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KUELBS-STEADMAN SPACES ON SEPARABLE BANACH SPACES

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Abstract. The purpose of this paper is to construct a new class of separable Banach spaces $\mathbb{K}^p[\mathbb{B}]$, $1 \le p \le \infty$ on separable Banach space $\mathbb{B}$. Each of these spaces contain the $\mathcal{L}^p[\mathbb{B}]$ spaces. These spaces are of interest because they also contain the Henstock-Kurzweil integrable functions on $\mathbb{B}$.
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1. Introduction and Preliminaries

T.L. Gill and T. Myers [5] introduced a new theory of Lebesgue measure on $\mathbb{R}^\infty$; the construction of which is virtually the same as the development of Lebesgue measure on $\mathbb{R}^n$. This theory can be useful in formulating a new class of spaces which will provide a Banach Space structure for Henstock-Kurzweil (HK) integrable functions. This later integral is interesting because it generalizes the Lebesgue, Bochner and Pettis integrals see for instance [6, 8, 9, 12, 16, 18]. However, fly in the ointment of HK-integrable function space is not naturally Banach space (see [1, 2, 6, 7, 9, 10, 12, 13, 15] references therein). In [20], Yeong broach a clue of wind up about the drawback, pointing about canonical construction. Gill and Zachary [3, 4], introduced a new class of Banach spaces $KS^p[\Omega], \forall\ 1 \le p \le \infty$ (Kuelbs-Steadman spaces) and $\Omega \subset \mathbb{R}^n$ which are canonical spaces (also see [11]). These spaces are separable and contain the corresponding $\mathcal{L}^p$ spaces as dense, continuous,

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compact embedding. They wanted to find these spaces containing Denjoy integrable function, also additive measures. They found that these spaces are perfect for distinctly vibrating functions that occur in quantum theory and non linear analysis.

Throughout the paper, we assume $J = [-\frac{1}{2}, \frac{1}{2}]$. We denote by $\mathcal{L}^1$, $\mathcal{L}^p$ the classical Lebesgue spaces. Our study focused on the main class of Banach spaces $\mathbb{K}^p[\mathbb{B}_J^\infty]$, $1 \leq p \leq \infty$. These spaces contain the HK-integrable functions, the $\mathcal{L}^p[\mathbb{B}_J^\infty]$ spaces, $1 \leq p \leq \infty$ as continuous dense and compact embedding.

Definition 1.1. [15] A function $f : [a, b] \to \mathbb{R}$ is Henstock integrable (HK integrable) if there exists a function $F : [a, b] \to \mathbb{R}$ and for every $\epsilon > 0$ there is a function $\delta(t) > 0$ such that for any $\delta-$fine partition $\mathsf{D} = \{[u, v], t\}$ of $I_0 = [a, b]$, we have

$$\left\| \sum [f(t)(v-u) - F(u, v)] \right\| < \epsilon,$$

where the sum $\sum$ is run over $\mathsf{D} = \{([u, v], t)\}$ and $F(u, v) = F(v) - F(u)$. We write $H \int_{I_0} f = F(I_0)$.

Definition 1.2. [5] If $\mathbb{A}_n = \mathbb{A} \times J_n$ and $\mathbb{B}_n = \mathbb{B} \times J_n$ (n^{th} box of order sets in $\mathbb{R}^\infty$). We consider

1. $\mathbb{A}_n \cup \mathbb{B}_n = (\mathbb{A} \cup \mathbb{B}) \times J_n$;
2. $\mathbb{A}_n \cap \mathbb{B}_n = (\mathbb{A} \cap \mathbb{B}) \times J_n$;
3. $\mathbb{B}_n^c = \mathbb{B}^c \times J_n$.

Definition 1.3. [4] Assume $\mathbb{R}_J^n = \mathbb{R}^n \times J_n$. If T is a linear transformation on $\mathbb{R}^n$ and $\mathbb{A}_n = \mathbb{A} \times J_n$, then T_J on $\mathbb{R}_J^n$ is denoted by $T_J[\mathbb{A}_n] = T[\mathbb{A}]$. We denote $B[\mathbb{R}_J^n]$ to be the Borel $\sigma-$algebra for $\mathbb{R}_J^n$, where the topology on $\mathbb{R}_J^n$ is define via the class of open sets $D_n = \{U \times J_n : U \text{ is open in } \mathbb{R}_J^n\}$. For any $\mathbb{A} \in B[\mathbb{R}^n]$, we define $\mu_{\mathbb{B}}(\mathbb{A}_n)$ on $\mathbb{R}_J^n$ by product measure $\mu_\infty(\mathbb{A}_n) = \mu\mathbb{A}_n(\mathbb{A}) \times \Pi_{i=n+1}^\infty \mu_J(J) = \mu\mathbb{A}_n(\mathbb{A})$.

Clearly $\mu_{\mathbb{R}}(.)$ is a measure on $B[\mathbb{R}_J^n]$, which is equivalent to n-dimensional Lebesgue measure on $\mathbb{R}_J^n$. The measure $\mu_{\mathbb{R}}(.)$ is both translationally and rotationally invariant on $(\mathbb{R}_J^n, B[\mathbb{R}_J^n])$ for each $n \in \mathbb{N}$. Recollecting the theory on $\mathbb{R}_J^n$ that completely paralleis that on $\mathbb{R}^n$. Since $\mathbb{R}_J^n \subset \mathbb{R}_J^{n+1}$, we have an increasing sequence, so we define $\widehat{\mathbb{R}}_J^\infty = \lim_{n\to\infty} \mathbb{R}_J^n = \bigcup_{k=1}^\infty \mathbb{R}_J^k$. Suppose $X_1 = \widehat{\mathbb{R}}_J^\infty$ and τ_1 is the topology induced by the class of open sets $D \subset X_1$ such that $D = \bigcup_{n=1}^\infty D_n = \bigcup_{n=1}^\infty \{U \times J_n : U \text{ is open in } \mathbb{R}^n\}$. Suppose $X_2 = \mathbb{R}^\infty \setminus \widehat{\mathbb{R}}_J^\infty$ and τ_2 is the discrete topology on X_2 induced by the discrete metric so that, for $x, y \in X_2$, $x \neq y$, $d_2(x, y) = 1$ and for $x = y$ $d_2(x, y) = 0$

Definition 1.4. [4] Let $(\mathbb{R}_J^\infty, \tau)$ be the co-product $(X_1, \tau_1) \otimes (X_2, \tau_2)$ of (X_1, τ_1) and (X_2, τ_2). Every open set in $(\mathbb{R}_J^\infty, \tau)$ is the disjoint union of two open sets $G_1 \cup G_2$ with G_1 in (X_1, τ_1) and G_2 in (X_2, τ_2).

