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ON MULTIPLICATIVE LIE n-HIGHER DERIVATIONS OF TRIANGULAR ALGEBRAS

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Abstract. Let R be a commutative ring with unity, A, B be R-algebras and M be an (A, B)-bimodule. Let $\mathfrak{T} = Tri(\mathrm{A}, \mathrm{M}, \mathrm{B})$ be a $(n-1)$-torsion free triangular algebra. In this article, we prove that every multiplicative Lie n-higher derivation on triangular algebras has the standard form. Also, the main result is applied to some classical examples of triangular algebras such as upper triangular matrix algebras and nest algebras.
Keywords: Triangular algebras, Lie type derivation.

1. Brief Historical Development

Many authors studied Lie type derivations on several rings and algebras [6, 7, 10, 12, 14–17, 19, 25]. In most of the cases, authors found that any Lie type derivation has the standard from on that particular ring or algebra under consideration. The first characterization of Lie derivations was obtained by Martindale [17] in 1964 who proved that every Lie derivation on a primitive ring can be written as a sum of derivations and an additive mapping of a ring to its center that maps commutators into zero, i.e, Lie derivation has the standard form.

Moreover, during last few decades, the multiplicative mappings on rings and algebras have been studied by many authors. Martindale [18] established a condition

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on a ring such that multiplicative bijective mappings on this ring are all additive. In particular, he proved that every multiplicative bijective mapping from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive. Daif [8] studied the additivity of derivable map on a 2-torsion free prime ring containing a nontrivial idempotent. In the year 1978, Miers [19] studied Lie triple derivations of von Neumann algebras and proved that if M is a von Neumann algebra with no central abelian summands then there exists an operator $A \in M$ such that $L(X) = [A, X] + \lambda(X)$ where $\lambda : M \to \mathrm{Z}(M)$ is a linear map which annihilates brackets of operators in M. In [7] Cheung initiated the study of Lie derivations of triangular algebras $\mathfrak{T}$ and gave a sufficient condition under which every Lie derivation on $\mathfrak{T}$ is a sum of derivations on $\mathfrak{T}$ and a mapping from $\mathfrak{T}$ to its center $\mathrm{Z}(\mathfrak{T})$. Further, Lie derivations on triangular algebras were studied in [15, 25], whereas the study of Lie triple derivations on triangular algebras can be found in [14, 16]. Yu and Zhang [25] proved that every nonlinear Lie derivation of triangular algebras is the sum of an additive derivation and a map from triangular algebra into its center sending commutators to zero. Ji et al. [14] proved the similar result for nonlinear Lie triple derivation of triangular algebras.

Benkovič and Eremita [6] discussed multiplicative Lie n-derivations of triangular rings, which in fact, generalized some results on nonlinear Lie (triple) derivations of triangular algebras (see [14, 25]).

Several authors have made important contributions to the related topics see for reference [5, 11, 13, 14, 16, 20, 23–25] where further references can be found. Xiao and Wei [24] considered the case of nonlinear Lie higher derivation on a triangular algebra and they proved that if $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ is a nonlinear Lie higher derivation on a triangular algebra, then $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ is of the standard form, i.e., $\mathrm{L}_r = d_r + \gamma_r$, where $\{d_r\}_{r\in\mathbb{N}}$ is an additive higher derivation and $\{\gamma_r\}_{r\in\mathbb{N}}$ is a functional vanishing on all commutators of triangular algebra. However, much less attention to the study of Lie n-higher derivations on operator algebras has been paid. To the best of our knowledge, there are very few articles dealing with Lie n-higher derivations on rings and algebras except for [9, 11]. The objective of this article is to describe the structure of multiplicative Lie n-higher derivations on triangular algebras.

2. Basic Definitions & Preliminaries

Let R be a commutative ring with unity and $\mathrm{Z}(\mathcal{A})$ be the center of an R-algebra $\mathcal{A}$. A map $\mathrm{L} : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) is called a multiplicative derivation (resp. multiplicative Lie derivation) on $\mathcal{A}$ if $\mathrm{L}(ab) = \mathrm{L}(a)b + a\mathrm{L}(b)$ (resp. $\mathrm{L}([a, b]) = [\mathrm{L}(a), b] + [a, \mathrm{L}(b)]$) holds for all $a, b \in \mathcal{A}$. In addition, if L is linear on $\mathcal{A}$, then L is said to be a derivation (resp. Lie derivation) on $\mathcal{A}$.

To explore a more approximate kind of maps. Define a sequence of polynomials:

$$\begin{aligned}
\mathfrak{p}_1(x_1) &= x_1, \\
\mathfrak{p}_2(x_1, x_2) &= [\mathfrak{p}_1(x_1), x_2] = [x_1, x_2], \\
&\vdots \\
\mathfrak{p}_n(x_1, x_2, \cdots, x_n) &= [\mathfrak{p}_{n-1}(x_1, x_2, \cdots, x_{n-1}), x_n].
\end{aligned}$$

The polynomial $\mathfrak{p}_n(x_1, x_2, \cdots, x_n)$ is called $(n-1)$-th commutator where $n \geq 2$. A map (not necessarily linear) $\mathrm{L} : \mathcal{A} \to \mathcal{A}$ is said to be a multiplicative Lie n-derivation on $\mathcal{A}$ if

$$\mathrm{L}(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) = \sum_{i=1}^{n} \mathfrak{p}_n(x_1, x_2, \cdots, x_{i-1}, \mathrm{L}(x_i), x_{i+1}, \cdots, x_n)$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{A}$. The concept of Lie n-derivation was first introduced by Abdullaev [1] on certain von Neumann algebras. Note that any multiplicative Lie 2-derivation is known as multiplicative Lie derivation and multiplicative Lie 3-derivation is said to be multiplicative Lie triple derivation. Thus multiplicative Lie derivation, multiplicative Lie triple derivation and multiplicative Lie n-derivation collectively known as multiplicative Lie type derivations on $\mathcal{A}$.

Apart from these, the concept of derivation were extended to higher derivation. Let us recall the basic facts about higher derivations. Let $\mathbb{N}$ be the set of nonnegative integers and $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ be a family of maps $\mathrm{L}_r : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) such that $\mathrm{L}_0 = I_{\mathcal{A}}$. Then $\mathfrak{L}$ is called

1. a multiplicative higher derivation if $\mathrm{L}_r(x_1x_2) = \sum\limits_{i_1+i_2=r} \mathrm{L}_{i_1}(x_1)\mathrm{L}_{i_2}(x_2)$,

2. a multiplicative Lie n-higher derivation if

$$\mathrm{L}_r(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) = \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\mathrm{L}_{i_1}(x_1), \mathrm{L}_{i_2}(x_2), \cdots, \mathrm{L}_{i_n}(x_n))$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{A}$ and for each $r \in \mathbb{N}$. Note that any multiplicative Lie 2-higher derivation is multiplicative Lie higher derivation and multiplicative Lie 3-higher derivation is multiplicative Lie triple higher derivation. Thus multiplicative Lie higher derivation, multiplicative Lie triple higher derivation and multiplicative Lie n-higher derivation collectively known as multiplicative Lie type higher derivations on $\mathcal{A}$. It is easy to observe that every higher derivation is a Lie higher derivation and every Lie higher derivation is a Lie triple higher derivation and so on but the converse need not be true in general.

Note that if $\mathfrak{D} = \{d_r\}_{r\in\mathbb{N}}$ is a higher derivation on $\mathcal{A}$ and for each $r \in \mathbb{N}$, $\mathrm{L}_r = d_r + f_r$ where $f_r : \mathcal{A} \to \mathrm{Z}(\mathcal{A})$ is a linear (resp. nonlinear) mapping, then it is easy to see that $\{\mathrm{L}_r\}_{r\in\mathbb{N}}$ is a Lie n-higher derivation (resp. nonlinear Lie n-higher derivation) if and only if $f_r(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) = 0$ for all $x_1, x_2, \cdots, x_n \in \mathcal{A}$. Lie

n-higher derivation (resp. nonlinear Lie n-higher derivation) of the above kind are called *standard*. The natural problem that one considers in this context is whether or not every Lie n-higher derivation (resp. nonlinear Lie n-higher derivation) is standard.

Throughout this paper, R will always denote a commutative ring with unity element. Let A and B be unital algebras over R and let M be a unital (A, B)-bimodule (i.e., $1_A \cdot m = m$ and $m \cdot 1_B = m$ for all $m \in M$.) which is faithful as a left A-module and also as a right B-module. The R-algebra

$$\mathfrak{T} = Tri(A, M, B) = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \middle| a \in A, m \in M, b \in B \right\}$$

under the usual matrix operations is called triangular algebra. The center of $\mathfrak{T}$ is

$$Z(\mathfrak{T}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| am = mb \ \forall \ m \in M \right\}.$$

Define two natural projections $\pi_A : \mathfrak{T} \to A$ and $\pi_B : \mathfrak{T} \to B$ by

$$\pi_A\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = a \text{ and } \pi_B\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = b.$$

Moreover, $\pi_A(Z(\mathfrak{T})) \subseteq Z(A)$ and $\pi_B(Z(\mathfrak{T})) \subseteq Z(B)$ and there exists a unique algebraic isomorphism $\tau : \pi_A(Z(\mathfrak{T})) \to \pi_B(Z(\mathfrak{T}))$ such that $am = m\tau(a)$ for all $a \in \pi_A(Z(\mathfrak{T})), m \in M$.

