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SOME FIXED POINT RESULTS ON RECTANGULAR $b-$METRIC SPACE

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Abstract. In this paper we have obtained some results on a complete rectangular $b-$metric space and these results generalized many existing results in this literature.
Keywords: rectangular $b-$metric space.

1. Introduction and Preliminaries

The Banach fixed point theorem in metric space has generalized by many researchers in various branches such as cone metric space, $b-$metric space, Generalized metric space, Fuzzy metric space etc. Many researchers such as Tiwary et al.[12], Sarkar et al.([10], [11]), S. Czerwik[3], H. Huang et al.[7], Ding et.al[5], Ozturk[9] and others have worked on Cone Banach Space, $b-$metric space, rectangular $b-$metric space. George et al.[6] have proved some results in rectangular $b-$metric space and have left two open problems for further investigations. Z. D. Mitrović and S. Radenović [8] has given a partial solutions of Reich and Kannan Type contraction in rectangular $b-$metric space. In this paper we have given partial solution of Cirić Type, Cirić almost contraction Type, Hardy Rogers Type contraction condition in rectangular $b-$metric space with some corollaries.

The following definitions are required to prove the main results.

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Definition 1.1. [1] Let X be a non-empty set $s \geq 1$ a real number. A function $d : X \times X \to \mathbb{R}$ is a said to be a $b-$ metric if for a distinct point $u \in X$, different from x and y, the following conditions holds:

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq s[d(x, u) + d(u, y)]$. □

The pair (X, d) is called a $b-$metric space (in short bMS) with coefficient $s \geq 1$.

Definition 1.2. [6] Let X be a non-empty set $s \geq 1$ a real number. A function $d : X \times X \to \mathbb{R}$ is a said to be a rectangular $b-$ metric if for all distinct points $u_1, u_2 \in X$, all are different from x and y, the following conditions holds:

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq s[d(x, u_1) + d(u_1, u_2) + d(u_2, y)]$.

The pair (X, d) is called a rectangular $b-$metric space (in short RbMS) with coefficient $s \geq 1$.

If $s = 1$ then (X, d) is called a rectangular metric space (in short RMS).

Definition 1.3. [6] Let (X, d) be a rectangular $b-$metric space, $\{x_n\}$ be a sequence in X and $x \in X$.
Then

i) the sequence $\{x_n\}$ is said to be convergent in (X, d) and converges to x if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$;

ii) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, d) if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \epsilon$ for all $n \geq n_0; p > 0$ or equivalently, if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$;

iii) (X, d) is said to be a complete rectangular $b-$metric space if every Cauchy sequence in X converges to some $x \in X$.

R. George et al. [6] has proved the result.

Theorem 1.1. *(**[6], Theorem 2.1**) Let (X, d) be a complete rectangular $b-$metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying*

$$d(Tx, Ty) < \lambda d(x, y)$$

for all $x, y \in X$ with $x \neq y$, where $\lambda \in [0, \frac{1}{s}]$. Then T has a unique fixed point.

2. Main Results

Our main resuts are as follows:

Theorem 2.1. *Let (X, d) be a complete rectangular $b-$metric space with coefficient $s > 1$ and $\{T^i\}$ be a sequence of self-maps satisfying the condition*

$d(T^ix, T^jy) \leq \alpha \max\{d(x, y), d(x, T^ix), d(y, T^jy), d(x, T^jy), d(y, T^ix)\}+Ld(y, T^ix),$ *where the constants $\alpha, L \geq 0$ and $\alpha + L < 1$. Then the sequence $\{T^i\}$ have unique common fixed point in X.*

Proof. *Let $x_0 \in X$ be an arbitrary. We construct a sequence for a fixed $i \in \mathbb{N}$ such that $x_n = T^ix_{n-1}$ where $n \in \mathbb{N}$.*

