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ON MULTIVALUED θ-CONTRACTIONS OF BERINDE TYPE WITH AN APPLICATION TO FRACTIONAL DIFFERENTIAL INCLUSIONS

Maroua Meneceur and Said Beloul

Operators theory and PDE Laboratory, Department of Mathematics, Exact Sciences Faculty, University of El Oued, P. O. Box 789 El Oued 39000, Algeria.

Abstract. In this paper, we discuss the existence of fixed points for Berinde type multivalued θ- contractions. An example is provided to demonstrate our findings and, as an application, the existence of the solutions for a nonlinear fractional inclusions boundary value problem with integral boundary conditions is given to illustrate the utility of our results.
Keywords: fixed point, θ contraction, α-admissible, fractional differential inclusions.

1. Introduction and preliminaries

Multivalued fixed point theory has been known some development, starting with the results of Nadler [21], where he proved the existence of multivalued fixed point using the Hausdorff metric, later, some generalizations were given in this way, for example, see [4, 10, 13, 27] and references therein.
Berinde [7] introduced the concept of almost contractions as a generalization to weak contractions notion in the context of single valued mappings, which was later extended to the multivalued case in [8, 9], and some results were obtained using this concept. .
Samet et al. [23] introduced a new concept called α-admissible and they obtained some fixed point results for $\alpha - \psi$-contractive mappings, later, some results were

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Corresponding Author: Said Beloul, Operators theory and PDE Laboratory, Department of Mathematics, Exact Sciences Faculty, University of El Oued, P. O. Box 789 El Oued 39000, Algeria. | E-mail: beloulsaid@gmail.com
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established in this direction, see for example [2, 14, 15, 20]. Recently, Jleli and Samet [18] introduced θ-contractions type and demonstrated the existence of fixed points for such contractions. It is worth noting here, that a Banach contraction is a particular case of θ contraction, whereas there are some θ-contractions that are not Banach contraction. Following that, several authors investigated various variants of θ-contraction for single-valued and multivalued mappings, for example, see [1, 11, 12, 28].
In this work, we combine the concept of α-admissible mappings with the concept of θ-contractions type in the context of multivalued mappings to demonstrate the existence of a fixed point for such new contractions type in complete metric spaces. Using our main results, we also deduce the existence of a fixed point in partially ordered metric spaces and in metric spaces endowed with a graph. Finally, to demonstrate the significance of the obtained results, we provide an example and an application of the existence of solutions for a fractional differential inclusion.
Denote by $CL(X)$ the family of nonempty and closed subsets of X, the family of nonempty, bounded and closed subsets of X is denoted by $CB(X)$ and the family of nonempty and compact subsets of X is denoted by $K(X)$.
Let (X, d) be a metric space, and the Pompeiu-Hausdorff metric is defined as a function $H: CL(X) \times CL(X) \to [0, \infty]$ which is defined by:

$$H(A,B) = \begin{cases} \max\left\{\sup\limits_{x \in A} d(x,B),\ \sup\limits_{y \in B} d(y,A)\right\} & \text{if the maximum exists;} \\ \infty, & \text{otherwise,} \end{cases}$$

where $d(a, B) = \inf\{d(a, b): b \in B\}$. Note that, if $A = \{a\}$ (singleton) and $B = \{b\}$, then $H(A, B) = d(a, b)$.

Lemma 1.1. *[21] Let (X, d) be a metric space and $A, B \in CL(X)$ with $H(A, B) > 0$. Then, for each $h > 1$ and for each $a \in A$, there exists $b = b(a) \in B$ such that $d(a, b) < hH(A, B)$.*

Now, we'll look at some fundamental definitions of α-admissibility and α-continuity concepts.

Definition 1.1. Let (X, d) be a metric space and $\alpha : X \times X \to [0, +\infty)$ be a given mapping. A mapping $T : X \to CL(X)$ is

- α-admissible [2], if for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$ we have $\alpha(y, z) \geq 1$, for all $z \in Ty$.
- α-lower semi-continuous [14], if for $x \in X$ and a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} d(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, implies

$$\lim_{n\to\infty} \inf d(x_n, Tx_n) \geq d(x, Tx).$$

Definition 1.2. [18] Let Θ be the set of all functions $\theta : (0, +\infty) \to (1, +\infty)$ satisfying:

(θ_1) : θ is non decreasing,

(θ_2) : for each sequence $\{t_n\}$ in $(0, +\infty)$, $\lim_{n\to\infty} t_n = 1$ if and only if $\lim_{n\to\infty} t_n = 0$,

(θ_3) : there exists $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0^+} \dfrac{\theta(t)-1}{t^r} = l$.