As a result $\mathbb{R}_J^\infty = \mathbb{R}^\infty$ as sets. However, since every point in X_2 is open and closed in $\mathbb{B}_J^\infty$ and no point is open and closed in $\mathbb{R}^\infty$. So, $\mathbb{R}_J^\infty \neq \mathbb{R}^\infty$ as topological spaces. It was shown in [5] that it can be extended the measure $\mu_{\mathbb{R}}(.)$ to $\mathbb{R}^\infty$.

Similarly, if $B[\mathbb{R}_J^n]$ is the Borel σ-algebra for $\mathbb{R}_J^n$, then $B[\mathbb{R}_J^n] \subset B[\mathbb{R}_J^{n+1}]$ defined by

$$\widehat{B}[\mathbb{R}_J^\infty] = \lim_{n\to\infty} B[\mathbb{R}_J^n] = \bigcup_{k=1}^{\infty} B[\mathbb{R}_J^k].$$

Suppose $B[\mathbb{R}_J^\infty]$ is the smallest σ-algebra restraining $\widehat{R}[\mathbb{R}_J^\infty] \cup P(\mathbb{R}^\infty \setminus \bigcup_{k=1}^{\infty} [\mathbb{R}_J^k])$, where $P(.)$ is the power set. It is obvious that the class $B[\mathbb{R}_J^\infty]$ coincides with the Borel $\sigma-$algebra generated by the $\tau-$topology on $\mathbb{R}_J^\infty$.

1.1. Measurable functions

We consider measurable function on $\mathbb{R}_J^\infty$ as follows. Suppose $x = (x_1, x_2, \ldots) \in \mathbb{B}_j^\infty$, $J_n = \Pi_{k=n+1}^{\infty}[-\frac{1}{2}, \frac{1}{2}]$ and $h_n(\widehat{x}) = \chi_{J_n}(\widehat{x})$, where $\widehat{x} = (x_i)_{i=n+1}^{\infty}$.

Definition 1.5. [4] Suppose M^n represents the class of measurable functions on $\mathbb{R}^n$. On condition that $x \in \mathbb{R}_j^\infty$ and $f^n \in M^n$, suppose $\overline{x} = (x_i)_{i=1}^n$ and define an essentially docile measurable function of order n(or e_n- docile) on $\mathbb{B}_j^\infty$ by

$$f(x) = f^n(\overline{x}) \otimes h_n(\widehat{x}).$$

We suppose $M_J^n = \{f(x) \ : f(x) = f^n(\overline{x}) \otimes h_n(\widehat{x}),\ x \in \mathbb{R}_J^\infty\}$ is the class of all e_n- docile function.

Definition 1.6. A function $f : \mathbb{R}_J^\infty \to \mathbb{R}$ is said to be measurable, written $f \in M_J$, if there is a sequence $\{f_n \in M_J^n\}$ of e_n- docile functions, such that

$$\lim_{n\to\infty} f_n(x) \to f(x)\ \mu_\infty - (a.e.).$$

This definition highlights our requirement that all functions on infinite dimensional space must be constructively defined as (essentially) finite dimensional limits. The existence of functions satisfying above definition is not obvious. So, we have the following theorem.

Theorem 1.1. (Existence) *Suppose that* $f : \mathbb{R}_J^\infty \to (-\infty, \infty)$ *and* $f^{-1}(a) \in B[\mathbb{R}_j^\infty]$ *for all* $a \in B[\mathbb{R}]$ *then there exists a family of functions* $\{f_n\},\ f_n \in M_J^n$ *such that* $f_n(x) \to f(x), \mu_\infty - (a.e.)$

Remark 1.1. Recalling that any set A, of non zero measure is concentrated in X_1 that is $\mu_\infty(A) = \mu_\infty(A \cap X_1)$ also follows that the essential support of the limit function $f(x)$ in Definition 1.6, i.e., $\{x :\ f(x) \neq 0\}$ is concentrated in $\mathbb{R}_J^n$ for some N.

1.2. Integration theory on $\mathbb{R}_J^\infty$

We deal with integration on $\mathbb{R}_J^\infty$ by using the known properties of integration on $\mathbb{R}_J^n$. This approach has the advantages that all the theorems for Lebesgue measure apply. Let $\mathcal{L}^1[\mathbb{R}_J^n]$ be the class of integrable functions on $\mathbb{R}_J^n$. Since $\mathcal{L}^1[\mathbb{R}_J^n] \subset \mathcal{L}^1[\mathbb{R}_J^{n+1}]$, we define $\mathcal{L}^1[\widehat{\mathbb{R}}_J^\infty] = \bigcup_{n=1}^{\infty} \mathcal{L}^1[\mathbb{R}_J^n]$.

Definition 1.7. A measurable function f is said to be in $\mathcal{L}^1[\mathbb{R}_J^\infty]$ if there is a Cauchy-sequence $\{f_n\} \subset \mathcal{L}^1[\widehat{\mathbb{R}}_J^\infty]$ with $f_n \in \mathcal{L}^1[\mathbb{R}_J^n]$ such that

$$\lim_{n\to\infty} f_n(x) = f(x),\ \mu_\infty - (a.e.).$$

Theorem 1.2. $\mathcal{L}^1[\mathbb{R}_J^\infty] = \mathcal{L}^1[\widehat{\mathbb{R}}_J^\infty]$.