Let, $d_n = d(x_n, x_{n+1})$ and $d^*_n = d(x_n, x_{n+2})$.
Then

$d(x_n, x_{n+1}) = d(T^ix_{n-1}, T^jx_n)$
$\leq \alpha \max\{d(x_{n-1}, x_n), d(x_{n-1}, T^ix_{n_1}), d(x_n, T^jx_n), d(x_{n-1}, T^jx_n), d(x_n, T^ix_{n-1})\}+$
$Ld(x_n, T^ix_{n-1})$
$\leq \alpha \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}+Ld(x_n, x_n).$

$$\leq \alpha \max\{d_{n-1}, d_n, d^*_{n-1}\}. \tag{2.1}$$

Suppose, $\{d_n\}$ is monotone increasing sequence. Then from equation (2.1) we get,

$$d_n \leq \alpha \max\{d_n, d^*_{n-1}\}.$$

If $d_n > d^*_{n-1}$, then from (2.1) we get, $d_n \leq \alpha d_n$ which implies, $1 \leq \alpha$, a contradiction.
Therefore,

$$d_n \leq d^*_{n-1}.$$

Then from (2.1), we get

$$d_n \leq \alpha d^*_{n-1} \leq \alpha^2 d^*_{n-2} \leq \ldots \leq \alpha^n d^*_0$$

implies, $d_n = 0$ as $n \to \infty$. Suppose, $\{d_n\}$ is monotone decreasing sequence. then from (2.1), we get

$$d_n \leq \alpha \max\{d_{n-1}, d^*_{n-1}\}. \tag{2.2}$$

If $d_{n-1} \leq d^*_{n-1}$, then from (2.2), we get

$$d_n = \alpha d^*_{n-1} \leq \alpha^2 d^*_{n-2} \leq \ldots \leq \alpha^n d^*_0$$

implies,

$$\lim_{n\to\infty} d_n = 0.$$

Again suppose $d^*_{n-1} \leq d_{n-1}$, then from (2.2) we have,

$$d_n = \alpha d_{n-1} \leq \alpha^2 d_{n-2} \leq \ldots \leq \alpha^n d_0$$

implies, $\lim_{n\to\infty} d_n = 0$.

Thus for all cases $\lim_{n\to\infty} d_n = 0$.

Now we show

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 \tag{2.3}$$

holds good by Mathematical Induction on $p \in \mathbb{N}$.

Clearly, (2.3) hold for $p = 1$.
Suppose it holds for p *i.e.,* $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$. *So* $\lim_{n\to\infty} d(x_{n+1}, x_{n+p+1}) = 0$.
We have to show

$\lim_{n\to\infty} d(x_n, x_{n+p+1}) = 0$.
Since

$$d(x_n, x_{n+p+1}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p}) + d(x_{n+p}, x_{n+p+1})].$$

Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+p+1}) \leq s \lim_{n\to\infty} d(x_{n+1}, x_{n+p}). \tag{2.4}$$

Case I: *If* $p = 2m, m \in \mathbb{N}$. *Then from (2.4) we get,*

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, x_{n+p+1}) &\leq s \lim_{n\to\infty} d(x_{n+1}, x_{n+2m}) \\
&\leq s^2 \lim_{n\to\infty} d(x_{n+1+1}, x_{n+2m-1}) \\
&\leq s^3 \lim_{n\to\infty} d(x_{n+1+2}, x_{n+2m-2}) \\
&\vdots \\
&\leq s^{m+1} \lim_{n\to\infty} d(x_{n+m}, x_{n+m+1}) \\
&= 0.
\end{aligned}$$

Case II: *If* $p = 2m+1, m \in \mathbb{N}$, *then from (2.4) we get,*

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, x_{n+2m+1+1}) &\leq s \lim_{n\to\infty} d(x_{n+1}, x_{n+2m+1}) \\
&\leq s^2 \lim_{n\to\infty} d(x_{n+1+1}, x_{n+2m-1}) \\
&\leq s^3 \lim_{n\to\infty} d(x_{n+1+2}, x_{n+2m-2})
\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&\leq s^m \lim_{n\to\infty} d(x_{n+m}, x_{n+m+1}) \\
&= 0.
\end{aligned}
$$

Thus

$$\lim_{n\to\infty} d(x_n, x_{n+p+1}) = 0.$$

Therefore, by Mathematical Induction $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ *for all* $p \in \mathbb{N}$. *So* $\{x_n\}$ *is a Cauchy sequence. Since* X *is complete, there exists an* $x \in X$ *such that* $\lim_{n\to\infty} x_n = x$. *So* $\lim_{n\to\infty} T^i x_n = \lim_{n\to\infty} x_{n+1} = x$ *i.e.,* $\lim_{n\to\infty} d(T^i x_n, x) = 0$.