Example 1.1. Let $\theta_i : (0,+\infty) \to (1,+\infty)$, $i \in \{1,2,3\}$, defined by:

1. $\theta_1(t) = e^t$.
2. $\theta_2(t) = e^{te^t}$.
3. $\theta_3(t) = e^{\sqrt{x}}$.
4. $\theta_4(t) = e^{\sqrt{t}e^t}$.

Then $\theta_i \in \Theta$, for each $i \in \{1,2,3\}$.

Throughout this paper, we will denote by Φ the set of all continuous functions $\psi : [0,+\infty) \to [0,+\infty)$ satisfying:

(1) : ψ is nondecreasing ,

(2) : $\sum_{i=1}^{\infty} \psi^n(t) < \infty$, for all $t \in [0,+\infty)$.

Clearly, if $\psi \in \Psi$, then $\psi(t) < t$, for all $t \in [0,+\infty)$.

2. Main results

Definition 2.1. Let (X,d) be a metric space and $\alpha : X \times X \to \mathbb{R}$. A mapping $T : X \to CL(X)$ is called a generalized almost (α,ψ,θ,k) contraction, if there exists a function $\theta \in \Theta$, $\psi \in \Psi$, $L \geq 0$ and $k : (0,\infty) \to [0,1)$ satisfies $\lim_{t\to s^+} \sup k(t) < 1$ for all $s \in (0,\infty)$ such that

$$\theta(H(Tx,Ty)) \leq \Big[\theta(\psi(M(x,y))\Big]^{k(M(x,y))} + LN(x,y), \tag{2.1}$$

for all $x, y \in X$ with $\alpha(x,y) \geq 1$ and $H(Tx,Ty) > 0$, where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}$$

and $N(x,y) = min\{d(x,Ty), d(y,Tx)\}$.

Theorem 2.1. *Let (X,d) be a complete metric space and $T : X \to K(X)$ be a generalized almost (α,ψ,θ,k) contraction, with $\theta \in \Theta$. Assume that the following conditions are satisfied:*

1. *T is α-admissible.*
2. *There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$.*
3. *T is α-lower semi-continuous, or X is α-regular, that is, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. From (2) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$, then $H(Tx_0, Tx_1) \geq d(x_1, Tx_1) > 0$, otherwise $x_1 \in Tx_1$, or, $x_0 = x_1$, which implies x_1 is a fixed point and the proof completes. For $H(Tx_0, Tx_1) > 0$ using (2.1) we get:

$$\theta(d(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1))$$

$$\leq \Big[\theta(\psi(d(x_0, x_1)))\Big]^{k(d(x_0,x_1)} + Ld(x_1, Tx_0) < [\theta(M(x_0, x_1))]^{k(M(x_0,x_1)}.$$

If $d(x_0, x_1) \leq d(x_1, Tx_1)$, we get

$$\theta(d(x_1, Tx_1)) \leq \Big[\theta(\psi(d(x_1, Tx_1)))\Big]^{k(d(x_1,Tx_1))} + LN(x_0, x_1) < \theta(d(x_1, Tx_1),$$

which is a contradiction. Then we have

$$\theta(d(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1)) \leq \Big[\theta(\psi(d(x_0, x_1)))\Big]^{k(d(x_0,x_1)}.$$

Since Tx_1 is compact, then there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1, x_2)) = \theta(d(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1))$$

$$\leq \Big[\theta(d(x_0, x_1))\Big]^{k(d(x_0,x_1)} < \theta(d(x_0, x_1)).$$

If $x_1 = x_2$, or $x_2 \in Tx_2$, then x_2 is a fixed point. Suppose $x_1 \neq x_2$ and $x_2 \notin Tx_2$, so $H(Tx_2, Tx_1) > 0$ and since T is α-admissible we have $\alpha(x_1, x_2) \geq 1$. Using (2.1) we get:

$$\theta(d(x_2, Tx_2)) \leq \theta(H(Tx_1, Tx_2)) \leq \Big[\theta(\psi(M(x_1, x_2)))\Big]^{k(M(x_1,x_2)} + LN(x_1, x_2)$$

$$= \Big[\theta(d(x_1, x_2))\Big]^{k(M(x_1,x_2))}.$$

If $d(x_1, x_2) \leq d(x_2, Tx_2)$, we get

$$\theta(d(x_2, Tx_2)) \leq \Big[\theta(\psi(d(x_2, Tx_2)))\Big]^{k(d(x_2,Tx_2))} + LN(x_1, x_2) < \theta(d(x_2, Tx_2),$$

which is a contradiction. Then we have

$$\theta(d(x_2, Tx_2)) \leq \theta(H(Tx_0, Tx_1)) \leq \Big[\theta(\psi(d(x_1, x_2)))\Big]^{k(d(x_1,x_2)}.$$