Proof. We know that $\mathcal{L}^1[\mathbb{R}_J^n] \subset \mathcal{L}^1[\widehat{\mathbb{R}}_J^\infty]$ for all n. It needs only to prove that $\mathcal{L}^1[\widehat{\mathbb{R}}_J^\infty]$ is closed. Suppose f is the limit point of $\mathcal{L}^1[\widehat{\mathbb{R}}_J^\infty]$ ($f \in \mathcal{L}^1[\mathbb{R}_J^\infty]$). On condition that $f = 0$ then the result is proved. So we consider $f \neq 0$. On condition that a_f is the support of f, then $\mu_{\mathbb{R}}(A_f) = \mu_\infty(A_f \cap X_1)$. Thus $A_f \cup X_1 \subset \mathbb{R}_J^n$ for some N. This means that there is a function $g \in \mathcal{L}^1[\mathbb{R}_J^{N+1}]$ with $\mu_\infty(\{x : f(x) \neq g(x)\}) = 0$. So, $f(x) = g(x)-$a.e. as $\mathcal{L}^1[\mathbb{R}_J^n]$ is a set of equivalence classes. So, $\mathcal{L}^1[\mathbb{R}_J^\infty] = \mathcal{L}^1[\widehat{\mathbb{R}}_J^\infty]$. $\square$

Definition 1.8. On condition that $f \in \mathcal{L}^1[\mathbb{R}_J^\infty]$, we define the integral of f by

$$\int_{\mathbb{R}_J^\infty} f(x)d\mu_\infty(x) = \lim_{n\to\infty} \int_{\mathbb{R}_J^\infty} f_n(x)d\mu_\infty(x),$$

where $\{f_n\} \subset \mathcal{L}^1[\mathbb{R}_J^\infty]$ is any Cauchy sequence converging to $f(x)-$a.e.

Theorem 1.3. *On condition that $f \in \mathcal{L}^1[\mathbb{R}_J^\infty]$ then the above integral exists and all theorems that are true for $f \in \mathcal{L}^1[\mathbb{R}_J^n]$, also hold for $f \in \mathcal{L}^1[\mathbb{R}_J^\infty]$.*

2. Class of $\mathbb{B}_j^\infty$, where $\mathbb{B}$ is a separable Banach space

As an application of $\mathbb{R}_J^\infty$, we can construct $\mathbb{B}_J^\infty$, where $\mathbb{B}$ is separable Banach space. The important fact is we can construct the measure $\mu_{\mathbb{B}}$ on separable Banach space $\mathbb{B}$ in similar fashion of μ_∞ of $\mathbb{R}^\infty$. Recalling a sequence $(e_n) \in \mathbb{B}$ is called a Schauder basis (S-basis) for $\mathbb{B}$, On condition that $||e_n||_{\mathbb{B}} = 1$ and for each $x \in \mathbb{B}$, there is a unique sequence (x_n) of scalars such that

$$x = \lim_{k\to\infty} \sum_{n=1}^{k} x_n e_n = \sum_{n=1}^{\infty} x_n e_n.$$

Any sequence (x_n) of scalars associated with a $x \in \mathbb{B}$, $\lim_{n\to\infty} x_n = 0$. Suppose

$$j_k = \left[\frac{-1}{2\ln(k+1)}, \frac{1}{2\ln(k+1)}\right]$$

and

$$j^n = \Pi_{k=n+1}^{\infty} j_k, \ j = \Pi_{k=1}^{\infty} J_k.$$

Suppose $\{e_k\}$ is an S-basis for $\mathbb{B}$, and suppose $x = \sum_{n=1}^{\infty} x_n e_n$, from $\mathcal{P}_n(x) = \sum_{k=1}^{n} x_k e_k$ and $\mathcal{Q}_n x = (x_1, x_2, \ldots, x_n)$, we define $\mathbb{B}_j^n$ as follows

$$\mathbb{B}_j^n = \{\mathcal{Q}_n x : x \in \mathbb{B}\} \times j^n$$

with norm

$$||(x_k)||_{\mathbb{B}_j^n} = \max_{1\le k\le n} \left\| \sum_{i=1}^{k} x_i e_i \right\|_{\mathbb{B}} = \max_{1\le k\le n} ||\mathcal{P}(x)||_{\mathbb{B}}.$$

As $\mathbb{B}_j^n \subset \mathbb{B}_j^{n+1}$ so we can set $\mathbb{B}_j^\infty = \bigcup_{n=1}^{\infty} \mathbb{B}_j^n$ and $\mathbb{B}_j$ is a subset of $\mathbb{B}_j^\infty$. We set $\mathbb{B}_j$ as $\mathbb{B}_j = \left\{(x_1, x_2, \ldots) : \sum_{k=1}^{\infty} x_k e_k \in \mathbb{B}\right\}$ and norm on $\mathbb{B}_j$ by

$$||x||_{\mathbb{B}_j} = \sup_n \|\mathcal{P}_n(x)\|_{\mathbb{B}} = ||x||_{\mathbb{B}}.$$

On condition that we consider $\mathbb{B}[\mathbb{B}_j^\infty]$ as the smallest σ-algebra restraining $\mathbb{B}_j^\infty$ and define $\mathbb{B}[\mathbb{B}_j] = \mathbb{B}[\mathbb{B}_j^\infty] \cap \mathbb{B}_J$ then by a known result

$$||x||_{\mathbb{B}} = \sup_n \left\| \sum_{k=1}^{n} x_k e_k \right\|_{\mathbb{B}} \tag{2.1}$$

is an equivalent norm on $\mathbb{B}$.

Proposition 2.1. *[4] When $\mathbb{B}$ carries the equivalent norm (2.1), the operator*

$$T : (\mathbb{B}, ||.||_{\mathbb{B}}) \to (\mathbb{B}_j, ||.||_{\mathbb{B}_j})$$

denoted by $T(x) = (x_k)$ is an isometric isomorphism from $\mathbb{B}$ onto $\mathbb{B}_j$.

