^p[p]_q})-(1-\frac{\lambda}{q^p})M(\theta)\}(1-\frac{1}{[p]_q})+\frac{1+q}{[p]_q}+q^2]z}{z^p(1-z)(1-qz)(1-q^2z)}$$
$$-\frac{\{(1-\frac{\lambda}{q^p})M(\theta)-(q+\frac{\lambda}{q^p[p]_q})\}(q+\frac{1}{[p]_q})qz^2}{z^p(1-z)(1-qz)(1-q^2z)}\Big]\neq 0$$

where $z\in\mathbb{U}^*$ and $M(\theta)$ is given by (2.2).

Proof. From (1.5) it follows that $f\in\mathcal{MK}^p_{q,\lambda}(b;A,B)$ if and only if $-\frac{q^p}{[p]_q}zD_qf\in\mathcal{MS}^p_{q,\lambda}(b;A,B)$. Then from Theorem 2.1, the function $-\frac{q^p}{[p]_q}zD_qf\in\mathcal{MS}^p_{q,\lambda}(b;A,B)$ if and only if

$$(2.8)\qquad z^p\big[-\frac{q^p}{[p]_q}zD_qf*g(z)\big]\neq 0,\qquad (z\in\mathbb{U}^*),$$

where

$$g(z)=\frac{1+\left\{\left(1-\frac{\lambda}{q^p}\right)M(\theta)-\left(q+\frac{\lambda}{q^p[p]_q}\right)\right\}z}{z^p(1-z)(1-qz)}.$$

On a basic computation we note that

$$D_qg(z)=\frac{g(qz)-g(z)}{(q-1)z}$$

$$=\frac{-[p]_q+[1+q+[p]_qq^2+\{(q+\frac{\lambda}{q^p[p]_q})-(1-\frac{\lambda}{q^p})M(\theta)\}([p]_q-1)]z}{q^pz^{p+1}(1-z)(1-qz)(1-q^2z)}$$

$$+\frac{\{(1-\frac{\lambda}{q^p})M(\theta)-(q+\frac{\lambda}{q^p[p]_q})\}(q+q^2[p]_q)z^2}{q^pz^{p+1}(1-z)(1-qz)(1-q^2z)}$$

and therefore

$$-\frac{q^p}{[p]_q}zD_qg(z)$$

$$=\frac{1-\left[\left\{(q+\frac{\lambda}{q^p[p]_q})-(1-\frac{\lambda}{q^p})M(\theta)\right\}\left(1-\frac{1}{[p]_q}\right)+\frac{1+q}{[p]_q}+q^2\right]z}{z^p(1-z)(1-qz)(1-q^2z)}$$

$$-\frac{\left\{(1-\frac{\lambda}{q^p})M(\theta)-(q+\frac{\lambda}{q^p[p]_q})\right\}(q+\frac{1}{[p]_q})qz^2}{z^p(1-z)(1-qz)(1-q^2z)}.$$

Using the above relation and the identity

$$\left(-\frac{q^p}{[p]_q}zD_qf(z)\right)*g(z)=f(z)*\left(-\frac{q^p}{[p]_q}zD_qg(z)\right),$$

it is simple to check that (2.8) is identical to (2.7). Thus, the proof of Theorem 2.2 is completed. □

Theorem 2.3. *A necessary and sufficient condition for the function f defined by (1.1) to be in the class $\mathcal{MS}^p_{q,\lambda}(b;A,B)$ is that*

$$1+\sum_{k=1}^{\infty}\frac{\left(1-\frac{\lambda}{q^p}\right)\left(e^{-\iota\theta}+B\right)[k]_q+\left(1-\frac{\lambda[k]_q}{q^p[p]_q}\right)(A-B)bq^p}{(A-B)bq^p}a_kz^k\neq 0\quad (z\in\mathbb{U}^*). \tag{2.9}$$