Now

$$\lim_{n\to\infty} d(T^i x_n, x) \leq \lim_{n\to\infty} s[d(T^i x_n, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, x)]$$

$$= s \lim_{n\to\infty} d(T^i x_n, x_{n+1}). \tag{2.5}$$

Again,

$$
\begin{aligned}
&\lim_{n\to\infty} d(T^i x, x_{n+1}) \\
&= \lim_{n\to\infty} d(T^i x, T^j x_n) \\
&\leq \lim_{n\to\infty} \alpha \max\{d(x, x_n), d(x, T^i x), d(x_n, T^j x_n), d(x, T^j x_n), d(x_n, T^i x)\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + Ld(x_n, T^i x),
\end{aligned}
$$

$$= \alpha \max\{0, \lim_{n\to\infty} d(x, T^i x), 0, 0, \lim_{n\to\infty} d(x_n, T^i x)\} + Ld(x_n, T^i x). \tag{2.6}$$

If

$$\lim_{n\to\infty} d(x, T^i x) \leq \lim_{n\to\infty} d(x_n, T^i x)\},$$

then from above (2.6) we get,

$$
\begin{aligned}
\lim_{n\to\infty} d(T^i x, x_{n+1}) &\leq \lim_{n\to\infty} (\alpha + L) d(x_n, T^i x)\} \\
&\leq \lim_{n\to\infty} (\alpha + L)^2 d(x_{n-1}, T^i x)\} \\
&\vdots \\
&\leq \lim_{n\to\infty} (\alpha + L)^{n+1} d(x_0, T^i x)\}
\end{aligned}
$$

implies,

$$\lim_{n\to\infty} d(T^i x, x_{n+1}) = 0[\textit{ since } \ \alpha + L < 1].$$

Again form (2.5) we get,

$$\lim_{n\to\infty} d(T^i x, x) \leq \lim_{n\to\infty} sd(T^i x, x_{n+1}) = 0.$$

Therefore, $d(T^i x, x) = 0$ *implies,* $T^i x = x$.
If $\lim_{n\to\infty} d(T^i x, x_n) \leq \lim_{n\to\infty} d(T^i x, x)$, *then from (2.6) we get,*

$$\lim_{n\to\infty} d(T^i x, x_{n+1}) \leq \lim_{n\to\infty} (\alpha + L) d(T^i x, x)\}.$$

Therefore from (2.5) we get,

$$d(T^i x, x) \leq \lim_{n\to\infty} (\alpha + L) d(T^i x, x)\} < d(T^i x, x),$$

a contradiction.
Thus x *is a common fixed point of* $\{T^i\}$.

Let, y *be another common fixed point.*
Then

$$\begin{aligned}
d(x,y) &= d(T^i x, T^j y) \\
&\leq \alpha \max\{d(x,y), d(x,T^i x), d(y,T^j y), d(x,T^j y), d(y,T^i x)\} + Ld(y,T^i x) \\
&= \alpha \max\{d(x,y), d(x,x), d(y,y), d(x,y), d(y,x)\} + Ld(y,x) \\
&= (\alpha + L) d(x,y) \\
&< d(x,y),
\end{aligned}$$

which is a contradiction.

Therefore, $d(x,y) = 0$ *implies,* $x = y$.
Hence $\{T^i\}$ *have unique common fixed point in* X. □

Note: The theorem is a partial solution of **Open Problem 2** of George et al.[6] another Cirić type [c.f [2]].