This shows that every Banach space with an S-basis has a natural embedding in $\mathbb{B}_j^\infty$. So, we call $\mathbb{B}_j$ the canonical representation of $\mathbb{B}$ in $\mathbb{B}_j^\infty$. With $\mathbb{B}[\mathbb{B}_j] = \mathbb{B}_j \cap \mathbb{B}[\mathbb{B}_j^\infty]$ we define $\sigma-$algebra generated by $\mathbb{B}$ and associated with $\mathbb{B}[\mathbb{B}_j]$ by

$$\mathbb{B}_j[\mathbb{B}] = \{T^{-1}(A) \ |A \in \mathbb{B}[\mathbb{B}_j]\} = T^{-1}\{\mathbb{B}[\mathbb{B}_j]\}.$$

Since $\mu_{\mathbb{B}}(A_j^n) = 0$ for $A_j^n \in \mathbb{B}[\mathbb{B}_j^n]$ with A_j^n compact, we see $\mu_{\mathbb{B}}(\mathbb{B}_j^n) = 0, \ n \in \mathbb{N}$. So, $\mu_{\mathbb{B}}(\mathbb{B}_j) = 0$ for every Banach space with an S-basis. Thus the restriction of $\mu_{\mathbb{B}}$ to $\mathbb{B}_j$ will not induce a non trivial measure on $\mathbb{B}$.

Definition 2.1. [4, 19] We define $\overline{v}_k, \overline{\mu}_k$ on $A \in B[\mathbb{R}]$ by

$$\overline{v}_k(A) = \frac{\mu(A)}{\mu(j_k)}, \overline{\mu}_k(A) = \frac{\mu(A \cap j_k)}{\mu(j_k)}$$

and for an elementary set $A = \pi_{k=1}^{\infty} A_k \in B[\mathbb{B}_j^n]$, define $\overline{V}_j^n$ by

$$\overline{V}_j^n = \pi_{k=1}^{n} \overline{v}_k(A) \times \pi_{k=n+1}^{\infty} \overline{\mu}_k(A).$$

Let V_j^n denote the Lebesgue extension of $\overline{V}_j^n$ to all $\mathbb{B}_j^n$ and $V_j(A) = \lim_{n\to\infty} V_j^n(A)$, $\forall A \in B[\mathbb{B}_j]$. We adopt a variation of method developed by Yamasaki [19], to define V_j^n to the Lebesgue extension of $\overline{V}_j^n$ for all $\mathbb{B}_j^n$ and define $V_j(\mathbb{B}) = \lim_{n\to\infty} V_j^n(\mathbb{B}), \forall \mathbb{B} \in \mathbb{B}[\mathbb{B}_j]$.

Remark 2.1. Let $\mathbb{B}_j$ be the image of $\mathbb{B}$ in $\mathbb{B}_j^{\infty}$, which can be endued with a norm via $||u_1, u_2, \ldots, u_n||_{\mathbb{B}_j} = ||u||_{\mathbb{B}}$.

2.1. Integration theory on $\mathbb{B}_j^{\infty}$

In this section, we study the integration on a separable Banach space $\mathbb{B}_j^{\infty}$ with an $S-$basis. Recalling $\mu_{\mathbb{B}}$ restricted to $B[\mathbb{B}_j^n]$ is equivalent to $\mu\mathbb{A}_n$. Assume that the integral restricted to $B[\mathbb{B}_j^n]$ is the integral on $\mathbb{R}^n$. Suppose $f : \mathbb{B} \to [0, \infty]$ is a measurable function and suppose $\mu_{\mathbb{B}}$ is constructed using the family $\{j_k\}$. If $\{j_n\} \subset M$ is an increasing family of non negative simple functions with $j_n \in M_j^n$, for each n and $\lim_{n\to\infty} j_n(x) = f(x)$, $\mu_{\mathbb{B}}-$a.e. We consider the integral of f over $\mathbb{B}_j^{\infty}$ by

$$\int_{\mathbb{B}_j^{\infty}} f(x) d\mu_{\mathbb{B}} = \lim_{n\to\infty} \int_{\mathbb{B}_j^{\infty}} \left[j_n(x) \prod_{i=1}^{n} \mu(j_i) \right] d\mu_{\mathbb{B}}(x).$$

Suppose $\mathcal{L}^1[\mathbb{B}_j^n]$ is the class of integrable functions on $\mathbb{B}_j^n$. Since $\mathcal{L}^1[\mathbb{B}_j^n] \subset \mathcal{L}^1[\mathbb{B}_j^{n+1}]$, we define $\mathcal{L}^1[\widehat{\mathbb{B}}_j^n] = \bigcup_{n=1}^{\infty} \mathcal{L}^1[\mathbb{B}_j^n]$.

1. We say that a measurable function $f \in \mathcal{L}^1[\mathbb{B}_j^{\infty}]$ if there exists a Cauchy sequence $\{f_m\} \subset \mathcal{L}^1[\widehat{\mathbb{B}}_j^{\infty}]$, such that

$$\lim_{m\to\infty} \int_{\mathbb{B}_j^{\infty}} |f_m(x) - f(x)| d\mu_{\mathbb{B}}(x) = 0.$$

   That is a measurable function $f \in \mathcal{L}^1[\mathbb{B}_j^{\infty}]$ if there exists a Cauchy sequence $\{f_m\} \subset \mathcal{L}^1[\widehat{\mathbb{B}}_j^{\infty}]$, with $f_m \in \mathcal{L}^1[\mathbb{B}_j^n]$ and

$$\lim_{m\to\infty} f_m(x) = f(x), \ \mu_{\mathbb{B}} - (a.e.).$$

2. We say that a measurable function $f \in C_0[\mathbb{B}_j^\infty]$, the space of continuous functions that vanish at infinity, if there exists a Cauchy sequence $\{f_m\} \subset C_0[\widehat{\mathbb{B}}_j^\infty]$, such that

$$\lim_{m\to\infty}\int_{\mathbb{B}_j^\infty}\sup_{x\in\mathbb{B}_j^\infty}|f_m(x)-f(x)|d\mu_{\mathbb{B}}(x)=0.$$

Theorem 2.1. $\mathcal{L}^1[\widehat{\mathbb{B}}_j^\infty]=\mathcal{L}^1[\mathbb{B}_j^\infty]$.