Let 1_A (resp.1_B) be the identity of the algebra A (resp. B) and let I be the unity of triangular algebra $\mathfrak{T}$. Throughout, this paper we shall use the following notions: $p = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}$, $q = I - p = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}$ and $A \cong p\mathfrak{T}p$, $M \cong p\mathfrak{T}q$, $B \cong q\mathfrak{T}q$. Thus, $\mathfrak{T} = p\mathfrak{T}p + p\mathfrak{T}q + q\mathfrak{T}q \cong A + M + B$. Also, $\pi_A(Z(\mathfrak{T}))$ and $\pi_B(Z(\mathfrak{T}))$ are isomorphic to $pZ(\mathfrak{T})p$ and $qZ(\mathfrak{T})q$ respectively. Then there is an algebra isomorphisms $\tau : pZ(\mathfrak{T})p \to qZ(\mathfrak{T})q$ such that $am = m\tau(a)$ for all $m \in p\mathfrak{T}q$.

Let us describe the result which is used subsequently in this article as :

Lemma 2.1. *[6, Theorem 5.9] Let* $\mathfrak{T} = Tri(A, M, B)$ *be a* $(n-1)$*-torsion free triangular ring. Suppose that* $\mathfrak{T}$ *satisfies the following conditions:*

1. $\pi_A(Z(\mathfrak{T})) = Z(A)$ *and* $\pi_B(Z(\mathfrak{T})) = Z(B)$,
2. $Z(A) = \{a \in A \mid [[a, x], y] = 0 \ \forall \ x, y \in A\}$
   or $Z(B) = \{b \in B \mid [[b, x], y] = 0 \ \forall \ x, y \in B\}$.

Then any multiplicative Lie n*-derivation* $L : \mathfrak{T} \to \mathfrak{T}$ *has the standard form.*

3. Multiplicative Lie n-higher derivation

In this section, we will prove the main result by a series of lemmas. It is clear that every Lie higher derivation is a Lie n-higher derivation for $n \geq 3$. Therefore, without loss of generality we assume $n \geq 3$ for convenience and for $n = 2$ we can look into [24].

Theorem 3.1. *Let $\mathfrak{T} = Tri(\mathrm{A}, \mathrm{M}, \mathrm{B})$ be a $(n-1)$-torsion free triangular algebra consisting of unital algebras A, B and a faithful unital (A, B)-bimodule M. Suppose that $\mathfrak{T}$ satisfies the following conditions:*

($\star$) $\pi_{\mathrm{A}}(\mathrm{Z}(\mathfrak{T})) = \mathrm{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathrm{Z}(\mathfrak{T})) = \mathrm{Z}(\mathrm{B})$,

($\sharp$) $\mathrm{Z}(\mathrm{A}) = \{a \in \mathrm{A} \mid [[a, x], y] = 0 \ \forall \ x, y \in \mathrm{A}\}$
or $\mathrm{Z}(\mathrm{B}) = \{b \in \mathrm{B} \mid [[b, x], y] = 0 \ \forall \ x, y \in \mathrm{B}\}$.

Then every multiplicative Lie n-higher derivation $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ on $\mathfrak{T}$ has the standard form. More precisely, there exists an additive higher derivation $\mathfrak{D} = \{d_r\}_{r\in\mathbb{N}}$ on $\mathfrak{T}$ and a sequence of functionals $\{h_r\}_{r\in\mathbb{N}}$ which annihilates all Lie n-product $\mathfrak{p}_n(x_1, x_2, \cdots, x_n)$ for all $x_1, x_2, \cdots, x_n \in \mathfrak{T}$ such that $\mathrm{L}_r(x) = d_r(x) + h_r(x)$ for all $x \in \mathfrak{T}$ and $r \in \mathbb{N}$.

In order to prove our main theorem, we apply an induction method for the component index r. For $r = 1$, L_1 is multiplicative Lie n-derivation on $\mathfrak{T}$. Hence by Lemma 2.1 it follows that there exists an additive derivation d_1 and a functional h_1 satisfying $h_1(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) = 0$ for all $x_1, x_2, \cdots, x_n \in \mathfrak{T}$ such that $\mathrm{L}_1(x) = d_1(x) + h_1(x)$ for all $x \in \mathfrak{T}$. Moreover, L_1 and d_1 satisfy the following properties:

$$\mathbf{C}_1 : \begin{cases} \mathrm{L}_1(0) = 0, & \mathrm{L}_1(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M} + \mathrm{Z}(\mathfrak{T}), \\ \mathrm{L}_1(\mathrm{M}) \subseteq \mathrm{M}, & \mathrm{L}_1(\mathrm{B}) \subseteq \mathrm{B} + \mathrm{M} + \mathrm{Z}(\mathfrak{T}), \\ \mathrm{L}_1(p) \in \mathrm{M} + \mathrm{Z}(\mathfrak{T}), & \mathrm{L}_1(q) \in \mathrm{M} + \mathrm{Z}(\mathfrak{T}), \\ d_1(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M}, & d_1(\mathrm{B}) \subseteq \mathrm{M} + \mathrm{B}, \\ d_1(\mathrm{M}) \subseteq \mathrm{M}, & d_1(p), d_1(q) \in \mathrm{M}. \end{cases}$$

We assume that the result holds for all $1 < s < r, r \in \mathbb{N}$. Then there exists an additive mapping d_s and a functional h_s satisfying $h_s(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) = 0$ for all $x_1, x_2, \cdots, x_n \in \mathfrak{T}$ such that $\mathrm{L}_s(x) = d_s(x) + h_s(x)$ for all $x \in \mathfrak{T}$. Thus the mapping L_s and d_s satisfy the following properties:

$$\mathbf{C}_s : \begin{cases} \mathrm{L}_s(0) = 0, & \mathrm{L}_s(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M} + \mathrm{Z}(\mathfrak{T}), \\ \mathrm{L}_s(\mathrm{M}) \subseteq \mathrm{M}, & \mathrm{L}_s(\mathrm{B}) \subseteq \mathrm{B} + \mathrm{M} + \mathrm{Z}(\mathfrak{T}), \\ \mathrm{L}_s(p) \in \mathrm{M} + \mathrm{Z}(\mathfrak{T}), & \mathrm{L}_s(q) \in \mathrm{M} + \mathrm{Z}(\mathfrak{T}), \\ d_s(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M}, & d_s(\mathrm{B}) \subseteq \mathrm{M} + \mathrm{B}, \\ d_s(\mathrm{M}) \subseteq \mathrm{M}, & d_s(p), d_s(q) \in \mathrm{M}. \end{cases}$$

Our aim is to show that above conditions also hold for r, it follows from the series of Lemmas:

Lemma 3.1. *Let $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ be a multiplicative Lie n-higher derivation on $(n-1)$-torsion free triangular algebra $\mathfrak{T}$. Then $\mathrm{L}_r(0) = 0$, and $\mathrm{L}_r(\mathrm{M}) \subseteq \mathrm{M}$ for each $r \in \mathbb{N}$.*

Proof. For each $r \in \mathbb{N}$, $\mathrm{L}_r(0) = 0$ is trivially true. For any $m \in \mathrm{M}$ using conditions C_s, we have

$$\begin{aligned}
\mathrm{L}_r(m) &= \mathrm{L}_r(\mathfrak{p}_n(m, q, \cdots, q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\mathrm{L}_{i_1}(m), \mathrm{L}_{i_2}(q), \cdots, \mathrm{L}_{i_n}(q)) \\
&= \mathfrak{p}_n(\mathrm{L}_r(m), q, \cdots, q) + \mathfrak{p}_n(m, \mathrm{L}_r(q), \cdots, q) + \cdots + \mathfrak{p}_n(m, q, \cdots, \mathrm{L}_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(r), \mathrm{L}_{i_2}(q), \cdots, \mathrm{L}_{i_n}(q)) \\
&= \mathfrak{p}_n(\mathrm{L}_r(m), q, \cdots, q) + \mathfrak{p}_n(m, \mathrm{L}_r(q), \cdots, q) + \cdots + \mathfrak{p}_n(m, q, \cdots, \mathrm{L}_r(q)) \\
&= p\mathrm{L}_r(m)q + (n-1)[m, \mathrm{L}_r(q)].
\end{aligned}$$

On multiplying the above equality from left by p and right by q, we get $(n-1)[\mathrm{M}, \mathrm{L}_r(q)] = 0$ and hence $\mathrm{L}_r(m) = p\mathrm{L}_r(m)q$. This implies that $\mathrm{L}_r(\mathrm{M}) \subseteq \mathrm{M}$. □

Lemma 3.2. *Let $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ be a multiplicative Lie n-higher derivation on $(n-1)$-torsion free triangular algebra $\mathfrak{T}$. Then $\mathrm{L}_r(p), \mathrm{L}_r(q) \in \mathrm{Z}(\mathfrak{T}) + \mathrm{M}$ for each $r \in \mathbb{N}$.*