The compactness of Tx_2 implies that there exists $x_3 \in Tx_2$ such that

$$\theta(d(x_2,x_3)) = \theta(d(x_2,Tx_2)) \le \theta(H(Tx_1,Tx_2))$$
$$\le \Big[\theta(d(x_1,x_2))\Big]^{k(d(x_1,x_2))} < \theta(d(x_1,x_2)).$$

Continuing in this manner we can construct a sequence (x_n) in X, if $x_n = x_{n+1}$ or $x_{n+1} \in Tx_{n+1}$, then x_{n+1} is a fixed point, otherwise we get

$$\theta(d(x_n,Tx_{n+1})) \le \Big[\theta(\psi(M(x_n,x_{n-1})))\Big]^{k(M(x_n,x_{n-1})} + LN(x_n,x_{n-1}).$$

As the same arguments in previous steps, we get

$$d(x_{n+1},Tx_{n+1}) \le d(x_n,x_{n+1}),$$

so we obtain

$$\theta(d(x_n,x_{n+1})) \le \theta(H(Tx_n,Tx_{n-1})) \le \Big[\theta(\psi(d(x_n,x_{n-1})))\Big]^{k(d(x_n,x_{n+1})}$$
$$= \Big[\theta(\psi(d(x_n,x_{n-1})))\Big]^{k(d(x_n,x_{n-1}))} < \theta(d(x_n,x_{n-1})).$$

Since θ is increasing, then the sequence $(d(x_n,x_{n+1}))_n$ is decreasing, further it is bounded at below so it is convergent. On the other hand, $\lim\limits_{t\to s^+}\sup k(t) < 1$, then there exists $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$ such that $k(d(x_n,x_{n+1})) < \delta$, for all $n \ge n_0$. Thus we have

$$1 < \theta(d(x_n,x_{n+1})) \le \Big[\theta(d(x_{n_0},x_{n_0+1}))\Big]^{\delta^{n-n_0}}, \tag{2.2}$$

for all $n \ge n_0$.
Letting $n \to \infty$ in (2.2), we get

$$\lim_{n\to\infty} \theta(d(x_n,x_{n+1})) = 1,$$

By (θ_2), we infer that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

Now, we prove $\{x_n\}$ is a Cauchy sequence, from (θ_3) there exist $r \in [0,1)$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(x_n,x_{n+1})) - 1}{(d(x_n,x_{n+1})^r} = l.$$

If $l < \infty$, let $2\varepsilon = l$, so from the definition of limit there exists $n_1 \in \mathbb{N}$ such that for all $n \ge n_1$, we have

$$\varepsilon = l - \varepsilon < \frac{\theta(d(x_n,x_{n+1})) - 1}{(d(x_n,x_{n+1})^r}$$
$$(d(x_n,x_{n+1}))^r < \frac{\theta(d(x_n,x_{n+1})) - 1}{\varepsilon}.$$

Then (2.2) gives

$$n(d(x_n, x_{n+1}))^r < \frac{n(\theta(d(x_0, x_1))^{\delta^{n-n_0}} - 1)}{\varepsilon}. \tag{2.3}$$

In the case where $l = \infty$, let A be an arbitrary positive real number, so from the definition of the limit there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{(d(x_n, x_{n+1}))^r} > A,$$

which implies that

$$n(d(x_n, x_{n+1}))^r \leq \frac{n(\theta(d(x_0, x_1))^{\delta^{n-n_0}} - 1)}{A}. \tag{2.4}$$

Letting $n \to \infty$ in (2.4)(or in (2.3), we obtain

$$\lim_{n\to\infty} n(d(x_n, x_{n+1}))^r = 0.$$

From the definition of the limit, there exists $n_2 \geq \max\{n_0, n_1\}$ such that for all $n \geq n_2$, we have

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{r}}},$$

This implies

$$\sum_{n=n_2}^{\infty} d(x_n, x_{n+1}) \leq \sum_{1}^{\infty} \frac{1}{n^{\frac{1}{r}}} < \infty.$$

Then $\{x_n\}$ is a Cauchy sequence.
The completness of (X, d) implies that $\{x_n\}$ converges to a some $x \in X$.
Now, we show that x is a fixed point of T. In fact, if T is α-lower continuous, then for all $n \in \mathbb{N}$ we have

$$0 \leq d(x_n, Tx_n) \leq d(x_n, x_{n+1}).$$

Letting $n \to +\infty$, we get

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

The α-lower semi continuity of T implies

$$0 \leq d(x, Tx) < \lim_{n\to\infty} \inf d(x_n, Tx_n) = 0.$$

Hence $d(x, Tx) = 0$ and x is a fixed point of T.
If X is regular, so $\alpha(x_n, x) \geq 1$ and $H(Tx_n, Tx) > 0$, by using (2.1) we get

$$1 < \theta(d(x_{n+1}, Tx)) \leq \theta(H(Tx_n, Tx)) < \Big[\theta(d(x_0, x_1))\Big]^{\delta^{n-n_0}}.$$