Corollary 2.1. *Let* (X,d) *be a complete rectangular* $b-$*metric space with coefficient* $s > 1$ *and* T_1 *and* T_2 *be two self-maps satisfying the condition*

$$d(T_1 x, T_2 y) \leq \alpha \max\{d(x,y), d(x,T_1 x), d(y,T_2 y), d(x,T_2 y), d(y,T_1 x)\} + Ld(y,T_1 x),$$

where the constants $\alpha, L \geq 0$ *and* $\alpha + L < 1$. *Then the sequence* T_1 *and* T_2 *have unique common fixed point in* X.

Proof. Putting $T^i = T_1$ *and* $T^j = T_2$ *in the above* ***Theorem 2.1*** *we get the result.* □

Corollary 2.2. *Let* (X,d) *be a complete rectangular* $b-$*metric space with coefficient* $s > 1$ *and* T *be a self-map satisfying the condition*

$$d(Tx, Ty) \leq \alpha \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} + Ld(y,Tx),$$

where the constants $\alpha, L \geq 0$ *and* $\alpha + L < 1$. *Then the sequence* T *have a unique fixed point in* X.

Proof. *Putting $T^i = T^j = T$ in the above* ***Theorem 2.1*** *we get the desired result.* □

Theorem 2.2. *Let (X, d) be a complete rectangular b-metric space with coefficient $s > 1$. Let $T : X \to X$ satisfying*

$$d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Tx) + d(y, Ty)]\}$$

where $k \in (0, 1)$. Then T has a unique fixed point.

Proof. *Let us consider x_0 in X as an initial point. Let $\{x_n\}$ be a sequence given by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_n = Tx_n$ i.e., $x_n = x_{n+1}$, then for all $n \in \mathbb{N}$, x_n is a fixed point of T. So we assume that $x_n \neq x_{n+1}$.*
Now

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq k \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\ &\qquad \frac{1}{2}[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)]\} \\ &\leq k \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &\leq k \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}).\} \end{aligned}$$

Suppose $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$. Then from above we get

$$d(x_n, x_{n+1}) \leq kd(x_n, x_{n+1}),$$

which is a contradiction.
Therefore, $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$. Thus $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. So it converges to a (say).
Then

$$\begin{aligned} a &= \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(Tx_{n-1}, Tx_n) \\ &\leq k \lim_{n\to\infty} \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \\ &\qquad \frac{1}{2}[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)]\} \\ &= k \lim_{n\to\infty} \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= k \lim_{n\to\infty} d(x_{n-1}, x_n) = k\ a \end{aligned}$$

implies, $a = 0$ i.e., $\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$.
Next, we show that $\{x_n\}$ is a Cauchy sequence i.e., $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$.
First we suppose that $p =$ odd i.e., $p = 2m + 1, m \in \mathbb{N}$.
Then

$$d(x_n, x_{n+2m+1}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})]$$

$$\leq 2sd(x_n, x_{n+1}) + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})]$$
$$\leq 2sd(x_n, x_{n+1}) + 2s^2 d(x_{n+2}, x_{n+3}) + \ldots + 2s^m \ d(x_{n+2m}, x_{n+2m+1})$$
$$\leq 2s[1 + s + s^2 + \ldots + s^{m-1}]d(x_n, x_{n+1})$$
$$= 2s(\frac{s^{m-1} - 1}{s - 1})d(x_n, x_{n+1}).$$

Therefore,

$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ *as* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$

Again suppose $p =$ *even* $= 2m, m \in \mathbb{N}.$

Then

$$d(x_n, x_{n+2m}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})]$$
$$\leq 2sd(x_n, x_{n+1}) + 2s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m})$$
$$\leq 2sd(x_n, x_{n+1}) + 2s^2 d(x_{n+2}, x_{n+3}) + \ldots + 2s^m \ d(x_{n+2m-1}, x_{n+2m})$$
$$\leq 2s[1 + s + s^2 + \ldots + s^{m-1}]d(x_n, x_{n+1})$$
$$= 2s(\frac{s^{m-1} - 1}{s - 1})d(x_n, x_{n+1}).$$

Therefore again we get,

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0.$$

Thus $\{x_n\}$ *is a Cauchy sequence. Since* X *is a complete space, there exists an* $x \in X$ *such that*

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

Now we show that x *is a fixed point of* T.

