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A GENERALIZATION OF ORDER CONVERGENCE IN THE VECTOR LATTICES

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Abstract. Let E be a sublattice of a vector lattice F. $(x_\alpha) \subseteq E$ is said to be $\tilde{o}$rder convergent to a vector x (in symbols $x_\alpha \xrightarrow{Fo} x$), whenever there exists another net (y_α) in F with the same index set satisfying $y_\alpha \downarrow 0$ in F and $|x_\alpha - x| \leq y_\alpha$ for all indexes α. If $F = E^{\sim\sim}$, this convergence is called b-order convergence and we write $x_\alpha \xrightarrow{bo} x$. In this manuscript, first we study some properties of $\tilde{o}$rder convergence nets and we extend same results to the general case. In the second part, we introduce b-order continuous operators and we investigate some properties of this new concept. An operator T between two vector lattices E and F is said to be b-order continuous, if $x_\alpha \xrightarrow{bo} 0$ in E implies $Tx_\alpha \xrightarrow{bo} 0$ in F.
Key words: order convergence, vector lattice, continuous operator.

1. Introduction

Generalizations of vector lattice notions by using subspaces of a vector lattice is rather common technique going back to [4] where it was applied to the notion of order ideal (see also [6], [5], and [8] where b-property and un-convergence were extended to this setting). To state our result, we need to fix some notation and recall some definitions. Let us say that a vector subspace G of an ordered vector space E is majorizing in E whenever for each $x \in E$ there exists some $y \in G$ with $x \leq y$. A vector sublattice G of vector lattice E is said to be order dense in E whenever for each $0 < x \in E$ there exists some $y \in G$ with $0 < y \leq x$. A Dedekind

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complete vector lattice E is said to be a Dedekind completion of the vector lattice G whenever E is lattice isomorphism to a majorizing order dense sublattice of E. A subset A of a vector lattice E is said to be order closed if it follows from $\{x_\alpha\} \subseteq A$ and $x_\alpha \xrightarrow{o} x$ in E that $x \in A$. A vector sublattice G of vector lattice E is said to be regular, if the embedding mapping of E into F preserves arbitrary suprema and infima. Let E, F be vector lattices. An operator $T : E \to F$ is said to be order bounded if it maps each order bounded subset of E into order bounded subset of F. The collection of all order bounded operators from a vector lattice E into a vector lattice F will be denoted by $\mathcal{L}_b(E, F)$. The vector space $E^\sim$ of all order bounded linear functionals on vector lattice E is called the order dual of E, i.e., $E^\sim = L_b(E, \mathbb{R})$. Let A be a subset of vector lattice E and Q_E be the natural mapping from E into $E^{\sim\sim}$. If $Q_E(A)$ is order bounded in $E^{\sim\sim}$, then A is said to $b-$order bounded E, see [3]. It is clear that every order bounded subset of E is $b-$order bounded. However, the converse is not true in general. For example, $A = \{e_n \mid n \in \mathbb{N}\}$ $b-$order bounded in c_0 but A is not order bounded in c_0. A linear operator between two vector lattices is order continuous (resp. σ-order continuous) if it maps order null nets (resp. sequences) to order null nets (resp. sequences). The collection of all order continuous (resp. σ-order continuous) linear operators from vector lattice E into vector lattice F will be denoted by $\mathcal{L}_n(E, F)$ (resp. $\mathcal{L}_c(E, F)$). For unexplained terminology and facts on Banach lattices and positive operators, we refer the reader to [1, 2].

2. $\tilde{o}$rder convergence on vector lattices

In all parts of this section E is a vector sublattice of vector lattice F. Let $A \subseteq E$. We say that $\inf A$ exists in E with respect to F, if $\inf A$ exits in F and $\inf A \in E$, in this case we write $\inf_F A$ exists,. For a net $(x_\alpha)_\alpha \subseteq E$ and $x \in E$, the notation $x_\alpha \downarrow_F x$ means that $x_\alpha \downarrow$ and $\inf (x_\alpha) = x$ holds in F. The meanings of $x_\alpha \uparrow$ and $x_\alpha \uparrow_F x$ are analogous. Obviously if $x_\alpha \downarrow_F 0$, then $x_\alpha \downarrow 0$, but as following example the converse in general not holds.