Letting $n \to +\infty$, we get

$$\lim_{n\to\infty} \theta(d(x_n, Tx)) = 1,$$

so (θ_2) gives

$$\lim_{n\to\infty} d(x_n, Tx) = 0,$$

which implies that $x \in Tx$. $\square$

Theorem 2.2. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a generalized almost (α, ψ, θ) contraction, with θ is right continuous. Assume that the following conditions are satisfied:*

(H_1) : *T is α-admissible,*

(H_2) : *there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$,*

(H_3) : *for every sequence $\{x_n\}$ in X converging to $x \in X$ with $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$, for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. From (H_2) there are $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$, if $x_0 = x_1$, or, $x_1 \in Tx_1$, so x_1 is a fixed point. Suppose the contrary, then $H(Tx_0, Tx_1) \geq d(x_1, Tx_1) > 0$ and by using (2.1) we get

$$\theta(d(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1)) \leq \Big[\theta(\psi(M(x_0, x_1)))\Big]^{k(M(x_0,x_1)} + LN(x_0, x_1)$$

$$< \Big[\theta(M(x_0, x_1))\Big]^{k(M(x_0,x_1)} + LN(x_0, x_1).$$

By right continuity of θ, there exists $h > 1$ such that

$$\theta(hH(Tx_0, Tx_1)) \leq \Big[\theta(\psi(M(x_0, x_1)))\Big]^{k(M(x_0,x_1))} + LN(x_0, x_1).$$

As in proof of Theorem 2.1 we get $M(x_0, x_1) = d(x_0, x_1)$ and $N(x_0, x_1) = 0$, then by using Lemma 1.1, there exist $x_2 \in Tx_1$ and $h_1 > 1$ such that

$$\theta(d(x_1, x_2)) \leq \theta(h_1 H(Tx_0, Tx_1)) \leq \Big[\theta(\psi(d(x_0, x_1)))\Big]^{k(d(x_0,x_1))}$$

$$< \Big[\theta(\psi(d(x_0, x_1)))\Big]^{k(d(x_0,x_1))} < \theta(d(x_0, x_1)).$$

Since T is α-admissible, then $\alpha(x_1, x_2) \geq 1$. Assume that $x_1 \neq x_2$ and $x_2 \in Tx_2$, so $H(Tx_1, Tx_2) \geq d(x_2, Tx_2) > 0$ and using (2.1), we obtain

$$1 < \theta(d(x_2, Tx_2)) \leq \theta(H(Tx_1, Tx_2)) \leq \Big[\theta(\psi(M(x_1, x_2)))\Big]^{k(M(x_1,x_2))} + LN(x_1, x_2)$$

$$< \Big[\theta(d(x_1, x_2))\Big]^{k(d(x_1,x_2))}.$$

As in previous step, we have $M(x_1, x_2) = d(x_1, x_2)$, so we get

$$\theta(d(x_2, Tx_2)) \le \theta(H(Tx_1, Tx_2)) \le \Big[\theta(\psi(d(x_1, x_2)))\Big]^{k(d(x_1,x_2))}$$
$$< \Big[\theta(d(x_1, x_2))\Big]^{k(d(x_1,x_2))}.$$

Since θ is right continuous and from Lemma 1.1, there exists $h_2 > 1$ and $x_3 \in Tx_2$ such that

$$\theta(d(x_2, x_3)) \le \theta(h_2 H(Tx_1, Tx_2)) \le \Big[\theta(\psi(d(x_1, x_2)\Big]^{k(d(x_1,x_2))}.$$
$$< \Big[\theta(d(x_1, x_2))\Big]^{k(d(x_1,x_2))} < \theta(d(x_1, x_2)).$$