Since

$$\lim_{n\to\infty} d(x_{n+1}, Tx) = \lim_{n\to\infty} d(Tx_n, Tx)$$
$$\leq k \lim_{n\to\infty} \max\{d(x_n, x), d(x_n, Tx_n), d(x, Tx), \frac{1}{2}[d(x_n, Tx_n) + d(x, Tx)]\}$$
$$\leq k \lim_{n\to\infty} \max\{d(x_n, x), d(x_n, x_{n+1}), d(x, Tx)\}$$
$$\leq k \lim_{n\to\infty} d(x, Tx)$$

which implies, $d(x, Tx) = 0$ *i.e.,* x *is a fixed point of* T.

To show the uniqueness, let x' *be another fixed point of* T.

Then

$d(x, x') = d(Tx, Tx')$

$\leq k \max\{d(x, x'), d(x, Tx), d(x', Tx'), \frac{1}{2}[d(x, Tx) + d(x', Tx')]\}$

$\leq k \max\{d(x, x'), d(x, x), d(x', x'), \frac{1}{2}[d(x, x) + d(x', x')]\}$

$= kd(x, x')$

which implies, $d(x, x') = 0$ *i.e.,* x *is unique.*

Hence the result. □

Note: This theorem is a partial solution of the **Open Problem 2** of George et al.[6] of Cirić type.

The next theorem is also a partial solution of **Open Problem 2** of George et al.[6] of Hardy-Rogers Type contraction.

Theorem 2.3. *Let (X, d) be a complete rectangular $b-$metric space with coefficient $s > 1$. Let $T : X \to X$ be a self-map satisfying the relation*

$$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Ty) + \alpha_5 d(y, Tx) \tag{2.7}$$

where $\alpha_i \geq 0, \forall i = 1, 2, 3, 4, 5$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < \frac{1}{s}$. Then T has a unique fixed point.

Proof. *Let $x_0 \in X$ be an initial approximation. We construct a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Suppose $d_n(x_n, x_{n+1})$ and $d^*_n(x_n, x_{n+2})$. Then byn the given condition (2.7) we get*

$$\begin{aligned} d_n = d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, Tx_{n-1}) + \alpha_3 d(x_n, Tx_n) + \alpha_4 d(x_{n-1}, Tx_n) \\ &\qquad + \alpha_5 d(x_n, Tx_{n-1}) \\ &= \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x_n, x_{n+1}) + \alpha_4 d(x_{n-1}, x_{n+1}) \\ &\qquad + \alpha_5 d(x_n, x_n) \\ &= (\alpha_1 + \alpha_2) d_{n-1} + \alpha_3 d_n + \alpha_4 d^*_{n-1} \end{aligned}$$

$$\text{implies, } (1 - \alpha_3) d_n \leq (\alpha_1 + \alpha_2) d_{n-1} + \alpha_4 d^*_{n-1}. \tag{2.8}$$

If $d_{n-1} \leq d^*_{n-1}$, then from (2.8) we get,
$(1 - \alpha_3) d_n \leq (\alpha_1 + \alpha_2 + \alpha_4) d^*_{n-1}$
implies,

$$d_n \leq (\frac{\alpha_1 + \alpha_2 + \alpha_4}{1 - \alpha_3}) d^*_{n-1} = k d^*_{n-1} \leq k^2 d^*_{n-2} \leq \quad \ldots \quad \leq k^n d^*_0 \ [k = \frac{\alpha_1 + \alpha_2 + \alpha_4}{1 - \alpha_3} < 1\,]$$

implies, $d_n \to 0$ as $n \to \infty$.
If $d_{n-1^*} \leq d_{n-1}$, then from (2.8) ,we get

$$(1 - \alpha_3) d_n \leq (\alpha_1 + \alpha_2 + \alpha_4) d_{n-1}$$

implies,

$$d_n \leq (\frac{\alpha_1 + \alpha_2 + \alpha_4}{1 - \alpha_3}) d_{n-1}$$

from which we get as above $d_n \to 0$ as $n \to \infty$.
Now we show that $\{x_n\}$ isa a Cauchy sequence. We show this by Marthematical Induction on $p \in \mathbb{N}$ to established

$$\lim_{n \to \infty} d(x_n, x_{n+p}) = 0. \tag{2.9}$$

Clearly (2.9) holds for $p = 1$. *Suppose it holds for* p *i.e.,* $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$.
So $\lim_{n\to\infty} d(x_{n+1}, x_{n+p+1}) = 0$.
Thus