Example 2.1. Assume that F is a set of real valued functions on $[0, 1]$ of form $f = g + h$ where g is continuous and h vanishes except at finitely many point. Let $E = C([0, 1])$ and $f_n(t) = t^n$ where $t \in [0, 1]$. It is clear that $f_n \downarrow 0$ in E and $\inf_F f_n$ not exists in E, but we have $f_n \downarrow \chi_{\{1\}}$ in F.

It is obvious that if E is regular in F, then for each net $(x_\alpha)_\alpha \subseteq E$ and $x \in E$, $x_\alpha \downarrow_F x$ if and only if $x_\alpha \downarrow x$.

The notation $x_\alpha \downarrow_b x$ means that $x_\alpha \downarrow$ and $\inf (x_\alpha) = x$ holds in $E^{\sim\sim}$. The meanings of $x_\alpha \uparrow_b x$ is analogous.

Definition 2.1. $(x_\alpha) \subseteq E$ is said to be $\tilde{o}$rder convergent (in short **$\tilde{o}$-convergent**) to a vector $x \in F$ (in symbols $x_\alpha \xrightarrow{Fo} x$), whenever there exists another net (y_α) in F with the same index set satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all indexes α.

In the same way, a net (x_α) of E is said to be b-order convergent (in short ***bo*-convergent**) to a vector x (in symbols $x_\alpha \xrightarrow{bo} x$), whenever there exists another net (y_α) in $E^{\sim\sim}$ with the same index set satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all indexes α.

It is clear that every order convergent net in vector lattice E is õrder convergent, but as following example the converse in general not holds.

Example 2.2. Suppose that $E = c_0$ and (e_n) is the standard basis of c_0. We know that (e_n) is not order convergent to zero, but (e_n) is õrder (or b-order) convergent to zero in ℓ^∞.

It can easily be seen that a net in vector lattice E can have at most one õrder limit. The basic properties of õrder convergent are summarized in the next theorem.

Theorem 2.1. *Assume that the nets (x_α) and (y_β) of a vector lattice E satisfy $x_\alpha \xrightarrow{Fo} x$ and $y_\beta \xrightarrow{Fo} y$. Then we have*

1. $|x_\alpha| \xrightarrow{Fo} |x|$; $x_\alpha^+ \xrightarrow{Fo} x^+$ *and* $x_\alpha^- \xrightarrow{Fo} x^-$.
2. $\lambda x_\alpha + \mu y_\beta \xrightarrow{Fo} \lambda x + \mu y$ *for all* $\lambda, \mu \in \mathbb{R}$.
3. $x_\alpha \vee y_\beta \xrightarrow{Fo} x \vee y$ *and* $x_\alpha \wedge y_\beta \xrightarrow{Fo} x \wedge y$.
4. *For each* $z \in F$, *if* $x_\alpha \leq z$ *for all* $\alpha \geq \alpha_0$, *then* $x \leq z$.
5. *If* $0 \leq x_\alpha \leq y_\alpha$, *then* $0 \leq x \leq y$.
6. *If* P *is order projection, then* $Px_\alpha \xrightarrow{Fo} Px$.

Definition 2.2. $A \subseteq E$ is said to be F-order closed whenever $(x_\alpha) \subseteq A$ and $x_\alpha \xrightarrow{Fo} x$ imply $x \in A$.

The set $A \subseteq E$ is F-order closed means that A is order closed with respect to vector lattice F. If $A \subseteq E$ is order closed in E, then it is clear that A is F-order closed, but the converse in general not holds. For example c_0 is order closed, but is not ℓ^∞-order closed.

Lemma 2.1. *Let $A \subseteq E$ be a solid subset of F. Then A is F-order closed if and only if $(x_\alpha) \subseteq A$ and $0 \leq x_\alpha \uparrow_F x$ imply $x \in A$.*

Proof. Suppose that A is a F-order closed, and $(x_\alpha) \subseteq A$ and $0 \leq x_\alpha \uparrow_F x$. Therefore $0 \leq |x_\alpha - x| \leq x - x_\alpha \downarrow_F 0$. It follows that $x_\alpha \xrightarrow{Fo} x$, and since A is order closed, it implies that $x \in A$.