Continuing in this manner, we can construct two sequences $\{x_n\} \subset X$ and $(h_n) \subset (1, \infty)$ such that $x_n \ne x_{n+1}$, $x_{n+1} \in Tx_n$, $\alpha(x_n, x_{n+1}) \ge 1$ and

$$1 < \theta(d(x_n, x_{n+1})) \le \theta(h_n H(Tx_{n-1}, Tx_n))$$
$$\le \Big[\theta(d(x_n, x_{n-1}))\Big]^{k(d(x_n,x_{n-1}))} + LN(x_n, Tx_{n-1})$$
$$< \theta(d(x_n, x_{n-1})),$$

which implies that $(d(x_n, x_{n+1}))_n$ is a decreasing sequence and bounded at below, so there exist $\delta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that $k(d(x_n, x_{n+1})) < \delta$, for all $n \ge n_0$. Thus we have

$$1 < \theta(d(x_n, x_{n+1})) < \Big[\theta(d(x_0, x_1))\Big]^{\delta^{n-n_0}}, \tag{2.5}$$

for all $n \ge n_0$.
On taking the limit as $n \to \infty$, we get $\lim_{n\to\infty} \theta(d(x_n, x_{n+1})) = 1$, (θ_2) gives

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

The rest of the proof is like in the proof of Theorem 2.1. □

Corollary 2.1. *Let (X, d) be a complete metric space, $\alpha: X \times X \to [0, +\infty)$ be a function and $T: X \to K(X)$ (resp $CB(X)$ with θ is right continuous) be an α-admissible multivalued mapping and the following assertions hold:*

- *(i) T is α-admissible.*
- *(ii) There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \ge 1$.*
- *(iii) T is α-lower semi-continuous, or, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $\alpha(x_n, x_{n+1}) \ge 1$, for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \ge 1$, for all $n \in \mathbb{N}$.*

(*iv*) *There exist* $\theta \in \Theta$, $\psi \in \Psi$ *and a function* $k : (0,\infty) \to [0,1)$ *satisfying* $\lim_{t\to s^+} \sup k(t) < 1$ *such for* $x, y \in X$ $H(Tx,Ty) > 0$ *implies*

$$\alpha(x,y)\theta\big(H(Tx,Ty)\big) \le \theta\Big[(\psi(M(x,y)))\Big]^{k(M(x,y))} + LN(x,y), \tag{2.6}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Tx))\}$$

and $N(x,y) = \min\{d(x,Ty), d(y,Tx)\}$.

Then T *has a fixed point.*

Proof. Let $x, y \in X$ be such that $\alpha(x,y) \ge 1$ and $H(Tx,Ty) > 0$. So from (2.7) we get

$$\begin{aligned}\theta\big(H(Tx,Ty)\big) &\le \alpha(x,y)\theta\big(H(Tx,Ty)\big)\\ &\le \theta\Big[(\psi(M(x,y)))\Big]^{k(M(x,y))} + LN(x,y),\end{aligned}$$

which implies that the inequality (2.1) holds. Thus, the rest of proof is like in the proof of Theorem 2.2 (resp. Theorem 2.1). □

If $\alpha(x,y) = 1$, for all $x, y \in X$, we get the following corollary.

Corollary 2.2. *Let* (X,d) *be a complete metric space and* $T: X \to K(X)$ *(resp.* $CB(X)$ *with* θ *is right continuous) be a multivalued mapping such that there exists* $\theta \in \Theta$, $\psi \in \Psi$ *and a function* $k : (0,\infty) \to [0,1)$ *satisfying* $\lim_{t\to s^+} \sup k(t) < 1$ *for all* $s \in (0,\infty)$ *such that*

$$\theta\big(H(Tx,Ty)\big) \le \theta\Big[(\psi(M(x,y)))\Big]^{k(M(x,y))} + LN(x,y), \tag{2.7}$$

for $x, y \in X$ *with* $H(Tx,Ty) > 0$ *where*

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Tx))\}$$

and $N(x,y) = \min\{d(x,Ty), d(y,Tx)\}$. *Then* T *has a fixed point in* X.

Example 2.1. Let $X = \{1,2,3\}$ and $d(x,y) = |x-y|$. Define $T: X \to CB(X)$ and $\alpha : X \times X \to [0,\infty)$ by

$$Tx = \begin{cases} \{1\}, & x \in \{1,2\} \\ \{2\}, & x = 3 \end{cases}$$

and $\alpha(x,y) = e^{|x-y|}$. Taking $\theta(t) = e^t$, $\psi(t) = \frac{4}{5}t$ and $k(t) = \frac{1}{2}$.
Now, we show that the contractive condition holds.
For $x, y \in X$, we have $|x-y| \ge 0$, which implies $e^{|x-y|} \ge 1$. Then T is α-admissible.
On other hand, $H(Tx,Ty) > 0$ and $\alpha(x,y) \ge 1$ for all $(x,y) \in \{(1,3),(3,1),(2,3),(3,2)\}$.
Then we have the following cases:

1. for $x = 1$ and $y = 3$, we have

$$H(T1, T3) = 1, \quad d(1, 3) = 2, \quad \psi(d(1, 3)) = \frac{8}{5} \quad \text{and} \quad d(3, T1) = 2,$$

then

$$\begin{aligned} e = e^{H(T1,T3)} &< (e^{\psi(d(1,3))})^{\frac{1}{2}} + d(3, T1) \\ &= e^{\frac{4}{5}} + 2. \end{aligned}$$

2. For $x = 2$ and $y = 3$, we have

$$H(T2, T3) = 1, \quad d(2, 3) = 1, \quad \psi(d(1, 3)) = \frac{4}{5} \quad \text{and} \quad d(3, T2) = 2,$$

then

$$\begin{aligned} e = e^{H(T2,T3)} &< (e^{\psi(d(1,3))})^{\frac{1}{2}} + d(3, T2) \\ &= e^{\frac{2}{5}} + 2. \end{aligned}$$

There exists $x_0 = 2$ and $x_1 = 1 \in Tx_0$ such that $\alpha(2, 1) \geq 1$.

It is clear that T is α- lower semi continuous. Consequently, all conditions of Theorem 2.1 are satisfied. Then T has a fixed point which is 1.

3. Fixed point on partially ordered metric spaces

Now, we give an existence theorem of fixed point in a partially order metric space, by using the results provided in previous section.

Theorem 3.1. *Let $(X, \preceq, d)$ be a complete ordered metric space and $T: X \to CB(X)$ be a multivalued mapping. Assume that the following assertions hold:*

1. *For each $x \in X$ and $y \in Tx$ with $x \preceq y$, we have $y \preceq z$ for all $z \in Ty$;*
2. *There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$.*
3. *For every nondecreasing sequence $\{x_n\}$ in X such that $x_n \to x \in X$, we have $x_n \preceq x$, for all $n \in \mathbb{N}$.*
4. *There exists a right continuous function $\theta \in \Theta$, $\psi \in \Psi$ and $k : (0, \infty) \to [0, 1)$ satisfies $\lim\limits_{t \to s^+} \sup k(t) < 1$ for all $s \in (0, \infty)$ such that*

$$\theta(H(Tx, Ty)) \leq \Big[\theta(\psi(M(x, y)))\Big]^{k(M(x,y)} + LN(x, y), \tag{3.1}$$

for all $x, y \in X$ with $x \preceq y$ and $H(Tx, Ty) > 0$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))$$

and $N(x, y) = \min\{d(x, Ty), d(y, Tx)\}$.

Then T has a fixed point.

Proof. Define $\alpha: X \times X \to [0, +\infty)$ as follows:

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

From (1), for each $x \in X$ and $y \in Tx$ with $x \preceq y$, i.e., $\alpha(x,y) = 1 \geq 1$, we have $z \preceq y$, for all $z \in Ty$, i.e., $\alpha(x,y) = 1 \geq 1$. Thus T is α-admissible.
From (2), there exit $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$, i.e., $\alpha(x_0, x_1) = 1 \geq 1$.
Condition (3) implies α- lower semi continuity of T, or regularity of X.
From (4), for $x \preceq y$, we have $\alpha(x,y) = 1 \geq 1$ then the inequality (2.1) holds, which implies that T is a generalized almost $(\alpha, \psi, \theta, k)$ contraction. □

4. Fixed point on metric spaces endowed with a graph

In this section, as a consequence of our main results, we present an existence theorem of fixed point for a multivalued mapping in a metric space X, endowed with a graph, into the space of nonempty closed and bounded subsets of the metric space. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$, where $\Delta = \{(x,x) : x \in X\}$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$.

Theorem 4.1. *Let (X,d) be a complete metric space endowed with a graph G and $T: X \to CB(X)$ be a multivalued mapping. Assume that the following conditions are satisfied:*

1. *For each $x \in X$ and $y \in Tx$ with $(x,y) \in E(G)$, we have $(y,z) \in E(G)$ for all $z \in Ty$;*
2. *There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;*
3. *T is G-lower semi-continuous, that is, for $x \in X$ and a sequence $\{x_n\}$ in X with*
   $\lim_{n\to\infty} d(x_n, x) = 0$ *and* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in \mathbb{N}$, *implies*
   $$\liminf_{n\to\infty} d(x_n, Tx_n) \geq d(x, Tx)$$
   or, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$;
4. *There exists a right continuous function $\theta \in \Theta$, $\psi \in \Psi$ and $k : (0,\infty) \to [0,1)$ satisfing $\lim_{t\to s^+} \sup k(t) < 1$ for all $s \in (0,\infty)$ such that*
   $$\theta(H(Tx,Ty)) \leq \Big[\theta(\psi(M(x,y)))\Big]^{k(M(x,y))} + LN(x,y), \tag{4.1}$$
   for all $x, y \in X$ with $(x,y) \in E(G)$ and $H(Tx,Ty) > 0$, where
   $$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Tx))$$
   and $N(x,y) = \min\{d(x,Ty), d(y,Tx)\}$.