Proof. From the proof of Lemma 3.1, we have seen that $(n-1)[\mathrm{M}, \mathrm{L}_r(q)] = 0$. Since $\mathfrak{T}$ is $(n-1)$-torsion free, we have $[\mathrm{M}, \mathrm{L}_r(q)] = 0$ and hence $p\mathrm{L}_r(q)p + q\mathrm{L}_r(q)q \in \mathrm{Z}(\mathfrak{T})$. Therefore, we have $\mathrm{L}_r(q) \in \mathrm{Z}(\mathfrak{T}) + \mathrm{M}$. Also, for any arbitrary $m \in \mathrm{M}$, we obtain that

$$\begin{aligned}
\mathrm{L}_r(m) &= \mathrm{L}_r(\mathfrak{p}_n(p, m, q, \cdots, q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\mathrm{L}_{i_1}(p), \mathrm{L}_{i_2}(m), \mathrm{L}_{i_3}(q), \cdots, \mathrm{L}_{i_n}(q)) \\
&= \mathfrak{p}_n(\mathrm{L}_r(p), m, q, \cdots, q) + \mathfrak{p}_n(p, \mathrm{L}_r(m), q, \cdots, q) \\
&\quad + \cdots + \mathfrak{p}_n(p, m, q, \cdots, \mathrm{L}_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(p), \mathrm{L}_{i_2}(m), \mathrm{L}_{i_3}(q), \cdots, \mathrm{L}_{i_n}(q)) \\
&= \mathfrak{p}_{n-1}([\mathrm{L}_r(p), m], q, \cdots, q) + \mathfrak{p}_{n-1}([p, \mathrm{L}_r(m)], q, \cdots, q) \\
&= p[\mathrm{L}_r(p), m]q + p[p, \mathrm{L}_r(m)]q.
\end{aligned}$$

Therefore, we get

$$\mathrm{L}_r(m) = p[\mathrm{L}_r(p), m]q + p\mathrm{L}_r(m)q \text{ for all } m \in \mathrm{M}. \tag{3.1}$$

Hence, $p\mathrm{L}_r(m)q = p[\mathrm{L}_r(p), m]q + p\mathrm{L}_r(m)q$, which implies that $[\mathrm{L}_r(p), \mathrm{M}] = 0$. Then $\mathrm{L}_r(p) \in \mathrm{Z}(\mathfrak{T}) + \mathrm{M}$. □

Lemma 3.3. *Let $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ be a multiplicative Lie n-higher derivation on $(n-1)$-torsion free triangular algebra $\mathfrak{T}$. Then for any $a \in \mathrm{A}, b \in \mathrm{B}$ and $m \in \mathrm{M}$, the following hold true:*

1. $p\mathrm{L}_r(b)p \in \mathrm{Z}(\mathrm{A})$ *and* $q\mathrm{L}_r(a)q \in \mathrm{Z}(\mathrm{B})$,

2. $\mathrm{L}_r(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M} + \mathrm{Z}(\mathfrak{T})$ *and* $\mathrm{L}_r(\mathrm{B}) \subseteq \mathrm{B} + \mathrm{M} + \mathrm{Z}(\mathfrak{T})$

for each $r \in \mathbb{N}$.

Proof. Let $a \in \mathrm{A}, b \in \mathrm{B}, m \in \mathrm{M}$. Using the condition $\mathbf{C}_s$ and the fact that $[a, b] = 0$, we have

$$\begin{aligned}
0 &= \mathrm{L}_r(\mathfrak{p}_n(a, b, m, q, \cdots, q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\mathrm{L}_{i_1}(a), \mathrm{L}_{i_2}(b), \mathrm{L}_{i_3}(m), \mathrm{L}_{i_4}(q), \cdots, \mathrm{L}_{i_n}(q)) \\
&= \mathfrak{p}_n(\mathrm{L}_r(a), b, m, q, \cdots, q) + \mathfrak{p}_n(a, \mathrm{L}_r(b), m, q, \cdots, q) + \mathfrak{p}_n(a, b, \mathrm{L}_r(m), \cdots, q) \\
&\quad + \mathfrak{p}_n(a, b, m, \mathrm{L}_r(q), q, \cdots, q) + \cdots + \mathfrak{p}_n(a, b, m, q, \cdots, \mathrm{L}_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\leq i_1, i_2, \cdots, i_n < r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(a), \mathrm{L}_{i_2}(b), \mathrm{L}_{i_3}(m), \mathrm{L}_{i_4}(q), \cdots, \mathrm{L}_{i_n}(q)) \\
&= \mathfrak{p}_{n-2}([[\mathrm{L}_r(a), b], m], q, \cdots, q) + \mathfrak{p}_{n-2}([[a, \mathrm{L}_r(b)], m], q, \cdots, q) \\
&= [[\mathrm{L}_r(a), b], m] + [[a, \mathrm{L}_r(b)], m].
\end{aligned}$$

Hence, $[q\mathrm{L}_r(a)q, b] + [a, p\mathrm{L}_r(b)p] \in \mathrm{Z}(\mathfrak{T})$. Now multiplying from right as well as left side by p and q respectively and on applying the assumptions $(\star)$ and $(\sharp)$, we get

$$p\mathrm{L}_r(b)p \in \mathrm{Z}(\mathrm{A}) \text{ and } q\mathrm{L}_r(a)q \in \mathrm{Z}(\mathrm{B}).$$

Then we obtain

$$\mathrm{L}_r(a) = (p\mathrm{L}_r(a)p - \tau^{-1}(q\mathrm{L}_r(a)q)) + p\mathrm{L}_r(a)q + (\tau^{-1}(q\mathrm{L}_r(a)q) + q\mathrm{L}_r(a)q)$$

and

$$\mathrm{L}_r(b) = (p\mathrm{L}_r(b)p + \tau(p\mathrm{L}_r(b)p)) + p\mathrm{L}_r(b)q + (q\mathrm{L}_r(b)q - \tau(p\mathrm{L}_r(b)p)$$

which gives $\mathrm{L}_r(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M} + \mathrm{Z}(\mathfrak{T})$ and $\mathrm{L}_r(\mathrm{B}) \subseteq \mathrm{B} + \mathrm{M} + \mathrm{Z}(\mathfrak{T})$. □

Remark 3.1. We define $f_{r_1}(a) = q\mathrm{L}_r(a)q$ and $f_{r_2}(b) = p\mathrm{L}_r(b)p$ for any $a \in \mathrm{A},\ b \in \mathrm{B}$. By Lemma 3.3 follows that $f_{r_1} : \mathrm{A} \to q\mathrm{Z}(\mathfrak{T})q$ is a mapping such that $f_{r_1}(\mathfrak{p}_n(\mathrm{A}, \mathrm{A}, \cdots, \mathrm{A})) = 0$ and $f_{r_2} : \mathrm{B} \to p\mathrm{Z}(\mathfrak{T})p$ is a mapping such that $f_{r_2}(\mathfrak{p}_n(\mathrm{B}, \mathrm{B}, \cdots, \mathrm{B})) = 0$. Define the maps $\delta_r : \mathfrak{T} \to \mathfrak{T}$ and $f_r : \mathfrak{T} \to \mathrm{Z}(\mathfrak{T})$ by $\delta_r = \mathrm{L}_r - f_r$ and

$$f_r(x) = f_{r_1}(pxp) + \tau^{-1}(f_{r_1}(pxp)) + f_{r_2}(qxq) + \tau(f_{r_2}(qxq)) \text{ for all } x \in \mathfrak{T}.$$

Obviously, $f_r(\mathrm{M}) = 0$. Hence $\delta_r(\mathrm{M}) = \mathrm{L}_r(\mathrm{M})$. We claim that $f_r(\mathfrak{p}_n(\mathfrak{T}, \mathfrak{T}, \cdots, \mathfrak{T})) = 0$.

Assume $x_1, x_2, \cdots, x_n \in \mathfrak{T}$. Since $f_r(x) = f_r(pxp + qxq)$ for each $x \in \mathfrak{T}$, we find that

$$\begin{aligned} f_r(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) &= f_r(p(\mathfrak{p}_n(x_1, x_2, \cdots, x_n))p + q(\mathfrak{p}_n(x_1, x_2, \cdots, x_n))q) \\ &= q\mathrm{L}_r(p(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)p)q \\ &\quad +\tau^{-1}(q\mathrm{L}_r(p(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)p)q) \\ &\quad +p\mathrm{L}_r(q(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)q)p \\ &\quad +\tau(p\mathrm{L}_r(q(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)q)p) \\ &= q\mathrm{L}_r(\mathfrak{p}_n(px_1p, px_2p, \cdots, px_np))q \\ &\quad +\tau^{-1}(q\mathrm{L}_r(\mathfrak{p}_n(px_1p, px_2p, \cdots, px_np)q) \\ &\quad +p\mathrm{L}_r(\mathfrak{p}_n(qx_1q, qx_2q, \cdots, qx_nq)p \\ &\quad +\tau(p\mathrm{L}_r(\mathfrak{p}_n(qx_1q, qx_2q, \cdots, qx_nq)p). \end{aligned}$$