Conversely assume that $(x_\alpha) \subseteq A$ and $x_\alpha \xrightarrow{Fo} x$. Set a net (y_α) in F with same index net satisfying $y_\alpha \downarrow_F 0$ and $|x_\alpha - x| \leq y_\alpha$ for each α. Since $(|x| - y_\alpha)^+ \leq |x_\alpha|$ for each α and A is solid, follows that $((|x| - y_\alpha)^+)_\alpha \subseteq A$. Obviously that $0 \leq (|x| - y_\alpha)^+ \uparrow_b |x|$, and so by hypothesis we have $x \in A$. It follows that A is F-order closed. $\square$

Definition 2.3. 1. E is said to be F-Dedekind (or F-order) complete, if every nonempty $A \subseteq E$ that is bounded from above in F has supermum in E. In case $F = E^{\sim\sim}$, we say that E is b-Dedekind complete.

2. A subset A of E is called F-order bounded, if A is order bounded in F. In case $F = E^{\sim\sim}$, we say that A is b-order bounded.

3. If each F-order bounded subset of E is order bounded in E, then E is said to have the F-property. In case $F = E^{\sim\sim}$, we say that E has b-property.

Remark 2.1. Every majorizing sublattice E of F has the F-property. Since $E^{\sim}$ has b-property, $E^{\sim}$ is b-Dedekind complete. If E is F-Dedekind complete, then E is Dedekind complete, but the converse in general not holds, of course c_0 is Dedekind complete, but is not ℓ^{∞}-Dedekind complete. Let K be a compact Hausdorff space and let $C(K)$ and $B(K)$ be vector lattices of real valued continuous and bounded functions on K, respectively, under pointwise order and algebric operations. By easy calculation, it is obvious that $C(K)$ is both $B(K)$-Dedekind complete and b-Dedekind complete. It is clear that E is F-Dedekind complete if and only if E is Dedekind complete with F-property. It is easy to show that a vector lattice E has F-property if and only if for each net (x_α) in E with $x_\alpha \uparrow\leq y$ for some $y \in F$, (x_α) is order bounded in E.

Theorem 2.2. *Assume that E is a Banach lattice with order continuous norm and vector subspace of a vector lattice F. E is KB-space iff E has property (F).*

Proof. Let E be a KB-space and net $(x_\alpha) \subseteq E$ such that $0 \leq x_\alpha \uparrow\leq x$ and $x \in F$. We have $x - x_\alpha \downarrow 0$. Since E has order continuous norm, therefore $\lim_\alpha x_\alpha = x$. On the other hand E is Dedekind complete, hence there exists $y \in E$ that $y = \sup_\alpha x_\alpha$. So $y = \sup_\alpha x_\alpha = \lim_\alpha x_\alpha = x$. Hence $x \in E$. Thus E has prperty (F).
Conversely, by assumption it is clear that E has property (b). By Proposition 2.1 of [3], it follows that E is a KB-space, and proof follows. □

Corollary 2.1. *If E is a Banach lattice with order continuous and property (b), then it has property (F) for each vector lattice F that E is a sublattice of F.*

Note that c_0 has property (c_0^u)(universal completion). Since c_0 has order continuous norm and it is not KB-space, therefore it has not property (b).

Let E be a vector sublattice of F and I be an ideal in E. In general, I is not an ideal in F. For example, set $F = \mathbb{R}^3$ and define the order on F in the following way:

$$x = (x_1, x_2, x_3) < y = (y_1, y_2, y_3)$$

whenever one of the following relations hold

1. $x_1 < y_1$ or;

2. $x_1 = y_1,\ x_2 < y_2$ or;

3. $x_1 = y_1,\ x_2 = y_2,\ x_3 < y_3$.

It is clear F with this order is a vector lattice. Now if we take $E = \{(x, y, 0) :\ x, y \in \mathbb{R}\}$ and $I = \{(0, y, 0) :\ y \in \mathbb{R}\}$, then obviously that I is an ideal in E, but is not ideal in F.
The above example shows that the property of being ideal thus depends on the space in which I is embedded. Now if E is F-Dedekind complete and order dense in F, the following theorem shows that I is an ideal in E if and only if I is an ideal in F.