Definition 2.2. If $f \in \mathcal{L}^1[\mathbb{B}_j^\infty]$, we define the integral of f by

$$\lim_{m\to\infty}\int_{\mathbb{B}_j^\infty} f_m(x)d\mu_{\mathbb{B}}(x)=\int_{\mathbb{B}_j^\infty} f(x)d\mu_{\mathbb{B}}(x), \mu_{\mathbb{B}}-(a.e.),$$

where $\{f_m\} \subset \mathcal{L}^1[\mathbb{B}_j^\infty]$ is any Cauchy sequence converging to $f(x)-$a.e.

Theorem 2.2. *If $f \in L^1[\mathbb{B}_j^\infty]$, then the above integral exists and all theorems that are true for $f \in \mathcal{L}^1[\mathbb{B}_j^n]$, also hold for $f \in \mathcal{L}^1[\mathbb{B}_j^\infty]$.*

Lemma 2.1. (Kuelbs Lemma) *[11] Let $\mathbb{B}$ be a separable Banach space. Then there exists a separable Hilbert space such that $\mathbb{B} \hookrightarrow H$ is a continuous dense embedding.*

3. The Kuelbs-Steadman space $\mathbb{K}^p[\mathbb{B}]$

In this section, we study the Kuelbs-Steadman space $\mathbb{K}^p[\mathbb{B}]$, where $\mathbb{B}$ is a separable Banach space. We proceed for the construction of the canonical space $\mathbb{K}^p[\mathbb{B}_J^\infty]$. Suppose $\mathbb{B}_j^n$ is a separable Banach space with $S-$basis, $\mathbb{K}^p[\widehat{\mathbb{B}}_j^n]=\bigcup_{k=1}^\infty \mathbb{K}^p[\mathbb{B}_j^k]$, and $C_0[\widehat{\mathbb{B}}_j^n]=\bigcup_{n=1}^\infty C_0[\mathbb{B}_j^n]$.

Definition 3.1. A measurable function f is said to be in $\mathbb{K}^p[\mathbb{B}_j^n]$ if there exists a Cauchy sequence $\{f_m\} \subset \mathbb{K}^p[\widehat{\mathbb{B}}_j^n]$, with $f_m \in \mathbb{K}^p[\widehat{\mathbb{B}}_j^n]$ such that

$$\lim_{m\to\infty} f_m(x)=f(x),\ \mu_{\mathbb{B}}-(a.e.)$$

Theorem 3.1. $\mathbb{K}^p[\widehat{\mathbb{B}}_j^n]=\mathbb{K}^p[\mathbb{B}_j^n]$.

Definition 3.2. Let $f \in \mathbb{K}^p[\mathbb{B}_j^n]$. The integral of f is defined by

$$\lim_{m\to\infty}\int_{\mathbb{B}_j^n} f_m(x)d\mu_{\mathbb{B}}(x)=\int_{\mathbb{B}} f(x)d\mu_{\mathbb{B}}(x),\ \mu_{\mathbb{B}}-(a.e.),$$

where $\{f_m\} \subset \mathbb{K}^p[\mathbb{B}_j^n]$ is any Cauchy sequence converging to $f(x)-$a.e.

Theorem 3.2. *If $f \in \mathbb{K}^p[\mathbb{B}_j^n]$, then the above integral exists.*

3.1. The construction of $\mathbb{K}^p[\mathbb{B}_j^n]$

We start with $\mathcal{L}^1[\mathbb{B}_j^n]$, picking a countable dense set of sequences $\{\mathcal{E}_n(x)\}_{n=1}^{\infty}$ on the unit ball of $\mathcal{L}^1[\mathbb{B}_j^n]$ and assume $\{\mathcal{E}_n^*\}_{n=1}^{\infty}$ is any corresponding set of duality mapping in $\mathcal{L}^\infty[\mathbb{B}]$, also on condition that $\mathbb{B}$ is $\mathcal{L}^1[\mathbb{B}_j^n]$, using Kuelbs Lemma, it is clear that the Hilbert space $\mathbb{K}^2[\mathbb{B}_j^n]$ will contain some non absolute integrable functions. From [17], we confirm that the non absolute integral is Henstock-Kurzweil integral (HK). Let $\overline{\mathcal{E}}_k(x)$ be the characteristic function of $\mathbb{B}_k$, so that $\overline{\mathcal{E}}_k(x) \in \mathcal{L}^p[\mathcal{B}_j^n] \cap \mathcal{L}^\infty[\mathbb{B}_j^n]$ for $1 \le p < \infty$. Define $F_k(f)$ on $\mathcal{L}^1[\mathbb{B}_j^n]$ by

$$F_k(f) = \int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x) f(x) d\mu_{\mathbb{B}}(x).$$

Since each $\mathbb{B}_k$ is a cube with sides parallel to the co-ordinate axes, $F_k(.)$ is well defined for all HK-integrable functions, and is a bounded linear functional on $\mathcal{L}^p[\mathbb{B}_j^n]$ for $1 \le p \le \infty$. Let $\mathbf{b}_k > 0$ be such that $\sum_{k=1}^{\infty} \mathbf{b}_k = 1$ and denote the inner product $(.)$ on $\mathcal{L}^1[\mathbb{B}_j^n]$ by

$$(f,g) = \sum_{k=1}^{\infty} \mathbf{b}_k \left[\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x) f(x) d\mu_{\mathbb{B}}(x) \right] \left[\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(y) g(y) d\mu_{\mathbb{B}}(y) \right]^c .$$

The completion of $\mathcal{L}^1[\mathbb{B}_j^n]$ in the inner product is the space $\mathbb{K}^2[\mathbb{B}_j^n]$. We can see directly that $\mathbb{K}^2[\mathbb{B}_j^n]$ contains the HK-integrable functions. We call the completion of $\mathcal{L}^1[\mathbb{B}_j^n]$ with the above inner product, the Kuelbs-Steadman space $\mathbb{K}^2[\mathbb{B}_j^n]$.