Proof. From Theorem 2.1, we find that $f\in\mathcal{MS}^p_{q,\lambda}(b;A,B)$ if and only if (2.1) holds.
Since

$$\frac{1}{z^p(1-z)(1-qz)}=\frac{1}{z^p}+(1+q)z^{1-p}+(1+q+q^2)z^{2-p}+(1+q+q^2+q^3)z^{3-p}+\cdots,\quad (z\in\mathbb{U}^*),$$

hence

$$\frac{1+\Big\{\big(1-\frac{\lambda}{q^p}\big)M(\theta)-\big(q+\frac{\lambda}{q^p[p]_q}\big)\Big\}z}{z^p(1-z)(1-qz)} =$$

$$\frac{1}{z^p}+\sum_{k=1}^{\infty}\Big(1+\Big\{(1-\frac{\lambda}{q^p})M(\theta)-\frac{\lambda}{q^p[p]_q}\Big\}[k]_q\Big)z^{k-p},$$

where $M(\theta)$ is given by (2.2).
Now a simple computation shows that (2.1) is identical to (2.9). Thus, the proof of Theorem 2.3 is completed. □

Theorem 2.4. *A necessary and sufficient condition for the function f defined by (1.1) to be in the class $\mathcal{MK}^p_{q,\lambda}(b;A,B)$ is that*

$$(2.10)\ 1+\sum_{k=1}^{\infty}\frac{\big(1-\frac{\lambda}{q^p}\big)\big(e^{-\iota\theta}+B\big)[k]_q+\big(1-\frac{\lambda[k]_q}{q^p[p]_q}\big)(A-B)bq^p}{(A-B)bq^p}\left(1-\frac{[k]_q}{[p]_q}\right)a_kz^k\neq 0$$
$$(z\in\mathbb{U}^*).$$

Proof. From Theorem 2.2, we find that $f\in\mathcal{MK}^p_{q,\lambda}(b;A,B)$ if and only if (2.7) holds.

Since

$$\frac{1}{z^p(1-z)(1-qz)(1-q^2z)}=\frac{1}{z^p}+(1+q+q^2)z^{1-p}+(1+q+2q^2+q^3+q^4)z^{2-p}$$
$$+(1+q+2q^2+2q^3+2q^4+q^5+q^6)z^{3-p}+\cdots,$$
$$(z\in\mathbb{U}^*),$$

hence

$$\frac{1-\Big[\Big\{\big(q+\frac{\lambda}{q^p[p]_q}\big)-(1-\frac{\lambda}{q^p})M(\theta)\Big\}\big(1-\frac{1}{[p]_q}\big)+\frac{1+q}{[p]_q}+q^2\Big]z-\Big\{(1-\frac{\lambda}{q^p})M(\theta)-\big(q+\frac{\lambda}{q^p[p]_q}\big)\Big\}\big(q+\frac{1}{[p]_q}\big)qz^2}{z^p(1-z)(1-qz)(1-q^2z)}$$
$$=\frac{1}{z^p}+\sum_{k=1}^{\infty}\Big(1+\Big\{(1-\frac{\lambda}{q^p})M(\theta)-\frac{\lambda}{q^p[p]_q}\Big\}[k]_q\Big)\left(1-\frac{[k]_q}{[p]_q}\right)z^{k-p},$$
$$(z\in\mathbb{U}^*)$$

where $M(\theta)$ is given by (2.2).
Now a simple computation shows that (2.7) is identical to (2.10). Thus, the proof of Theorem 2.4 is completed. □

Theorem 2.5. *If $f\in\Sigma_p$ satisfies the inequality*

$$(2.11)\qquad \sum_{k=1}^{\infty}\Big[\big|1-\frac{\lambda}{q^p}\big|[k]_q\big(1+|B|\big)+\big|b\big(1-\frac{\lambda[k]_q}{q^p[p]_q}\big)\big|(A-B)q^p\Big]|a_k|<(A-B)|b|q^p$$

then $f\in\mathcal{MS}^p_{q,\lambda}(b;A,B)$.