Theorem 2.3. *Assume that E is F-Dedekind complete. The following statements hold.*

1. *Each F-order convergent net in E is order convergent in E.*
2. *If E is order dense in F, then B is a band in E if and only if B is a band in F.*
3. *By assumption (2), if $F = E^{\sim\sim}$, then $E^{\delta} = E^{\sim\sim}$.*
4. *If $F = E^{\sim\sim}$ and $E^{\sim} = E^{\sim}_{n}$, then E is perfect.*

Proof. 1. Assume that $(x_\alpha)_\alpha \subseteq E$ is F-order convergent to x in E. Then there exists a net $(y_\alpha)_\alpha \subseteq F$ such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leqslant y_\alpha$ for all indexes α. Set $z_\alpha = |x_\alpha - x|$ and take $w_\alpha = \bigvee_{\beta \geqslant \alpha} z_\beta$. Since E is F-Dedekind complete, $(w_\alpha)_\alpha \subseteq E$. It follows that $|x_\alpha - x| \leqslant w_\alpha \leqslant y_\alpha$ and $w_\alpha \downarrow 0$ in E. Thus $(x_\alpha)_\alpha$ is order convergence to x in E.

2. If B is a band in F, it is clear that B is a band in E. Now assume that B is a band in E. First we prove that I is an ideal in F. Let $x \in F$ and $y \in I$ such that $0 < |x| < |y|$. Since E is order dense in F, there is a $z \in E$ such that $0 < z \leqslant x^+ \leqslant |x|$, which follows that $z \in B$. Put $\sup\{z \in B :\ 0 < z \leqslant x^+\} = w$. By F-Dedekind completeness of E, we have $w \in E$, and so $w \in B$. If $w < x^+$, then $0 < x^+ - w$. Since E is order dense in F, there is $v \in E$ such that $0 < v < x^+ - w$, and so $0 < w + v < x^+$. It follows that $w + v \in B$, which is impossible. Thus $w = x^+$ belong to B. In the same way $x^- \in B$, and so $x \in B$. This shows that B is an ideal in F. Now since E is F-Dedekind complete, by using Lemma 2.1, B is order closed in F and proof follows.

3. By Proposition 8 from [6], proof follows.

4. By Corollary 10 from [6], proof follows.

□

3. b-order continuous operators

Let E and F be two vector lattices. An operator $T : E \to F$ is called b-order bounded operator if it maps b-order bounded subset of E into b-order bounded subset of F. The collection of b-order bounded operators will be denoted by:

$$L_{b^{\sim}}(E,F) := \{T \in L(E,F) : T \text{ is } b-\text{order bounded operator}\}.$$

An order bounded operator between two vector lattices is b-order bounded, but as Example 2.4, [3], the converse, in general, not holds.

Proposition 3.1. *Let T be an order bounded operator from a vector lattice E into Dedekind complete vector lattice F. Then $T \in L_{b^{\sim}}(E,F)$ if and only if $|T^{\sim\sim}| \in L_{b^{\sim}}(E^{\sim\sim}, F^{\sim\sim})$*

Proof. Assume that $T \in L_{b^{\sim}}(E,F)$, we shows that $|T| \in L_{b^{\sim}}(E,F)$. Let $(x_\alpha)_\alpha$ be a net in E^+ with $x_\alpha \uparrow x''$ for some $x'' \in E^{\sim\sim}$. Let A be the solid hull of $(x_\alpha)_\alpha$ in E. Since $(x_\alpha)_\alpha$ is order bounded in $E^{\sim\sim}$, A is b-order bounded in E. Then $T(A)$ is b-order bounded in F. Since $F^{\sim\sim}$ is Dedekind complete, $\sup T(A)$ exists in $F^{\sim\sim}$. Let $y \in E$ with $|y| \leq x_\alpha$ for fix $\alpha \in I$. Then $y \in A$ and $T(y) \leq \sup T(A)$. It follows that $|T|(x_\alpha) \leq \sup T(A)$. By Dedekind completeness of $F^{\sim\sim}$, $\sup_\alpha |T|(x_\alpha)$ exists in $F^{\sim\sim}$. This shows that $|T^{\sim\sim}| \in L_{b^{\sim}}(E^{\sim\sim}, F^{\sim\sim})$. The converse by easy calculation follows. □

By the conditions of above proposition, $L_{b^{\sim}}(E^{\sim\sim}, F^{\sim\sim})$ is a lattice, and so is a vector lattice. So it is easy to shows that $L_{b^{\sim}}(E^{\sim\sim}, F^{\sim\sim})$ is an ideal in $L_b(E^{\sim\sim}, F^{\sim\sim})$.