Since

$$\begin{aligned} & p\mathrm{L}_r(\mathfrak{p}_n(qx_1q, qx_2q, \cdots, qx_nq)p \\ &= p(\mathfrak{p}_n(\mathrm{L}_r(qx_1q), qx_2q, \cdots, qx_nq))p \\ &\quad +p(\mathfrak{p}_n(qx_1q, \mathrm{L}_r(qx_2q), \cdots, qx_nq))p \\ &\quad +p(\mathfrak{p}_n(qx_1q, qx_2q, \cdots, \mathrm{L}_r(qx_nq)))p \\ &\quad +p\left(\sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(qx_1q), \mathrm{L}_{i_2}(qx_2q), \cdots, \mathrm{L}_{i_n}(qx_nq))\right)p \\ &= 0. \end{aligned}$$

Similarly, $q\mathrm{L}_r(\mathfrak{p}_n(px_1p, px_2p, \cdots, px_np))q = 0$, and hence $f_r(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) = 0$ for all $x_1, x_2, \cdots, x_n \in \mathfrak{T}$. Consequently,

$$\begin{aligned} & \delta_r(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) \\ &= \mathrm{L}_r(\mathfrak{p}_n(x_1, x_2, \cdots, x_n)) \\ &= \mathfrak{p}_n(\mathrm{L}_r(x_1), x_2, \cdots, x_n) + \mathfrak{p}_n(x_1, \mathrm{L}_r(x_2), \cdots, x_n) + \cdots + \mathfrak{p}_n(x_1, x_2, \cdots, \mathrm{L}_r(x_n)) \\ &\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(x_1), \mathrm{L}_{i_2}(x_2), \cdots, \mathrm{L}_{i_n}(x_n)) \\ &= \mathfrak{p}_n(\mathrm{L}_r(x_1) - f_r(x_1), x_2, \cdots, x_n) + \mathfrak{p}_n(x_1, \mathrm{L}_r(x_2) - f_r(x_2), \cdots, x_n) \\ &\quad + \cdots + \mathfrak{p}_n(x_1, x_2, \cdots, \mathrm{L}_r(x_n) - f_r(x_n)) \\ &\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(x_1) - d_{i_1}(x_1), \mathrm{L}_{i_2}(x_2) - d_{i_2}(x_2), \cdots, \mathrm{L}_{i_n}(x_n) - d_{i_n}(x_n)) \\ &= \mathfrak{p}_n(\delta_r(x_1), x_2, \cdots, x_n) + \mathfrak{p}_n(x_1, \delta_r(x_2), \cdots, x_n) + \mathfrak{p}_n(x_1, x_2, \cdots, \delta_r(x_n)) \\ &\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(d_{i_1}(x_1), d_{i_2}(x_2), \cdots, d_{i_n}(x_n)) \end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in \mathfrak{T}$. Thus $\{\delta_r\}_{r\in\mathbb{N}}$ is a multiplicative Lie n-higher derivation on $\mathfrak{T}$.

Since $\mathfrak{p}_n(p, x, q, \cdots, q) = \mathfrak{p}_n(x, q, q, \cdots, q)$ for all $x \in \mathfrak{T}$, we find that

$$\begin{aligned} &\mathfrak{p}_n(\mathrm{L}_r(p), x, q, \cdots, q) + \mathfrak{p}_n(p, \mathrm{L}_r(x), q, \cdots, q) + \mathfrak{p}_n(p, x, q, \cdots, \mathrm{L}_r(q)) \\ &\qquad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1,i_2,\cdots,i_n<r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(p), \mathrm{L}_{i_2}(x), q, \cdots, \mathrm{L}_{i_n}(q)) \\ &= \mathfrak{p}_n(\mathrm{L}_r(x), q, \cdots, q) + \mathfrak{p}_n(x, \mathrm{L}_r(q), \cdots, q) + \mathfrak{p}_n(x, q, \cdots, \mathrm{L}_r(q)) \\ &\qquad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1,i_2,\cdots,i_n<r}} \mathfrak{p}_n(\mathrm{L}_{i_1}(x), \mathrm{L}_{i_2}(q), \cdots, \mathrm{L}_{i_n}(q)). \end{aligned}$$

Considering the induction hypothesis, the above equation becomes

$$[\delta_r(p), x] + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} [d_{i_1}(p), d_{i_2}(x)] = [x, \delta_r(q)] + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} [d_{i_1}(x), d_{i_2}(q)].$$

Note that d_i is additive and $d_i(I) = 0$ for all $0 < i < r$. Thus we arrive $[\delta_r(p), x] = [x, \delta_r(q)]$. That is $\delta_r(p) + \delta_r(q) \in \mathrm{Z}(\mathfrak{T})$. On the other hand, $\delta_r(p) = \mathrm{L}_r(p) - f_r(p) \in \mathrm{M}$ by Lemma 3.2 and $\delta_r(q) \in \mathrm{M}$. By the characterization of the centre of $\mathfrak{T}$, we can calculate that $\delta_r(p) + \delta_r(q) = 0$.

Now from Lemma 3.1 and Lemma 3.2, it is clear that

Lemma 3.4. *For $r \in \mathbb{N}$, we have the following:*

1. $\delta_r(0) = 0$,
2. $\delta_r(\mathrm{M}) \subseteq \mathrm{M}$,
3. $\delta_r(p), \delta_r(q) \in \mathrm{M}$ *and* $\delta_r(p) + \delta_r(q) = 0$,
4. $\delta_r(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M}$ *and* $\delta_r(\mathrm{B}) \subseteq \mathrm{B} + \mathrm{M}$.

Lemma 3.5. *For any $a \in \mathrm{A}, m \in \mathrm{M}$ and $b \in \mathrm{B}$, we have*

1. $\delta_r(am) = \delta_r(a)m + a\delta_r(m) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a)d_{i_2}(m)$,
2. $\delta_r(mb) = \delta_r(m)b + m\delta_r(b) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(m)d_{i_2}(b)$

for $r \in \mathbb{N}$.

Proof. Using the fact that $\delta_s(q) \in \mathrm{M}$ for all $0 < s \leq r$, we get

$$\begin{aligned}
\delta_r(am) &= \mathrm{L}_r([a,m]) \\
&= \mathrm{L}_r(\mathfrak{p}_n(a,m,q,\cdots,q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\delta_{i_1}(a),\delta_{i_2}(m),\delta_{i_3}(q),\cdots,\delta_{i_n}(q)) \\
&= \mathfrak{p}_n(\delta_r(a),m,q,\cdots,q)+\mathfrak{p}_n(a,\delta_r(m),q,\cdots,q) \\
&\quad +\mathfrak{p}_n(a,m,\delta_r(q),q,\cdots,q)+\cdots+\mathfrak{p}_n(a,m,q,\cdots,\delta_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\leq i_1,i_2,\cdots,i_n<r}} \mathfrak{p}_n(d_{i_1}(a),d_{i_2}(m),d_{i_3}(q),\cdots,d_{i_n}(q)) \\
&= \mathfrak{p}_{n-1}([\delta_r(a),m],q,\cdots,q)+\mathfrak{p}_{n-1}([a,\delta_r(m)],q,\cdots,q) \\
&\quad + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} \mathfrak{p}_{n-1}([d_{i_1}(a),d_{i_2}(m)],q,\cdots,q) \\
&= \delta_r(a)m+a\delta_r(m)+\sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a)d_{i_2}(m).
\end{aligned}$$

for $a \in \mathrm{A}, m \in \mathrm{M}$. Likewise, $\delta_r(mb) = \delta_r(m)b + m\delta_r(b) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(m)d_{i_2}(b)$ for all $b \in \mathrm{B}, m \in \mathrm{M}$. □

Lemma 3.6. *For any $a_1, a_2 \in \mathrm{A}$ and $b_1, b_2 \in \mathrm{B}$, we have*

1. $\delta_r(a_1a_2) = \delta_r(a_1)a_2 + a_1\delta_r(a_2) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2)$;
2. $\delta_r(b_1b_2) = \delta_r(b_1)b_2 + b_1\delta_r(b_2) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(b_1)d_{i_2}(b_2)$

for $r \in \mathbb{N}$.