Theorem 3.3. *The space $\mathbb{K}^2[\mathbb{B}_j^n]$ contains $\mathcal{L}^p[\mathbb{B}_j^n]$ (for each p, $1 \le p < \infty$) as a dense subspace.*

Proof. We know $\mathbb{K}^2[\mathbb{B}_j^n]$ contains $\mathcal{L}^1[\mathbb{B}_j^n]$ densely. Thus we need only to show $L^q[\mathbb{B}_j^n] \subset \mathbb{K}^2[\mathbb{B}_j^n]$ for $q \neq 1$. Suppose $f \in L^q[\mathbb{B}_j^n]$ and $q < \infty$. Since $|\mathcal{E}(x)| = \mathcal{E}(x) \le 1$ and $|\mathcal{E}(x)|^q \le \mathcal{E}(x)$, we have

$$\begin{aligned} ||f||_{\mathbb{K}^2} &= \left[\sum_{n=1}^{\infty} \mathbf{b}_k \left| \int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x) f(x) d\mu_{\mathbb{B}}(x) \right|^{\frac{2q}{q}} \right]^{\frac{1}{2}} \\ &\le \left[\sum_{n=1}^{\infty} \mathbf{b}_k \left(\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x) |f(x)|^q d\mu_{\mathbb{B}}(x) \right)^{\frac{2}{q}} \right]^{\frac{1}{2}} \\ &\le \sup_k \left(\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x) |f(x)|^q d\mu_{\mathbb{B}}(x) \right)^{\frac{1}{q}} \le ||f||_q. \end{aligned}$$

Therefore $f \in \mathbb{K}^2[\mathbb{B}_j^n]$. $\square$

We can construct the norm of $\mathbb{K}^p[\mathbb{B}_j^n]$, which is defined by

$$||f||_{\mathbb{K}^p[\mathbb{B}_j^n]} = \begin{cases} \left(\sum_{k=1}^{\infty} \mathbf{b}_k \left| \int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x) f(x) d\mu_{\mathcal{B}}(x) \right|^p \right)^{\frac{1}{p}}, \text{ for } 1 \leq p < \infty; \\ \sup_{k \geq 1} \left| \int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x) f(x) d\mu_{\mathcal{B}}(x) \right|, \text{ for } p = \infty \end{cases}$$

It is easy to see that $||.||_{\mathbb{K}^p[\mathbb{B}_j^n]}$ is a norm on $\mathcal{L}^p[\mathbb{B}_j^n]$. If $\mathbb{K}^p[\mathbb{B}]$ is the completion of $\mathcal{L}^p[\mathbb{B}]$ with respect to this norm, we have the following theorem.

Theorem 3.4. *For each* q, $1 \leq q < \infty$, $L^q[\mathbb{B}_j^n] \hookrightarrow \mathbb{K}^p[\mathbb{B}_j^n]$ *is a densely continuous embedding.*

Proof. We know from Theorem 3.3, and by the construction of $\mathbb{K}^p[\mathbb{B}_j^n]$ contains $\mathcal{L}^p[\mathbb{B}_j^n]$ densely. Thus we need only to show $L^q[\mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{B}_j^n]$ for $q \neq p$. Suppose $f \in L^q[\mathbb{B}_j^n]$ and $q < \infty$. Since $|\mathcal{E}(x)| = \mathcal{E}(x) \leq 1$ and $|\mathcal{E}(x)|^q \leq \mathcal{E}(x)$, we have

$$\begin{aligned} ||f||_{\mathbb{K}^p} &= \left[\sum_{n=1}^{\infty} \mathbf{b}_k \left| \int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x) f(x) d\mu_{\mathbb{B}}(x) \right|^{\frac{qp}{q}} \right]^{\frac{1}{p}} \\ &\leq \left[\sum_{n=1}^{\infty} \mathbf{b}_k \left(\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x) |f(x)|^q d\mu_{\mathbb{B}}(x) \right)^{\frac{p}{q}} \right]^{\frac{1}{p}} \\ &\leq \sup_k \left(\int_{\mathbb{B}} \overline{\mathcal{E}}_k(x) |f(x)|^q d\mu_{\mathbb{B}}(x) \right)^{\frac{1}{q}} \leq ||f||_q. \end{aligned}$$

Therefore $f \in \mathbb{K}^p[\mathbb{B}_j^n]$. □

Corollary 3.1. $\mathcal{L}^\infty[\mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{B}_j^n]$.

Theorem 3.5. $\mathbb{C}_c[\mathbb{B}_j^n]$ *is dense in* $\mathbb{K}^2[\mathbb{B}_j^n]$.

Proof. As $\mathbb{C}_c[\mathbb{B}_j^n]$ is dense in $\mathcal{L}^p[\mathbb{B}_j^n]$ and $\mathcal{L}^p[\mathbb{B}_j^n]$ is densely contained in $\mathbb{K}^2[\mathbb{B}_j^n]$, the conclusion follows. □

Remark 3.1. As Hölder and generalized Hölder inequalities for $\mathcal{L}^p[\mathbb{B}_j^n]$ are valid for $1 \leq p < \infty$ (see [4, P. 83]). As $\mathbb{K}^p[\mathbb{B}_j^n]$ is completion of $\mathcal{L}^p[\mathbb{B}_j^n]$, the Hölder and generalized Hölder inequalities hold in $\mathbb{K}^p[\mathbb{B}_j^n]$ for $1 \leq p < \infty$.