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, x_{n+p+1}) &= \lim_{n\to\infty} d(Tx_{n-1}, Tx_{n+p}) \\
&\leq \lim_{n\to\infty} [\alpha_1 d(x_{n-1}, x_{n+p}) + \alpha_2 d(x_{n-1}, Tx_{n-1}) + \alpha_3 d(x_{n+p}, Tx_{n+p}) \\
&\qquad + \alpha_4 d(x_{n-1}, Tx_{n+p}) + \alpha_5 d(x_{n+p}, Tx_{n-1})] \\
&\leq \lim_{n\to\infty} [\alpha_1 d(x_{n-1}, x_{n+p}) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(x_{n+p}, x_{n+p+1}) \\
&\qquad + \alpha_4 d(x_{n-1}, x_{n+p+1}) + \alpha_5 d(x_{n+p}, x_n)] \\
&= \lim_{n\to\infty} \alpha_1 d(x_{n-1}, x_{n+p}) + \lim_{n\to\infty} \alpha_4 d(x_{n-1}, x_{n+p+1}) \\
&\leq \lim_{n\to\infty} \alpha_1 s[d(x_{n-1}, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, x_{n+p})] \\
&\qquad + \lim_{n\to\infty} \alpha_4 s[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p+1})] \\
&= \lim_{n\to\infty} \alpha_1 s d^*_{n-1} + \lim_{n\to\infty} \alpha_4 s.0 \\
&= \lim_{n\to\infty} s\alpha_1 d^*_{n-1}. \qquad (2.10)
\end{aligned}$$

Again,

$$\begin{aligned}
\lim_{n\to\infty} d^*_{n-1} &= \lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = \lim_{n\to\infty} d(Tx_{n-2}, Tx_n) \\
&\leq \lim_{n\to\infty} [\alpha_1 d(x_{n-2}, x_n) + \alpha_2 d(x_{n-2}, Tx_{n-2}) + \alpha_3 d(x_n, Tx_n) \\
&\qquad + \alpha_4 d(x_{n-2}, Tx_n) + \alpha_5 d(x_n, Tx_{n-2})] \\
&= \lim_{n\to\infty} [\alpha_1 d(x_{n-2}, x_n) + \alpha_2 d(x_{n-2}, x_{n-1}) + \alpha_3 d(x_n, x_{n+1}) \\
&\qquad + \alpha_4 d(x_{n-2}, x_{n+1}) + \alpha_5 d(x_n, x_{n-1})] \\
&= \lim_{n\to\infty} \alpha_1 d(x_{n-2}, x_n) + \lim_{n\to\infty} \alpha_4 s[d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
&= \lim_{n\to\infty} \alpha_1 d^*_{n-2} \\
&\leq \lim_{n\to\infty} \alpha_1^2 d^*_{n-3} \\
&\quad \vdots \\
&\leq \lim_{n\to\infty} \alpha_1^{n-1} d^*_0 \\
&= 0.
\end{aligned}$$

Thus from (2.10) we get, $\lim_{n\to\infty} d(x_n, x_{n+p+1}) = 0$.