Proof. For any $a_1, a_2 \in \mathrm{A}$ and $m \in \mathrm{M}$.

$$\begin{aligned}
\delta_r(a_1a_2m) &= \delta_r((a_1a_2)m) \\
&= \delta_r(a_1a_2)m + a_1a_2\delta_r(m) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a_1a_2)d_{i_2}(m) \\
&= \delta_r(a_1a_2)m + a_1a_2\delta_r(m) + \sum_{\substack{i_1+i_2+i_3=r \\ 0\leq i_1,i_2<r \\ 0<i_3<r}} d_{i_1}(a_1)d_{i_2}(a_2)d_{i_3}(m).
\end{aligned}$$

On the other way,

$$\begin{aligned}\delta_r(a_1a_2m) &= \delta_r(a_1(a_2m))\\ &= \delta_r(a_1)a_2m + a_1\delta_r(a_2m) + \sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2m)\\ &= \delta_r(a_1)a_2m + a_1\delta_r(a_2)m + a_1a_2\delta_r(m)\\ &\quad + \sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2)m + \sum_{\substack{i_1+i_2+i_3=r\\ 0\leq i_1,i_2<r\\ 0<i_3<r}} d_{i_1}(a_1)d_{i_2}(a_2)d_{i_3}(m).\end{aligned}$$

By the condition $\mathbf{C}_s$, the above expression becomes

$$\delta_r(a_1a_2)m = \big\{\delta_r(a_1)a_2 + a_1\delta_r(a_2) + \sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2)\big\}m.$$

Since $\delta_r(\mathrm{A}) \subseteq \mathrm{A}+\mathrm{M}$ and M is faithful as left A-module, the above relation implies that

$$\delta_r(a_1a_2)p = \big\{\delta_r(a_1)a_2 + a_1\delta_r(a_2) + \sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2)\big\}p. \tag{3.2}$$

Also, $[a_1, q] = 0$ for all $a_1 \in \mathrm{A}$

$$\begin{aligned}0 &= \mathrm{L}_r(\mathfrak{p}_n(a_1, q, q, \cdots, q))\\ &= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\delta_{i_1}(a_1), \delta_{i_2}(q), \delta_{i_3}(q), \cdots, \delta_{i_n}(q))\\ &= \mathfrak{p}_n(\delta_r(a_1), q, q, \cdots, q) + \mathfrak{p}_n(a_1, \delta_r(q), q, \cdots, q)\\ &\quad +\mathfrak{p}_n(a_1, q, \delta_r(q), q, \cdots, q) + \cdots + \mathfrak{p}_n(a_1, q, q, \cdots, \delta_r(q))\\ &\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r\\ 0\leq i_1,i_2,\cdots,i_n<r}} \mathfrak{p}_n(d_{i_1}(a_1), d_{i_2}(q), d_{i_3}(q), \cdots, d_{i_n}(q))\\ &= \mathfrak{p}_{n-1}([\delta_r(a_1), q], q, \cdots, q) + \mathfrak{p}_{n-1}([a_1, \delta_r(q)], q, \cdots, q)\\ &\quad + \sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} \mathfrak{p}_{n-1}([d_{i_1}(a_1), d_{i_2}(q)], q, \cdots, q).\end{aligned}$$

Since $\delta_r(\mathrm{A}) \subseteq \mathrm{A}+\mathrm{M}, \delta_r(q) \in \mathrm{M}$. The above equation implies that

$$0 = \delta_r(a_1)q + a_1\delta_r(q) + \sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(q) \text{ for all } a_1 \in \mathrm{A}. \tag{3.3}$$

On substituting a_1 by a_2 and a_1a_2 in (3.3) respectively, we get

$$0 = \delta_r(a_2)q + a_2\delta_r(q) + \sum_{\substack{i_1+i_2=r\\ 0<i_1,i_2<r}} d_{i_1}(a_2)d_{i_2}(q) \tag{3.4}$$

and

$$0 = \delta_r(a_1a_2)q + a_1a_2\delta_r(q) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1a_2)d_{i_2}(q). \tag{3.5}$$

Now left multiplying a_1 in (3.4) and combining it with (3.5) gives

$$\delta_r(a_1a_2)q + \sum_{\substack{i_1+i_2+i_3=r\\0<i_1,i_2,i_3<r}} d_{i_1}(a_1)d_{i_2}(a_2)d_{i_3}(q) = a_1\delta_r(a_2)q$$

which implies that

$$\delta_r(a_1a_2)q + \sum_{i_1=1}^{r-1} d_{i_1}(a_1) \sum_{\substack{i_2+i_3=r\\0<i_2,i_3<r}} d_{i_2}(a_2)d_{i_3}(q) = a_1\delta_r(a_2)q.$$

Now using the condition $\mathbf{C}_s$, we find that

$$\delta_r(a_1a_2)q - \sum_{i_1=1}^{r-1} d_{i_1}(a_1)d_{r-i_1}(a_2)q = a_1\delta_r(a_2)q$$

gives us

$$\delta_r(a_1a_2)q = a_1\delta_r(a_2)q + \sum_{i_1=1}^{r-1} d_{i_1}(a_1)d_{r-i_1}(a_2)q.$$

Hence,

$$\delta_r(a_1a_2)q = \big\{\delta_r(a_1)a_2 + a_1\delta_r(a_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2)\big\}q. \tag{3.6}$$

Now adding the (3.2) and (3.6), we have

$$\delta_r(a_1a_2) = \delta_r(a_1)a_2 + a_1\delta_r(a_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2).$$

Similarly, we can obtain that

$$\delta_r(b_1b_2) = \delta_r(b_1)b_2 + b_1\delta_r(b_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(b_1)d_{i_2}(b_2)$$

for all $b_1, b_2 \in \mathrm{B}$. $\square$

Lemma 3.7. *For any $a \in \mathrm{A}, m \in \mathrm{M}$ and $b \in \mathrm{B}$, we have*

1. $\delta_r(a+m) - \delta_r(a) - \delta_r(m) \in \mathrm{Z}(\mathfrak{T})$;

2. $\delta_r(b+m)-\delta_r(b)-\delta_r(m)\in \mathrm{Z}(\mathfrak{T})$

for $r\in\mathbb{N}$.

Proof. Let $a\in \mathrm{A}$ and $m,m_1\in \mathrm{M}$. Since $[a,m_1]=[a+m,m_1]$, we find

$$\mathrm{L}_r(\mathfrak{p}_n(a,m_1,q,\cdots,q)) \quad=\quad \mathrm{L}_r(\mathfrak{p}_n(a+m,m_1,q,\cdots,q)). \tag{3.7}$$

Using induction hypothesis, Lemma 3.3 and (3.7) reduces to

$$\mathfrak{p}_n(\delta_r(a),m_1,q,\cdots,q)=\mathfrak{p}_n(\delta_r(a+m),m_1,q,\cdots,q).$$

Therefore, $[\delta_r(a),m_1]=[\delta_r(a+m),m_1]$ and hence $[\delta_r(a+m)-\delta_r(a),\mathrm{M}]=0$. Hence, we get that

$$\begin{aligned}&\delta_r(a+m)-\delta_r(a)-p(\delta_r(a+m)-\delta_r(a))q\\ &\quad= p(\delta_r(a+m)-\delta_r(a))p+q(\delta_r(a+m)-\delta_r(a))q\in \mathrm{Z}(\mathfrak{T})\end{aligned} \tag{3.8}$$

for all $a\in \mathrm{A}, m\in \mathrm{M}$. Applying Lemma 3.2, 3.4 and Remark 3.1, we have

$$\begin{aligned}&p(\delta_r(a+m)-\delta_r(a))q\\ &\quad= [p,\delta_r(a+m)-\delta_r(a)]\\ &\quad= [p,\mathrm{L}_r(a+m)]-[p,\mathrm{L}_r(a)]\\ &\quad= \mathrm{L}_r(\mathfrak{p}_n(p,a+m,q,\cdots,q))-\mathfrak{p}_n(\mathrm{L}_r(p),a+m,q,\cdots,q)\\ &\qquad -\mathfrak{p}_n(p,a+m,\mathrm{L}_r(q),\cdots,q)-\cdots-\mathfrak{p}_n(p,a+m,q,\cdots,\mathrm{L}_r(q))\\ &\qquad -\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ 0\le i_1,i_2,\cdots,i_n<r}}\mathfrak{p}_n(\delta_{i_1}(p),\delta_{i_2}(a+m),\delta_{i_3}(q),\cdots,\delta_{i_n}(q))\\ &\qquad -\mathrm{L}_r(\mathfrak{p}_n(p,a,q,\cdots,q))+\mathfrak{p}_n(\mathrm{L}_r(p),a,q,\cdots,q)\\ &\qquad +\mathfrak{p}_n(p,a,\mathrm{L}_r(q),\cdots,q)+\cdots+\mathfrak{p}_n(p,a,q,\cdots,\mathrm{L}_r(q))\\ &\qquad +\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ 0\le i_1,i_2,\cdots,i_n<r}}\mathfrak{p}_n(\delta_{i_1}(p),\delta_{i_2}(a),\delta_{i_3}(q),\cdots,\delta_{i_n}(q))\\ &\quad= \mathrm{L}_r(\mathfrak{p}_n(p,m,q,\cdots,q))\\ &\quad= \mathrm{L}_r(m)=\delta_r(m).\end{aligned}$$

From (3.8), it follows that $\delta_r(a+m)-\delta_r(a)-\delta_r(m)\in \mathrm{Z}(\mathfrak{T})$ for all $a\in \mathrm{A}, m\in \mathrm{M}$. Similarly, we can prove $\delta_r(b+m)-\delta_r(b)-\delta_r(m)\in \mathrm{Z}(\mathfrak{T})$ for all $b\in \mathrm{B}, m\in \mathrm{M}$. □

Lemma 3.8. δ_r *is additive on* A, M *and* B.