Theorem 3.6. (The Minkowski Inequality) *Suppose* $1 \leq p < \infty$ *and* $f, g \in \mathbb{K}^p[\mathbb{B}_j^n]$. *Then* $f + g \in \mathbb{K}^p[\mathbb{B}_j^n]$ *and*

$$||f + g||_{\mathbb{K}^p[\mathbb{B}_j^n]} \leq ||f||_{\mathbb{K}^p[\mathbb{B}_j^n]} + ||g||_{\mathbb{K}^p[\mathbb{B}_j^n]}.$$

Proof. The proof follows from the Lemma 2 of [14]. $\square$

Theorem 3.7. *For $1 \leq p \leq \infty$, we have*

1. *If $f_n \to f$ weakly in $\mathcal{L}^p[\mathbb{B}_j^n]$, then $f_n \to f$ strongly in $\mathbb{K}^p[\mathbb{B}_j^n]$.*
2. *If $1 < p < \infty$, then $\mathbb{K}^p[\mathbb{B}_j^n]$ is uniformly convex.*
3. *If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the dual space of $\mathbb{K}^p[\mathbb{B}_j^n]$ is $\mathbb{K}^q[\mathbb{B}_j^n]$.*
4. *$\mathbb{K}^\infty[\mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{B}_j^n]$, for $1 \leq p < \infty$.*

Proof. (1) If $\{f_n\}$ is weakly convergence sequence in $\mathcal{L}^p[\mathbb{B}_j^n]$ with limit f. Then $\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x)[f_n(x) - f(x)]d\mu_{\mathbb{B}}(x) \to 0$ for each k.
Now for $\{f_n\} \in \mathbb{K}^p[\mathbb{B}_j^n]$ we find the following:

$$\lim_{n\to\infty} \int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x)[f_n(x) - f(x)]d\mu_{\mathbb{B}}(x) \to 0.$$

So, $\{f_n\}$ is converges strongly in $\mathbb{K}^p[\mathbb{B}_j^n]$.
(2) We know $\mathcal{L}^p[\mathbb{B}_j^n]$ is uniformly convex and that is dense and compactly embedded in $\mathbb{K}^q[\mathbb{B}_j^n]$ for all $q,\ 1 \leq q \leq \infty$. So, $\bigcup\limits_{n=1}^{\infty} \mathcal{L}^p[\mathbb{B}_j^n]$ is uniformly convex for each n and that is dense and compactly embedded in $\bigcup\limits_{n=1}^{\infty} \mathbb{K}^q[\mathbb{B}_j^n]$ for all $q,\ 1 \leq q \leq \infty$. However $\mathcal{L}^p[\widehat{\mathbb{B}_j^n}] = \bigcup\limits_{n=1}^{\infty} \mathcal{L}^p[\mathbb{B}_j^n]$. That is $\mathcal{L}^p[\widehat{\mathbb{B}_j^n}]$ is uniformly convex, dense and compactly embedded in $\mathbb{K}^q[\widehat{\mathbb{B}_j^n}]$ for all $q,\ 1 \leq q \leq \infty$ as $\mathbb{K}^q[\mathbb{B}_j^n]$ is the closure of $\mathbb{K}^q[\widehat{\mathbb{B}_j^n}]$. Therefore $\mathbb{K}^q[\mathbb{B}_j^n]$ is uniformly convex.
(3) From (2), that $\mathbb{K}^p[\mathbb{B}_j^n]$ is reflexive for $1 < p < \infty$ as

$$\{\mathbb{K}^p[\mathbb{B}_j^n]\}^* = \mathbb{K}^q[\mathbb{B}_j^n],\ \frac{1}{p} + \frac{1}{q} = 1,\ \forall n$$

and

$$\mathbb{K}^p[\mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{B}_J^{n+1}],\ \forall n\ \Rightarrow \bigcup_{n=1}^{\infty} \{\mathbb{K}^p[\mathbb{B}_j^n]\}^* = \bigcup_{n=1}^{\infty} \mathbb{K}^q[\mathbb{B}_j^n],\ \frac{1}{p} + \frac{1}{q} = 1.$$

Since each $f \in \mathbb{K}^p[\mathbb{B}_j^n]$ is the limit of a sequence $\{f_n\} \subset \mathbb{K}^p[\widehat{\mathbb{B}_j^n}] = \bigcup\limits_{n=1}^{\infty} \mathbb{K}^p[\mathbb{B}_j^n]$, we see that $\{\mathbb{K}^p[\mathbb{B}_j^n]\}^* = \mathbb{K}^q[\mathbb{B}_j^n]$, for $\frac{1}{p} + \frac{1}{q} = 1$.
(4) Suppose $f \in \mathbb{K}^\infty[\mathbb{B}_j^n]$. This implies $|\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x)f(x)d\mu_{\mathbb{B}}(x)|$ is uniformly bounded for all k. It follows that $|\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x)f(x)d\mu_{\mathbb{B}}(x)|^p$ is uniformly bounded for all $1 \leq p < \infty$. It is clear from the definition of $\mathbb{K}^p[\mathbb{B}_j^n]$ that

$$\left[\sum \left|\int_{\mathbb{B}_j^n} \overline{\mathcal{E}}_k(x)f(x)d\mu_{\mathbb{B}}(x)\right|^p\right]^{\frac{1}{p}} \quad \leq \quad M||f||_{\mathbb{K}^p[\mathbb{B}_j^n]} < \infty.$$

So, $f \in \mathbb{K}^p[\mathbb{B}_j^n]$. This completes the proof. □

Theorem 3.8. *$C_c^\infty[\mathbb{B}_j^n]$ is a dense subset of $\mathbb{B}\mathbb{K}^2[\mathbb{B}_j^n]$.*

Proof. As $C_c^\infty[\mathbb{B}_j^n]$ is dense in $\mathcal{L}^p[\mathbb{B}_j^n], \forall p$. Moreover $\mathcal{L}^p[\mathbb{B}_j^n]$ is a dense subset of $\mathbb{K}^2[\mathbb{B}_j^n]$. So, $C_c^\infty[\mathbb{B}_j^n]$ is a dense subset of $\mathbb{K}^2[\mathbb{B}_j^n]$. □

Corollary 3.2. *The embedding $C_0^\infty[\mathbb{B}_j^n] \hookrightarrow \mathbb{K}^p[\mathbb{B}_j^n]$ is dense.*

Remark 3.2. Since $\mathcal{L}^1[\mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{B}_j^n]$ and $\mathbb{K}^p[\mathbb{B}_j^n]$ is reflexive for $1 < p < \infty$. We see the second dual $\{\mathcal{L}^1[\mathbb{B}_j^n]\}^{**} = \mathfrak{M}[\mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{B}_j^n]$, where $\mathfrak{M}[\mathbb{B}_j^n]$ is the space of bounded finitely additive set functions define on the Borel sets $B[\mathbb{B}_j^n]$.