Therefore, $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ *for all* $p \in \mathbb{N}$.
Thus $\{x_n\}$ *is a Cauchy sequence in* X. *Since* X *is a complete RbMS, there exists*

an $x \in x$ *such that* $\lim_{n\to\infty} x_n = x$.
Now

$$\begin{aligned} d(Tx,x) &\le s[d(Tx,x_{n+1}) + d(x_{n+1},x_n) + d(x_n,x)] \\ &= s[d(Tx,Tx_n) + d(x_{n+1},x_n) + d(x_n,x)] \\ &\le s[\alpha_1 d(x,x_n) + \alpha_2 d(x,Tx) + \alpha_3 d(x_n,Tx_n) \\ &\qquad +\alpha_4 d(x,Tx_n) + \alpha_5 d(x_n,Tx) + d(x_{n+1},x_n) + d(x_n,x)] \end{aligned}$$

$$= s[\alpha_1 d(x,x_n) + \alpha_2 d(x,Tx) + \alpha_3 d(x_n,x_{n+1}) + \alpha_4 d(x,x_{n+1}) +\alpha_5 d(x_n,Tx) + d(x_{n+1},x_n) + d(x_n,x)]. \tag{2.11}$$

Again,

$$d(x_n,Tx) = d(Tx_{n-1},Tx)$$

$$\le \alpha_1 d(x_{n-1},x)+\alpha_2 d(x_{n-1},Tx_{n-1})+\alpha_3 d(x,Tx)+\alpha_4 d(x_{n-1},Tx)+\alpha_5 d(x,Tx_{n-1})$$

$$= \alpha_1 d(x_{n-1},x) + \alpha_2 d(x_{n-1},x_n) + \alpha_3 d(x,Tx) + \alpha_4 d(x_{n-1},Tx) + \alpha_5 d(x,x_n). \tag{2.12}$$

Suppose, $d(x,Tx) \le d(x_{n-1},Tx)$. *Then from (2.12) we get,*

$$d(x_n,Tx) \le \alpha_1 d(x_{n-1},x) + \alpha_2 d(x_{n-1},x_n) + (\alpha_3+\alpha_4) d(x_{n-1},Tx) + \alpha_5 d(x,x_n)$$

implies,

$$\begin{aligned} \lim_{n\to\infty} d(x_n,Tx) &\le \lim_{n\to\infty} (\alpha_3+\alpha_4) d(x_{n-1},Tx) \\ &\le \lim_{n\to\infty} (\alpha_3+\alpha_4)^2 d(x_{n-2},Tx) \\ &\vdots \\ &\le \lim_{n\to\infty} (\alpha_3+\alpha_4)^n d(x_0,Tx) = 0. \end{aligned}$$

Thus from (2.11) we get,

$$\lim_{n\to\infty} d(Tx,x) \le s\alpha_2 \lim_{n\to\infty} d(Tx,x)$$

implies, $d(Tx,x) = 0$

implies, $Tx = x$.

Again suppose, $d(x_{n-1},Tx) \le d(x,Tx)$. *Then from (2.12) we get,*

$$d(x_n,Tx) \le \alpha_1 d(x_{n-1},x) + \alpha_2 d(x_{n-1},x_n) + (\alpha_3+\alpha_4) d(x,Tx) + \alpha_5 d(x,x_n).$$

Therefore,

$\lim_{n\to\infty} d(x_n, Tx) \leq \lim_{n\to\infty}(\alpha_3 + \alpha_4)d(x, Tx)$.
From (2.11) we get,

$$d(Tx, x) \leq s[\alpha_2 d(x, Tx) + \lim_{n\to\infty} \alpha_5 d(x_n, Tx)]$$

$$\leq s\alpha_5(\alpha_3 + \alpha_5)(\alpha_3 + \alpha_4)d(x, Tx)$$

$$\leq s\alpha_5 d(Tx, x)$$

$$\textit{implies,} \quad d(Tx, x) = 0.$$

Therefore, x a fixed point of T.

Suppose, y be another fixed point of T.
Then

$$d(x, y) = d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Ty) + \alpha_5 d(y, Tx)$$

$$= \alpha_1 d(x, y) + \alpha_2 d(x, x) + \alpha_3 d(y, y) + \alpha_4 d(x, y) + \alpha_5 d(y, x)$$

$$= (\alpha_1 + \alpha_4 + \alpha_5)d(x, y),$$

implies, $[1 - (\alpha_1 + \alpha_4 + \alpha_5)]d(x, y) = 0$ *i.e.,* $x = y$.

Thus x is a unique fixed point of T.
Hence the theorem. □

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