Proof. Using $m_1 + m_2 = \mathfrak{p}_n(p + m_1, m_2 + q, q, \cdots, q)$ and Lemma 3.4, we find that

$$\begin{aligned}
\delta_r(m_1 + m_2) &= \mathrm{L}_r(\mathfrak{p}_n(p + m_1, m_2 + q, q, \cdots, q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\delta_{i_1}(p + m_1), \delta_{i_2}(m_2 + q), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&= \mathfrak{p}_n(\delta_r(p + m_1), m_2 + q, q, \cdots, q) \\
&\quad + \mathfrak{p}_n(p + m_1, \delta_r(m_2 + q), q, \cdots, q) \\
&\quad + \mathfrak{p}_n(p + m_1, m_2 + q, \delta_r(q), q, \cdots, q) \\
&\quad + \cdots + \mathfrak{p}_n(p + m_1, m_2 + q, q, \cdots, \delta_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1,i_2,\cdots,i_n<r}} \mathfrak{p}_n(\delta_{i_1}(p + m_1), \delta_{i_2}(m_2 + q), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&= \mathfrak{p}_{n-1}([\delta_r(p + m_1), m_2 + q], q, \cdots, q) \\
&\quad + \mathfrak{p}_{n-1}([p + m_1, \delta_r(m_2 + q)], q, \cdots, q) \\
&= \mathfrak{p}_{n-1}([\delta_r(p) + \delta_r(m_1), m_2 + q], \cdots, q) \\
&\quad + \mathfrak{p}_{n-1}([p + m_1, \delta_r(m_2) + \delta_r(q)], \cdots, q) \\
&= \delta_r(p) + \delta_r(m_1) + \delta_r(m_2) + \delta_r(q) \\
&= \delta_r(m_1) + \delta_r(m_2)
\end{aligned}$$

for all $m_1, m_2 \in \mathrm{M}$. Now,

$$\begin{aligned}
\delta_r((a_1 + a_2)m) &= \delta_r(a_1 m) + \delta_r(a_2 m) \\
&= \delta_r(a_1)m + a_1\delta_r(m) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(m) \\
&\quad + \delta_r(a_2)m + a_2\delta_r(m) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a_2)d_{i_2}(m) \qquad (3.9)
\end{aligned}$$

for all $a_1, a_2 \in \mathrm{A}$ and $m \in \mathrm{M}$. On the other hand,

$$\begin{aligned}
\delta_r((a_1 + a_2)m) &= \delta_r(a_1 + a_2)m + (a_1 + a_2)\delta_r(m) \\
&\quad + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a_1 + a_2)d_{i_2}(m). \qquad (3.10)
\end{aligned}$$

Combining (3.9), (3.10) and applying condition $\mathbf{C}_s$, we have

$$\delta_r(a_1 + a_2)m = \delta_r(a_1)m + \delta_r(a_2)m. \qquad (3.11)$$

Since $\delta_r(\mathrm{A}) \subseteq \mathrm{A} + \mathrm{M}$ and M is faithful as left A. Then (3.11) implies that

$$\delta_r(a_1 + a_2)p = \delta_r(a_1)p + \delta_r(a_2)p. \qquad (3.12)$$

Replace a_1 for $a_1 + a_2$ in (3.3), we get

$$0 = \delta_r(a_1 + a_2)q + (a_1 + a_2)\delta_r(q) + \sum_{\substack{i_1+i_2=r \\ 0<i_1,i_2<r}} d_{i_1}(a_1 + a_2)d_{i_2}(q) \qquad (3.13)$$

for all $a_1 \in \mathrm{A}$. Combining (3.13) with (3.3) and (3.4), we obtain

$$\delta_r(a_1+a_2)q = \delta_r(a_1)q + \delta_r(a_2)q. \tag{3.14}$$

Addition of (3.12) and (3.14) implies that $\delta_r(a_1+a_2) = \delta_r(a_1)+\delta_r(a_2)$ for all $a_1, a_2 \in \mathrm{A}$.

Similarly, we can deduce that $\delta_r(b_1+b_2) = \delta_r(b_1)+\delta_r(b_2)$ for all $b_1, b_2 \in \mathrm{B}$. □

Lemma 3.9. *$\delta_r(a+m+b) - \delta_r(a) - \delta_r(m) - \delta_r(b) \in \mathrm{Z}(\mathfrak{T})$ for all $a \in \mathrm{A}, m \in \mathrm{M}, b \in \mathrm{B}$.*

Proof. Using induction hypothesis and fact $\delta_r(q) \in \mathrm{M}$. On one hand, we have

$$\begin{aligned}
&\mathrm{L}_r(\mathfrak{p}_n(a+m+b), m_1, q, \cdots, q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\delta_{i_1}(a+m+b), \delta_{i_2}(m_1), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&= \mathfrak{p}_n(\delta_r(a+m+b), m_1, q, \cdots, q) + \mathfrak{p}_n(a+m+b, \delta_r(m_1), q, \cdots, q) \\
&\quad +\mathfrak{p}_n(a+m+b, m_1, \delta_r(q), q, \cdots, q) + \cdots + \mathfrak{p}_n(a+m+b, m_1, q, \cdots, \delta_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\delta_{i_1}(a+m+b), \delta_{i_2}(m_1), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&= \mathfrak{p}_{n-1}([\delta_r(a+m+b), m_1], q, \cdots, q) + \mathfrak{p}_{n-1}([a+m+b, \delta_r(m_1)], q, \cdots, q) \\
&= [\delta_r(a+m+b), m_1] + [a+m+b, \delta_r(m_1)] \\
&= [\delta_r(a+m+b), m_1] + [a, \delta_r(m_1)] + [b, \delta_r(m_1)]
\end{aligned} \tag{3.15}$$

for all $a \in \mathrm{A}, m, m_1 \in \mathrm{M}, b \in \mathrm{B}$. On the other hand, using Lemma 3.8, we obtain

$$\begin{aligned}
&\mathrm{L}_r(\mathfrak{p}_n(a+m+b, m_1, q, \cdots, q)) \\
&= \mathrm{L}_r([a, m_1] + [b, m_1]) \\
&= \mathrm{L}_r(\mathfrak{p}_n(a, m_1, q, \cdots, q)) + \mathrm{L}_r(\mathfrak{p}_n(b, m_1, q, \cdots, q)) \\
&= \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\delta_{i_1}(a), \delta_{i_2}(m_1), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&\quad + \sum_{i_1+i_2+\cdots+i_n=r} \mathfrak{p}_n(\delta_{i_1}(b), \delta_{i_2}(m_1), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&= \mathfrak{p}_n(\delta_r(a), m_1, q, \cdots, q) + \mathfrak{p}_n(a, \delta_r(m_1), q, \cdots, q) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\delta_{i_1}(a), \delta_{i_2}(m_1), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&\quad +\mathfrak{p}_n(\delta_r(b), m_1, q, \cdots, q) + \mathfrak{p}_n(b, \delta_r(m_1), q, \cdots, q) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0\le i_1, i_2, \cdots, i_n<r}} \mathfrak{p}_n(\delta_{i_1}(b), \delta_{i_2}(m_1), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&= \mathfrak{p}_{n-1}([\delta_r(a), m_1], q, \cdots, q) + \mathfrak{p}_{n-1}([a, \delta_r(m_1)], q, \cdots, q) \\
&\quad +\mathfrak{p}_{n-1}([\delta_r(b), m_1], q, \cdots, q) + \mathfrak{p}_{n-1}([b, \delta_r(m_1)], q, \cdots, q) \\
&= [\delta_r(a), m_1] + [a, \delta_r(m_1)] + [\delta_r(b), m_1] + [b, \delta_r(m_1)]
\end{aligned} \tag{3.16}$$

for all $a \in \mathrm{A}, m, m_1 \in \mathrm{M}, b \in \mathrm{B}$. Combining (3.15) and (3.16), we get

$$[\delta_r(a+m+b) - \delta_r(a) - \delta_r(b), \mathrm{M}] = 0,$$

which in turn implies that

$$\delta_r(a+m+b) - \delta_r(a) - \delta_r(b) - p(\delta_r(a+m+b) - \delta_r(a) - \delta_r(b))q \in \mathrm{Z}(\mathfrak{T})$$

for all $a \in \mathrm{A}, m \in \mathrm{M}, b \in \mathrm{B}$.

$$\begin{aligned}
& p(\delta_r(a+m+b) - \delta_r(a) - \delta_r(b))q \\
&= [p, \delta_r(a+m+b) - \delta_r(a) - \delta_r(b)] \\
&= [p, \mathrm{L}_r(a+m+b)] - [p, \mathrm{L}_r(a)] - [p, \mathrm{L}_r(b)] \\
&= \mathrm{L}_r(\mathfrak{p}_n(p, a+m+b, q, \cdots, q)) - \mathfrak{p}_n(\mathrm{L}_r(p), a+m+b, q, \cdots, q) \\
&\quad -\mathfrak{p}_n(p, a+m+b, \mathrm{L}_r(q), \cdots, q) - \cdots - \mathfrak{p}_n(p, a+m+b, q, \cdots, \mathrm{L}_r(q)) \\
&\quad - \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0 \le i_1, i_2, \cdots, i_n < r}} \mathfrak{p}_n(\delta_{i_1}(p), \delta_{i_2}(a+m+b), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&\quad -\mathrm{L}_r(\mathfrak{p}_n(p, a, q, \cdots, q)) + \mathfrak{p}_n(\mathrm{L}_r(p), a, q, \cdots, q) \\
&\quad +\mathfrak{p}_n(p, a, \mathrm{L}_r(q), \cdots, q) + \cdots + \mathfrak{p}_n(p, a, q, \cdots, \mathrm{L}_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0 \le i_1, i_2, \cdots, i_n < r}} \mathfrak{p}_n(\delta_{i_1}(p), \delta_{i_2}(a), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&\quad -\mathrm{L}_r(\mathfrak{p}_n(p, b, q, \cdots, q)) + \mathfrak{p}_n(\mathrm{L}_r(p), b, q, \cdots, q) \\
&\quad +\mathfrak{p}_n(p, b, \mathrm{L}_r(q), \cdots, q) + \cdots + \mathfrak{p}_n(p, b, q, \cdots, \mathrm{L}_r(q)) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ 0 \le i_1, i_2, \cdots, i_n < r}} \mathfrak{p}_n(\delta_{i_1}(p), \delta_{i_2}(b), \delta_{i_3}(q), \cdots, \delta_{i_n}(q)) \\
&= \mathrm{L}_r(\mathfrak{p}_n(p, m, q, \cdots, q)) \\
&= \mathrm{L}_r(m) = \delta_r(m).
\end{aligned}$$