Then T has a fixed point.

Proof. This result is a direct consequence of results of Theorem 2.1 by taking the function $\alpha\colon X \times X \to [0, +\infty)$ defined by:

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G), \\ 0, & \text{otherwise,} \end{cases}$$

□

5. Application to fractional differential inclusions

Consider the following boundary value problem of fractional order differential inclusion with boundary integral conditions:

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \ 0 \le t \le 1, \ 1 < q \le 2 \\ ax(0) - bx'(0) = 0 \\ x(1) = \int_0^1 h(s)g(s, x(s))ds \end{cases} \tag{5.1}$$

where ${}^cD^q$, $1 < q \le 2$ is the Caputo fractional derivative, F, g, and h are given continuous functions, where
$F : [0, 1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{K}(\mathbb{R})$, $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$, $h \in L^1([0, 1])$, $a + b > 0$, $\frac{a}{a+b} < q - 1$ and $h_0 = \|h\|_{L^1}$.
Denote by $X = \mathcal{C}([0, 1], \mathbb{R})$ the Banach space of continuous functions $x : [0, 1] \longrightarrow \mathbb{R}$, with the supermum norm

$$\| x \|_\infty = \sup\{\| x(t) \|, \ t \in I = [0, 1]\}.$$

X can be endowed with the partial order relationship $\preceq$, that is, for all $x, y \in X$ $x \preceq y$ if and only if $x(t) \le y(t)$, so $(X, d_\infty, \preceq)$ is a complete order metric space.
x is a solution of problem (5.1) if there exists $v(t) \in F(t, x(t)))$, for all $t \in I$ such that

$$\begin{cases} {}^cD^q x(t) = v(t), \ 0 \le t \le 1, \ 1 < q \le 2 \\ ax(0) - bx'(0) = 0 \\ x(1) = \int_0^1 h(s)g(s))ds \end{cases} \tag{5.2}$$

Lemma 5.1. *Let $1 < q \le 2$ and $v \in \mathcal{AC}(I, \mathbb{R}) = \{v : I \to \mathbb{R}, f \text{ is absolutely continuous}\}$. A function x is a solution of (5.2) if and only if it is a solution of the integral equation:*

$$x(t) = \int_0^1 G(t, s)v(s)ds + \frac{at + b}{a + b} \int_0^1 h(s)g(s)ds,$$

where G is the Green function given by

$$G(t, s) = \begin{cases} \frac{(at+b)(1-s)^{q-1}}{(a+b)\Gamma(q)} - \frac{(t-s)^{q-1}}{\Gamma(q)}, & s \le t \\ \frac{(at+b)(1-s)^{q-1}}{(a+b)\Gamma(q)}, & t \le s. \end{cases} \tag{5.3}$$

Proof. The problem (5.2) can be reduced to an equivalent integral equation:

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s)ds + c_0 + c_1 t,$$

for some constants $c_0, c_1 \in X$.
Using the boundary conditions on (5.2), we get

$$ac_0 - bc_1 = 0,$$

$$\frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds + c_0 + c_1 = \int_0^1 h(s)g(s)ds.$$

Therefore

$$c_0 = \frac{b}{a+b} \left[\frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} g(s, x(s))ds + \int_0^1 h(s)g(s, x(s))ds \right].$$

$$c_1 = \frac{a}{a+b} \left[\frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds + \int_0^1 h(s)g(s, x(s))ds \right].$$

It means that

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s)ds + \frac{b}{a+b} \left[\frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds + \int_0^1 h(s)g(s, x(s))ds \right]$$

$$+ \frac{at}{a+b} \left[\frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s)ds + \int_0^1 h(s)g(s, x(s))ds \right]$$

$$= \int_0^t \left[\frac{(at+b)(1-s)^{q-1}}{(a+b)\Gamma(q)} - \frac{(t-s)^{q-1}}{\Gamma(q)} \right] v(s)ds + \int_t^1 \frac{(at+b)(1-s)^{q-1}}{(a+b)\Gamma(q)} v(s)ds$$

$$+ \frac{at+b}{a+b} \int_0^1 h(s)g(s, x(s))ds = \int_0^1 G(t,s)v(s)ds + \frac{at+b}{a+b} \int_0^1 h(s)g(s)ds.$$