Proof. Since

$$\left|1+\sum_{k=1}^{\infty}\frac{\left(1-\frac{\lambda}{q^p}\right)\left(e^{-\iota\theta}+B\right)[k]_q+\left(1-\frac{\lambda[k]_q}{q^p[p]_q}\right)(A-B)bq^p}{(A-B)bq^p}a_kz^k\right|$$

$$\geq 1-\left|\sum_{k=1}^{\infty}\frac{\left(1-\frac{\lambda}{q^p}\right)\left(e^{-\iota\theta}+B\right)[k]_q+\left(1-\frac{\lambda[k]_q}{q^p[p]_q}\right)(A-B)bq^p}{(A-B)bq^p}a_kz^k\right|$$

$$\geq 1-\sum_{k=1}^{\infty}\frac{\left|1-\frac{\lambda}{q^p}\right|\left(1+|B|\right)[k]_q+\left|b\left(1-\frac{\lambda[k]_q}{q^p[p]_q}\right)\right|(A-B)q^p}{(A-B)|b|q^p}|a_k|>0.$$

Thus, the inequality (2.11) holds and our result follows from Theorem 2.3. □

Using similar arguments to those in the proof of Theorem 2.5, we may also prove the next result.

Theorem 2.6. *If $f\in\Sigma_p$ satisfies the inequality*

$$\sum_{k=1}^{\infty}\left[\left|1-\frac{\lambda}{q^p}\right|\left(1+|B|\right)[k]_q+\left|b\left(1-\frac{\lambda[k]_q}{q^p[p]_q}\right)\right|(A-B)q^p\right]\left(1-\frac{[k]_q}{[p]_q}\right)|a_k|<(A-B)|b|q^p \tag{2.12}$$

then $f\in\mathcal{MK}^p_{q,\lambda}(b;A,B)$.

Remarks: Note that the results obtained in the present paper provide us a lot of interesting particular cases by assigning different values to the involved parameters, some illustration are given here:
(i) Taking $p=1$, $q\to 1^-$, $\lambda=0$, $b=1$ and $e^{\iota\theta}=x$ in Theorem 2.1 and 2.2 we get the results of Ponnusamy [13].
(ii) Taking $p=1$, $q\to 1^-$, $\lambda=0$, $b=(1-\alpha)e^{-\iota\mu}cos\mu$ ($\mu\in\mathbb{R}$, $|\mu|<\frac{\pi}{2}$, $0\leq\alpha<1$), A=1, B=-1 and $e^{\iota\theta}=x$ in Theorem 2.1 we get the result of Ravichandran et *al.* [14].
(iii) Taking $p=1$, $q\to 1^-$ and $\lambda=0$ in Theorem 2.1 and 2.2, our results matches with Aouf [3] and Bulboacă et *al.* [4].
(iv) Taking $p=1$ in Theorem 2.1 and 2.2 our results matches with Mostafa et *al.* [11].
(v) Taking $\lambda=0$ in Theorem 2.1 to 2.6 our results matches with Kant et *al.* [9].

3. Conclusions

By the use of Q-calculus, we have introduced two subclasses $\mathcal{MS}^p_{q,\lambda}(b;A,B)$ and $\mathcal{MK}^p_{q,\lambda}(b;A,B)$ of meromorphic multivalent functions by using q-derivative operator linked to a punctured unit disc. We learned about some key issues, such as convolution properties, the necessary and sufficient condition and coefficient estimates for the newly defined subclasses. We also pointed out several important correlations between our findings and those which were considered in previous studies. Recently, some published articles that deal with q-derivative operator have also attracted researchers. In these articles authors find interesting results by using the principles of q-derivative operator for meromorphic harmonic functions, partial sums of meromorphically starlike functions, normalized holomorphic and bi-univalent functions in the open unit disk. See ([10],[21],[23],[24]).

As pointed out in the survey-cum-expository review paper by Srivastava ([18], p. 340), any attempt to produce the so-called (p, q)-variation of the q-results, which we have presented in this paper, will be trivial and inconsequential because the additional parameter p is obviously redundant or superfluous. Also see ([19], p. 1511-1512, [20], p. 18).

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