This leads to $\delta_r(a+m+b) - \delta_r(a) - \delta_r(m) - \delta_r(b) \in \mathrm{Z}(\mathfrak{T})$ for all $a \in \mathrm{A}, m \in \mathrm{M}, b \in \mathrm{B}$. □

Remark 3.2. Now we establish a mapping $g_r : \mathfrak{T} \to \mathrm{Z}(\mathfrak{T})$ by

$$g_r(x) = \delta_r(x) - \delta_r(pxp) - \delta_r(pxq) - \delta_r(qxq) \text{ for all } x \in \mathfrak{T}.$$

Obviously, $g_r(\mathrm{A}) = g_r(\mathrm{M}) = g_r(\mathrm{B}) = 0$. Observe that $g_r(\mathfrak{p}_n(\mathfrak{T}, \mathfrak{T}, \cdots, \mathfrak{T})) = 0$. Define a mapping $d_r(x) = \delta_r(x) - g_r(x)$ for all $x \in \mathfrak{T}$. It is easy to verify for each $r \in \mathbb{N}$, d_r satisfies $d_r(a+m+b) = d_r(a) + d_r(m) + d_r(b)$. From the definition of d_r and g_r, it follows that

$$\mathrm{L}_r = \delta_r + f_r = d_r + g_r + f_r = d_r + h_r, \quad \text{where} \quad h_r = g_r + f_r.$$

Proof. [Proof of Theorem 3.1] Suppose $x, y \in \mathfrak{T}$ such that $x = a_1 + m_1 + b_1$ and $y = a_2 + m_2 + b_2$. Then

$$\begin{aligned}
d_r(x+y) &= d_r\big((a_1+m_1+b_1)+(a_2+m_2+b_2)\big)\\
&= d_r\big((a_1+a_2)+(m_1+m_2)+(b_1+b_2)\big)\\
&= \delta_r(a_1+a_2)+\delta_r(m_1+m_2)+\delta_r(b_1+b_2)\\
&= \delta_r(a_1)+\delta_r(a_2)+\delta_r(m_1)+\delta_r(m_2)+\delta_r(b_1)+\delta_r(b_2)\\
&= d_r(a_1+m_1+b_1)+d_r(a_2+m_2+b_2)\\
&= d_r(x)+d_r(y).
\end{aligned}$$

By Lemma 3.6 and Lemma 3.7, we have

$$\begin{aligned}
d_r(xy) &= d_r\big((a_1+m_1+b_1)(a_2+m_2+b_2)\big)\\
&= \delta_r(a_1a_2+a_1m_2+m_1b_2+b_1b_2)\\
&= \delta_r(a_1)a_2+a_1\delta_r(a_2)+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2)\\
&\quad +\delta_r(a_1)m_2+a_1\delta_r(m_2)+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(m_2)\\
&\quad +\delta_r(m_1)b_2+m_1\delta_r(b_2)+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(m_1)d_{i_2}(b_2)\\
&\quad +\delta_r(b_1)b_2+b_1\delta_r(b_2)+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(b_1)d_{i_2}(b_2). \qquad (3.17)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&d_r(x)y + xd_r(y) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(x)d_{i_2}(y)\\
=\ & (\delta_r(a_1)+\delta_r(m_1)+\delta_r(b_1))y + x(\delta_r(a_2)+\delta_r(m_2)+\delta_r(b_2))\\
&+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2)+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(m_2)\\
&+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(b_2)+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(m_1)d_{i_2}(a_2)\\
&+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(m_1)d_{i_2}(m_2)+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(m_1)d_{i_2}(b_2)\\
&+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(b_1)d_{i_2}(a_2)+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(b_1)d_{i_2}(m_2)\\
&+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(b_1)d_{i_2}(b_2).
\end{aligned}\tag{3.18}$$

By using condition $\mathbf{C}_s$, and from Lemma 3.6, we have

$$\begin{aligned}
&d_r(x)y + xd_r(y) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(x)d_{i_2}(y)\\
=\ & \delta_r(a_1)a_2 + a_1\delta_r(a_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(a_2) + \delta_r(a_1)m_2 + a_1\delta_r(m_2)\\
&+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(a_1)d_{i_2}(m_2) + \delta_r(m_1)b_2 + m_1\delta_r(b_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(m_1)d_{i_2}(b_2)\\
&+\delta_r(b_1)b_2 + b_1\delta_r(b_2) + \sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(b_1)d_{i_2}(b_2).
\end{aligned}\tag{3.19}$$

Combining (3.17) and (3.19) ,we get $d_r(xy) = d_r(x)y+xd_r(y)+\sum_{\substack{i_1+i_2=r\\0<i_1,i_2<r}} d_{i_1}(x)d_{i_2}(y)$.

This implies that, $\{d_r\}_{r\in\mathbb{N}}$ is an additive higher derivation on $\mathfrak{T}$. Finally, there exists a map $h_r : \mathfrak{T} \to Z(\mathfrak{T})$ such that $h_r(\mathfrak{p}_n(\mathfrak{T},\mathfrak{T},\cdots,\mathfrak{T})) = \mathrm{L}_r(\mathfrak{p}_n(\mathfrak{T},\mathfrak{T},\cdots,\mathfrak{T})) - d_r(\mathfrak{p}_n(\mathfrak{T},\mathfrak{T},\cdots,\mathfrak{T})) = 0$. This completes the proof. □

As a direct consequence of Theorem 3.1, we have the following result:

Corollary 3.1. *[3, Theorem 3.1] Let* $\mathfrak{T} = Tri(\mathrm{A},\mathrm{M},\mathrm{B})$ *be a 2-torsion free triangular algebra consisting of unital algebras* A,B *and a faithful unital* (A,B)*-bimodule* M*. Suppose that* $\mathfrak{T}$ *satisfies the following conditions:*

1. $\pi_{\mathrm{A}}(\mathrm{Z}(\mathfrak{T})) = \mathrm{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathrm{Z}(\mathfrak{T})) = \mathrm{Z}(\mathrm{B})$,

2. $\mathrm{Z}(\mathrm{A}) = \{a \in \mathrm{A} \mid [[a, x], y] = 0 \ \forall \ x, y \in \mathrm{A}\}$
or $\mathrm{Z}(\mathrm{B}) = \{b \in \mathrm{B} \mid [[b, x], y] = 0 \ \forall \ x, y \in \mathrm{B}\}$.

Then every multiplicative Lie triple higher derivation $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$ *on* $\mathfrak{T}$ *has the standard form. More precisely, there exists an additive higher derivation* $\mathfrak{D} = \{d_r\}_{r\in\mathbb{N}}$ *on* $\mathfrak{T}$ *and a sequence of functionals* $\{h_r\}_{r\in\mathbb{N}}$ *which annihilates all Lie triple product* $[[x_1, x_2], x_3]$ *for all* $x_1, x_2, x_3 \in \mathfrak{T}$ *such that* $\mathrm{L}_r(x) = d_r(x) + h_r(x)$ *for all* $x \in \mathfrak{T}$ *and* $r \in \mathbb{N}$.

4. Applications

In this section, we apply Theorem 3.1 to some triangular and related algebras, such as upper triangular matrix algebras, block upper triangular matrix algebras, nest algebras, incidence algebras.

Since an arbitrary derivation on $\mathcal{T}(\mathcal{N})$ is inner and in view of [23, Proposition 2.6], we know that an arbitrary higher derivation on $\mathcal{T}(\mathcal{N})$ is inner.

Corollary 4.1. *Let* X *be an infinite dimensional Banach space over the real or complex field* $\mathbb{F}$ *,* $\mathcal{N}$ *be a nest on* X *which contains a nontrivial element complemented in* X *and* $\mathcal{T}(\mathcal{N})$ *be a nest algebra. Then for every multiplicative Lie n-higher derivation* $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$*, there exists an inner higher derivation* $\{d_r\}_{r\in\mathbb{N}}$ *on* $\mathcal{T}(\mathcal{N})$ *and a sequence of functionals* $\{h_r\}_{r\in\mathbb{N}}$ *which annihilates all* $(n-1)$*-th commutators* $\mathfrak{p}_n(\mathcal{T}(\mathcal{N}), \mathcal{T}(\mathcal{N}), \cdots, \mathcal{T}(\mathcal{N}))$ *such that* $\mathrm{L}_r = d_r + h_r$*, where* $d_r : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ *and* $h_r : \mathcal{T}(\mathcal{N}) \to \mathbb{F}I$ *for* $r \in \mathbb{N}$.