3.2. The family of $\mathbb{K}^p[\mathbb{B}_j^\infty]$

We can now construct the spaces $\mathbb{K}^p[\mathbb{B}_j^\infty],\ 1 \leq p \leq \infty$, using the same approach that led to $\mathcal{L}^1[\mathbb{B}_j^\infty]$. Since $\mathbb{K}^p[\mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{B}_j^{n+1}]$. We define $\mathbb{K}^p[\widehat{\mathbb{B}}_j^\infty] = \bigcup_{n=1}^\infty \mathbb{K}^p[\mathbb{B}_j^n]$.

Definition 3.3. A measurable function f is said to be in $\mathbb{K}^p[\mathbb{B}_j^\infty]$, for $1 \leq p \leq \infty$, if there is a Cauchy sequence $\{f_n\} \subset \mathbb{K}^p[\widehat{\mathbb{B}}_j^\infty]$ with $f_n \in \mathbb{K}^p[\mathbb{B}_j^n]$ such that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ $\mu_{\mathbb{B}}$-a.e.

The functions in $\mathbb{K}^p[\widehat{\mathbb{B}}_j^\infty]$ differ from functions in its closure $\mathbb{K}^p[\mathbb{B}_j^\infty]$, by sets of measure zero.

Theorem 3.9. $\mathbb{K}^p[\widehat{\mathbb{B}}_j^\infty] = \mathbb{K}^p[\mathbb{B}_j^\infty]$.

Definition 3.4. If $f \in \mathbb{K}^p[\mathcal{B}_j^\infty]$, we define the integral of f by

$$\int_{\mathbb{B}_j^\infty} f(x)d\mu_{\mathbb{B}}(x) = \lim_{n\to\infty} \int_{\mathbb{B}_j^n} f_n(x)d\mu_{\mathbb{B}}(x),$$

where $f_n \in \mathbb{K}^p[\mathbb{B}_j^n]$ is any Cauchy sequence convergerging to $f(x)$.

Theorem 3.10. *If $f \in \mathbb{K}^p[\mathbb{B}_j^\infty]$, then the integral of f define in Definition 3.4 exists and is unique for every $f \in \mathbb{K}^p[\mathbb{B}_j^\infty]$.*

Proof. If in the consideration of the family of functions $\{f_n\}$ is Cauchy, it follows: On condition that the integral exists, it is unique. For existence considering $f(x) \geqslant 0$ with standard argument with the assumption of increasing sequence so that the integral exists. The general case now follows by the standard decomposition. □

Theorem 3.11. *If $f \in \mathbb{K}^p[\mathbb{B}_j^\infty]$, then all theorems that are true for $f \in \mathbb{K}^p[\mathbb{B}_j^n]$, also hold for $f \in \mathbb{K}^p[\mathbb{B}_j^\infty$.]*

Theorem 3.12. *$\mathbb{K}^p[\mathbb{B}_j^n]$ and $\mathbb{K}^p[\mathbb{B}_j]$ are equivalent spaces.*

Proof. Let $\mathbb{B}_j^n$ is a separable Banach space, T maps $\mathbb{B}_j^n$ onto $\mathbb{B}_j \subset \mathbb{B}_j^\infty$, where T is an isometric isomorphism so that $\mathbb{B}_j$ is an embedding of $\mathbb{B}_j^n$ into $\mathbb{R}_J^\infty$. This is how we able to define a Lebesgue integral on $\mathbb{B}_j^n$ using $\mathbb{B}_j$ and T^{-1}. Thus $\mathbb{K}^p[\mathbb{B}_j^n]$ and $\mathbb{K}^p[\mathbb{B}_j]$ are not different spaces. □

Theorem 3.13. *$\mathbb{K}^p[\mathbb{B}_j^\infty]$ can be embedded into $\mathbb{K}^p[\mathbb{R}_J^\infty]$ as a closed subspace.*

Proof. As every separable Banach space can be embedded in $\mathbb{R}_J^\infty$ as a closed subspace containing $\mathbb{B}_j^\infty$. So, $\mathbb{K}^p[\mathbb{B}_j^\infty] \subset \mathbb{K}^p[\mathbb{R}_J^\infty]$ embedding as a closed subspace. That is $\mathbb{K}^p[\bigcup_{n=1}^\infty \mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{R}_J^\infty]$ embedding as a closed subspace. So, $\mathbb{K}^p[\mathbb{B}_j^n] \subset \mathbb{K}^p[\mathbb{R}_J^\infty]$ embedding as a closed subspace. Finally we can conclude that $\mathbb{K}^p[\mathbb{B}_J^\infty] \subset \mathbb{K}^p[\mathbb{R}_J^\infty]$ embedding as closed subspace. □

3.3. Feynman path integral

The properties of $\mathbb{K}^2[\mathbb{B}_J^\infty]$ derived earlier suggests that it may be a better replacement of $\mathcal{L}^2[\mathbb{B}_J^\infty]$ in the study of the Path Integral formulation of quantum theory developed by Feynman. We see that position operator have closed densely define extensions to $\mathbb{K}^2[\mathbb{B}_J^\infty]$. Further Fourier and convolution insure that all of the Schrödinger and Heisenberg theories have a faithful representation on $\mathbb{K}^2[\mathbb{B}_J^\infty]$. Since $\mathbb{K}^2[\mathbb{B}_J^\infty]$ contains the space of measures, it follows that all the approximating sequences for Dirac measure convergent strongly in $\mathbb{K}^2[\mathbb{B}_J^\infty]$.

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