□

Moreover, we have

$$\int_0^1 G(t,s)ds = \frac{(1}{\Gamma(q)} \Big[\int_0^t (t-s)^{q-1} ds + \frac{at+b}{a+b} \int_0^1 (1-s)^{q-1} ds \Big]$$

$$\leq \frac{1}{\Gamma(q+1)} t^q + \frac{1}{\Gamma(q+1)} \leq \frac{2}{\Gamma(q+1)}.$$

Define a set valued mapping

$$Tx_1(t) = \{z \in X, z(t) = \int_0^1 G(t,s)v(s)ds + \frac{at+b}{a+b} \int_0^1 h(s)g(s, x_1(s)ds\}.$$

The problem (5.1) has a solution if and only if T has a fixed point. Assume that the following assumptions hold:

- (A_1) : For each $x_1 \in X$ and $x_2 \in Tx_1$ with $x_1 \preceq x_2$ we have $x_2 \preceq x_3$ for all $x_3 \in Tx_2$.
- (A_2) : There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$.
- (A_3) : There exists $K > 0$ and $L > 0$ such that for all $x_1, x_2 \in \mathbb{R}$, we have

$$H(F(t, x_1(t)) - F(t, x_2(t))) \leq K|x_1 - x_2|)$$

and

$$|g(t, x_1(t)) - g(t, x_2(t))| \leq L|x_1 - x_2|,$$

with $k_0 = \frac{2K}{\Gamma(q+1)} + h_0 L < \frac{1}{2}$.

Theorem 5.1. *Under the assumptions $(A_1)-(A_3)$ the problem (5.1) has a solution in X.*

Proof. Since F is continuous, it has a selection, i,e., there exists a continuous function $v_1 \in F(t, x_1(t))$ such that Tx_1 is nonempty and has compact values.
Let $x_1, x_2 \in X$ and $z_1 \in Tx_1$, then there exists $v_1 \in F(t, x_1(t))$ such that

$$z_1(t) = \int_0^1 G(t,s)v_1(s)ds + \frac{at+b}{a+b}\int_0^1 h(s)g(s, x_1(s))ds.$$

Then by using (A_2), we get

$$d(v_1, Fx_2) = \inf_{u \in Fx_2} |v_1 - u| \leq H(F(t, x_1(t)) - F(t, x_2(t)))$$

$$\leq K\|x_1 - x_2\|,$$

the compactness of $F(t, x_2(t))$ implies that there exists $u^* \in F(t, x_2(t))$ such that

$$d(v_1, Fx_2) = |v_1 - u^*| \leq K|x_1 - x_2|.$$

Define an operator $P(t) = \{u^* \in \mathbb{R}, |u_1(t) - u^*| \leq K|x_1(t) - x_2(t)|\}$. Clearly $P \cap F(t, x_2(t))$ is continuous, so it has a selection v_2 such that

$$|u_1 - u_2| \leq K|x_1 - x_2|.$$

Define

$$z_2 = \int_0^1 G(t,s)u_2(s)ds + \frac{at+b}{a+b}\int_0^1 h(s)g(s, x_2(s)ds.$$

For all $t \in I$, we have

$$|z_1 - z_2| \leq \int_0^1 |G(t,s)||u_1 - u_2|ds + \frac{at+b}{a+b}\int_0^1 |h(s)||g(s, x_1(s)) - g(s, x_2(s))|ds$$

$$\leq K|x_1 - x_2|) \int_0^1 |G(t,s)|ds + \frac{at+b}{a+b} h_0 L|x_1(s) - x_2(s)|$$

$$\leq (\frac{2K}{\Gamma(q+1)} + h_0 L)|x_1 - x_2| = k_0|x_1 - x_2|$$

Then, we have

$$\sup_{z_1 \in Tx_1} \big[\inf_{z_2 \in Tx_2} |z_1 - z_2| \big] \leq k_0 \|x_1 - x_2\|.$$

Hence, by interchanging the role of x_1 and x_2 we obtain

$$H(Tx_1, Tx_2) \leq k_0|x_1 - x_2|).$$

On taking the exponential of two sides, we get

$$e^{H(Tx_1,Tx_2)} \leq (e^{2k_0|x_1-x_2|})^{\frac{1}{2}}$$

$$\leq e^{k_0|x_1-x_2|} + d(x_2, Tx_1).$$

If $\{x_n\}$ is a nondecreasing sequence in X which converges to $x \in X$, so for all $t \in I$ and $n \in \mathbb{N}$ we have $x_n(t) \leq x(t)$, which implies that x is an upper bound for all terms x_n (see [22]), then $x_n \preceq x$.
Consequently, all the conditions of Theorem 3.1 are satisfied, with $\theta(t) = e^t$, $\psi(t) = 2k_0 t$ and $k(t) = k_0$.
Hence, T has a fixed point which is a solution of the problem (5.1). □

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