Corollary 4.2. *Let* $\mathcal{N}$ *be a nest of a Hilbert space* H *dimension greater than* 2 *and* $\mathcal{T}(\mathcal{N})$ *be a nontrivial nest algebra. Then for every multiplicative Lie n-higher derivation* $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$*, there exists an inner higher derivation* $\{d_r\}_{r\in\mathbb{N}}$ *on* $\mathcal{T}(\mathcal{N})$ *and a sequence of functionals* $\{h_r\}_{r\in\mathbb{N}}$ *which annihilates all* $(n-1)$*-th commutators* $\mathfrak{p}_n(\mathcal{T}(\mathcal{N}), \mathcal{T}(\mathcal{N}), \cdots, \mathcal{T}(\mathcal{N}))$ *such that* $\mathrm{L}_r = d_r + h_r$*, where* $d_r : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ *and* $h_r : \mathcal{T}(\mathcal{N}) \to \mathbb{F}I$ *for each* $r \in \mathbb{N}$.

If Hilbert space H is finite dimensional, then nest algebras are upper block triangular matrices algebras [7].

Corollary 4.3. *Let* R *be a* $(n-1)$*-torsion free commutative ring with unity and* $B_m^{\bar{k}}(\mathrm{R})(m \geq 3)$ *i.e. block upper triangular matrix algebra defined over* R *with* $B_m^{\bar{k}}(\mathrm{R}) \neq M_m(\mathrm{R})$*. Then for every multiplicative Lie n-higher derivation* $\mathfrak{L} = \{\mathrm{L}_r\}_{r\in\mathbb{N}}$*, there exist an inner higher derivation* $\{d_r\}_{r\in\mathbb{N}}$ *on* $B_m^{\bar{k}}(\mathrm{R})$ *and a sequence of functionals* $\{h_r\}_{r\in\mathbb{N}}$ *which annihilates all* $(n-1)$*-th commutators* $\mathfrak{p}_n(B_m^{\bar{k}}(\mathrm{R}), B_m^{\bar{k}}(\mathrm{R}), \cdots, B_m^{\bar{k}}(\mathrm{R}))$ *such that* $\mathrm{L}_r = d_r + h_r$*, where* $d_r : B_m^{\bar{k}}(\mathrm{R}) \to B_m^{\bar{k}}(\mathrm{R})$ *and* $h_r : B_m^{\bar{k}}(\mathrm{R}) \to \mathrm{R}I$ *for each* $r \in \mathbb{N}$.

Proof. It can be easily seen that conditions of Theorem 3.1 hold for block upper triangular matrix algebra and since all derivations of $B_m^{\bar{k}}(\mathrm{R})$ are inner. By [23, Proposition 2.6] we arrive at that any higher derivation of $B_m^{\bar{k}}(\mathrm{R})$ is inner. Hence the result follows. □

Note that $T_m(\mathrm{R}) \subseteq B_m^{\bar{k}}(\mathrm{R}) \subseteq M_m(\mathrm{R})(m \geq 3)$ is a proper block upper triangular matrix algebra over a commutative ring R.

Corollary 4.4. *Every multiplicative Lie n-higher derivation has standard form on upper triangular matrix algebra $T_m(\mathrm{R})$.*

Incidence algebra. Let R be a commutative ring with unity. Let X be a finite partially ordered set (poset) with the partial order $\leq$. We define the incidence algebra of X over R as $\mathrm{I}(X, \mathrm{R}) = \{f : X \times X \to \mathrm{R} \mid f(x, y) = 0 \ if \ x \nleq y\}$ with algebraic operation given by

1. $(f + g)(x, y) = f(x, y) + g(x, y)$,
2. $(f \star g)(x, y) = \sum\limits_{x \leq z \leq y} f(x, z)g(z, y)$,
3. $(r.f)(x, y) = r.f(x, y)$

for all $f, g \in \mathrm{I}(X, \mathrm{R}), r \in \mathrm{R}$ and $x, y, z \in X$. Obviously, f is an R-valued function on $\{(x, y) \in X \times X \mid x \leq y\}$. The product $\star$ is usually called convolution in function theory. If X is a partially ordered set (poset) with n elements, then $\mathrm{I}(X, \mathrm{R})$ is isomorphic to a subalgebra of the algebra $M_n(\mathrm{R})$ of square matrices over R with elements $[a_{ij}] \in M_n(\mathrm{R})$ satisfying $a_{ij} = 0$ if $i \nleq j$, for some partial order $\leq$ defined in the partial order set (poset) $\{1, \ldots, n\}$. This shows that $\mathrm{I}(X, \mathrm{R})$ is a triangular algebra.

The incidence algebra of a partially ordered set (poset) X is the algebra of functions from the segments of X into R, which extends the various convolutions in algebras of arithmetic functions. Incidence algebras, in fact, were first considered by Ward [22] as generalized algebras of arithmetic functions. Rota and Stanley [21] developed incidence algebras as the fundamental structures of enumerative combinatorial theory and allied areas of arithmetic function theory. The theory of Möbius functions, including the classical Möbius function of number theory and the combinatorial inclusion-exclusion formula, is established in the context of incidence algebras. For the later, we refer the reader to [21, Sections 2.1 and 3.7].

In the theory of operator algebras, incidence algebras of a finite poset X are referred as bigraph algebras or finite dimensional CSL algebras. If X is connected, then $\mathrm{Z}(\mathrm{I}(X, \mathrm{R})) = \mathrm{R}I$. Clearly, any incidence algebra $\mathrm{I}(X, \mathrm{R})$ is a triangular algebra and hence it satisfies the condition $(\sharp)$. Then we have

Corollary 4.5. *Let* R *be a $(n-1)$-torsion free commutative ring with unity, X be a connected finite partially ordered set (poset) with the partial order $\leq$ and $\mathrm{I}(X, \mathrm{R})$ an incidence algebra of X over* R. *Then every multiplicative Lie n-higher derivation has the standard form.*

5. For Future Discussions

In this section, we make an attempt to pull out attention of readers towards the obtainable research problem. Let us observe a more general class of maps. Note down the sequence of polynomials:

$$\begin{aligned} \mathfrak{q}_1(x_1) &= x_1, \\ \mathfrak{q}_2(x_1,x_2) &= \mathfrak{q}_1(x_1)\circ x_2 = x_1\circ x_2, \\ &\vdots \qquad\qquad \vdots \\ \mathfrak{q}_n(x_1,x_2,\cdots,x_n) &= \mathfrak{q}_{n-1}(x_1,x_2,\cdots,x_{n-1})\circ x_n. \end{aligned}$$

The polynomial $\mathfrak{q}_n(x_1,x_2,\cdots,x_n)$ is called $(n-1)$-th anti-commutator where $n \geq 2$. Let R be a commutative ring with unity and $\mathcal{A}$ be an R-algebra. A map (not necessarily linear) $\mathfrak{J}:\mathcal{A}\to\mathcal{A}$ is said to be a multiplicative Jordan n-derivation on $\mathcal{A}$ if

$$\mathfrak{J}(\mathfrak{q}_n(x_1,x_2,\cdots,x_n)) = \sum_{i=1}^{n}\mathfrak{q}_n(x_1,x_2,\cdots,x_{i-1},\mathfrak{J}(x_i),x_{i+1},\cdots,x_n)$$

for all $x_1,x_2,\cdots,x_n\in\mathcal{A}$.

Let $\mathbb{N}$ be the set of nonnegative integers and $\mathfrak{J}=\{\mathfrak{J}_r\}_{r\in\mathbb{N}}$ be a family of maps $\mathfrak{J}_r:\mathcal{A}\to\mathcal{A}$ (not necessarily linear) such that $\mathfrak{J}_0 = I_{\mathcal{A}}$. Then $\mathfrak{J}$ is called a multiplicative Jordan n-higher derivation if

$$\mathfrak{J}_r(\mathfrak{q}_n(x_1,x_2,\cdots,x_n)) = \sum_{i_1+i_2+\cdots+i_n=r}\mathfrak{q}_n(\mathfrak{J}_{i_1}(x_1),\mathfrak{J}_{i_2}(x_2),\cdots,\mathfrak{J}_{i_n}(x_n))$$

for all $x_1,x_2,\cdots,x_n\in\mathcal{A}$ and for each $r\in\mathbb{N}$. It is easy to see that any multiplicative Jordan 2-higher derivation is a multiplicative Jordan higher derivation and multiplicative Jordan 3-higher derivation is multiplicative Jordan triple higher derivation. Thus multiplicative Jordan higher/Jordan triple higher/$\cdots$/Jordan n-higher derivation collectively known as multiplicative Jordan type higher derivations on $\mathcal{A}$. At this point, in view of [2, 4], it is reasonable to raise the following open problem as:

Problem 5.1. What is the most general form of multiplicative Jordan type higher derivations on triangular algebras and which constraints are needed to apply on triangular algebras?

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