Theorem 3.1. *Let E be a Banach lattice. Then each operator T from E into a AL-space F is a b-order bounded.*

Proof. Assume that $A \subseteq E$ is b-order bounded. Since F is an AM-space, follows that F is a KB-space, and therefore by Theorem 4.60 of [2], c_0 is not embeddable in F. Hence by Proposition 2.2 of [7], T is b-weakly compact operator. So $T(A)$ is a relatively weakly compact subset of F and therefore by Theorem 4.27 of [2], it is a relatively weakly compact in $L^1(\mu)$ for some finite measures. We know that, $T(A)$ is a almost bounded in $L^1(\mu)$ if and only if $T(A)$ is relatively weakly compact. Therefore $T(A)$ is bounded, and so is b-order bounded in F. □

Definition 3.1. An operator $T : E \to F$ between two vector lattices is said to be b-order continuous, if $x_\alpha \xrightarrow{bo} 0$ in E implies $Tx_\alpha \xrightarrow{bo} 0$ in F.

Proposition 3.2. *If T is b-order continuous operator between two vector lattices E and F. Then T is b-order bounded.*

Proof. Suppose that $T : E \to F$ is a b-order continuous operator. Let $A = [0, x''] \cap E$ for some $x'' \in E^{\sim\sim}$. Let $\Lambda = \{\beta : 0 \leq \beta \leq x''\}$ and we write $\alpha \preceq \beta$ if and only if $\alpha \geq \beta$. We consider a net $(x_\alpha)_{\alpha \in \Lambda}$ as follows

$$x_\alpha = \begin{cases} \alpha & \alpha \in A; \\ 0 & \alpha \notin A \end{cases}$$

Therefore $x_\alpha \xrightarrow{bo} 0$, since if we set $y_\alpha = \alpha$ then $y_\alpha \downarrow_b 0$ and $|x_\alpha| \leq y_\alpha$. By the b-order continuity of T, there exists a net (z_α) of $F^{\sim\sim}$ with the same index Λ such that $|Tx_\alpha| \leq z_\alpha \downarrow_b 0$. Consequently, if $\alpha \in \Lambda$ then we have $|Tx_\alpha| \leq z_\alpha \leq z_{x''}$ that $z_{x''} \in F^{\sim\sim}$, and this show that T is a b-order bounded operator. □

As above proposition, the class of b-order continuous operators is a subspace of $L_{b^\sim}(E,F)$ and will be denoted by $L_{n^\sim}(E,F)$, that is

$$L_{n^\sim}(E,F) := \{T \in L_{b^\sim}(E,F) : T \text{ is } b-\text{ order continuous}\}.$$

Proposition 3.3. *Let E and F be vector lattices, T and S are operators from E into F and $0 \leq S \leq T$. If $T \in L_{n^\sim}(E,F)$, then $S \in L_{n^\sim}(E,F)$.*

Proof. Let (x_α) be net in E that $x_\alpha \xrightarrow{bo} 0$. Since T is b-order continuous, there exists $y_\alpha \in F^{\sim\sim}$ such that $|T|x_\alpha|| = T|x_\alpha| \leq y_\alpha \downarrow_b 0$. On the other hand, $|S(x_\alpha)| \leq S|x_\alpha| \leq T|x_\alpha| \leq y_\alpha \downarrow_b 0$ for every x_α in E, and this yields that S is b-order continuous. □

Lemma 3.1. *Let E and F be two vector lattices with Dedekind complete. Then $0 < T \in L_{n^\sim}(E,F)$ if and only if for each net (x_α) in E, $x_\alpha \downarrow_b 0$ implies $Tx_\alpha \downarrow_b 0$.*

Proof. Assume that $0 < T \in L_{n^\sim}(E,F)$ and $x_\alpha \downarrow_b 0$. It follows that $x_\alpha \xrightarrow{bo} 0$, and so $Tx_\alpha \xrightarrow{bo} 0$. Then there is a net (y_α) in $F^{\sim\sim}$ such that $Tx_\alpha = |Tx_\alpha| \leq y_\alpha \downarrow 0$, which follows that $Tx_\alpha \downarrow_b 0$.

Conversely, Let $(x_\alpha) \subseteq E$ such that $x_\alpha \xrightarrow{bo} 0$ in E. Then there is a net (y_α) in $E^{\sim\sim}$ such that $|x_\alpha| \leq y_\alpha \downarrow 0$ in $E^{\sim\sim}$. Set $w_\alpha = \bigvee_{\beta \geq \alpha} |x_\beta| < y_\alpha$. Then we have $w_\alpha \downarrow_b 0$, and so $Tw_\alpha \downarrow_b 0$. Since $|Tx_\alpha| \leq T|x_\alpha| \leq Tw_\alpha$, $Tx_\alpha \xrightarrow{bo} 0$ in F, and proof follows. □

As an application of Lemma 3.1, we have the following corollary, in which the techniques of this corollary has been similar argument like as Theorem 1.56 [1] and we omit its proof.

Corollary 3.1. *Let E and F be two vector lattices with Dedekind complete. Then the following assertions are equivalent.*

1. *$T \in L_{n^\sim}(E,F)$.*
2. *$x_\alpha \downarrow_b 0$ implies $Tx_\alpha \downarrow_b 0$.*
3. *$x_\alpha \downarrow_b 0$ implies $\inf_b |Tx_\alpha| = 0$.*
4. *T^-, T^+ and $|T|$ belong to $L_{n^\sim}(E,F)$.*

Proposition 3.4. *Let E and F be both vector lattices. Then we have the following assertions.*

1. *If F is a b-Dedekind complete, then $L_b(E,F) = L_{b^\sim}(E,F)$.*
2. *If E and F are both b-Dedekind complete, then $L_{n^\sim}(E,F) = L_n(E,F)$.*
3. *$L_{n^\sim}(E,F)$ is a band of $L_{b^\sim}(E,F)$.*

Proof. 1. Obviously that $L_b(E,F) \subseteq L_{b^\sim}(E,F)$. Now we prove $L_{b^\sim}(E,F) \subseteq L_b(E,F)$. Let A be an order bounded subset of E and $T \in L_{b^\sim}(E,F)$. Then $T(A)$ is an b-order bounded subset of F. Since F be b-Dedekind complete, follows that $\sup_b T(A)$ exists in E. It follows that $T(A)$ is order bounded in E, and so proof follows.

2. Assume that $T \in L_n(E,F)$. Let (x_α) be a net in E that $x_\alpha \xrightarrow{bo} 0$. Since E is b-Dedekind complete, by using Theorem 2.3, $x_\alpha \xrightarrow{o} 0$. By assumption, we have $Tx_\alpha \xrightarrow{o} 0$, which follows that $Tx_\alpha \xrightarrow{bo} 0$, and so $T \in L_{n^\sim}(E,F)$.
   Now let $T \in L_{n^\sim}(E,F)$ and $(x_\alpha) \subseteq E$ such that $x_\alpha \xrightarrow{o} 0$. It follows that $x_\alpha \xrightarrow{bo} 0$, and so $Tx_\alpha \xrightarrow{bo} 0$. By Dedekind completeness of F and another using Theorem 2.3, we have $Tx_\alpha \xrightarrow{o} 0$, which follows that $L_{n^\sim}(E,F) \subseteq L_n(E,F)$, and proof down.

3. Proof has the similar argument from Theorem 1.57 [1].
   □

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REFERENCES

1. Y. A. Abramovich and C. D. Aliprantis: *Locally Solid vector lattices with Application to Economics*: Mathematical Surveys, **105**, American Mathematical Society, Providence, RI, 2003.
2. C. D. Aliprantis and O. Burkinshaw: *Positive operators*, **119**, Springer Science & Business Media, 2006.
3. Ş. Alpay, B. Altin and C. Tonyali: *On property (b) of vector lattices*: Positivity. **7** (2003), 135–139.
4. Ş. Alpay; E. Yu. Emel'yanov and Z. Ercan: *A characterization of an order ideal in Riesz spaces*. Proc. Amer. Math. Soc. **132** (2004), 3627-3628.
5. Ş. Alpay and S. Gorokhova: *b-property of sublattices in vector lattices*: Turkish J. Math. **45** (2021), 1555-1563.
6. Ş. Alpay and Z. Ercan: *Characterizations of Riesz spaces with b-property*: Positivity. **13** (2009), 21–30.
7. B. Aqzzouz, A. Elbour and J. Hmichane: *The duality problem for the class of b-weakly compact operators*: Positivity. **13** (2009), 683–692.
8. M. Kandic, H. Li and V. G. Troitsky: *Unbounded norm topology beyond normed lattices*: Positivity. **22** (2018), 745-760.