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ON CARTAN NULL BERTRAND CURVES IN MINKOWSKI 3-SPACE

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Abstract. In this paper, we consider Cartan null Bertrand curves in Minkowski 3-space. Since the principal normal vector of a null curve is a spacelike vector, the Bertrand mate curve of a null curve can be a timelike curve and a spacelike curve with spacelike principal normal. We give the necessary and sufficient conditions for these cases to be Bertrand curves and we also give the related examples.
Keywords: Bertrand curve, Minkowski 3-space, Cartan null curve, non-null curve.

1. Introduction

In the theory of curves in Euclidean space, one of the important and interesting problem is characterization of a regular curve. In the solution of the problem, the curvature functions κ_1 (or $\varkappa$) and κ_2 (or τ) of a regular curve have an effective role. For example: if $\kappa_1 = 0 = \kappa_2$, then the curve is a geodesic or if $\kappa_1 = \text{constant} \neq 0$ and $\kappa_2 = 0$, then the curve is a circle with radius $(1/\kappa_1)$, etc. Another way in the solution of the problem is the relationship between the Frenet vectors and Frenet planes of the curves ([8],[13]). *Mannheim* curves is an interesting examples for such classification. If there exists a corresponding relationship between the space curves α and β such that, at the corresponding points of the curves, the principal normal lines of α coincides with the binormal lines of β, then α is called a Mannheim curve, β is called Mannheim partner curve of α. Mannheim partner curves was studied by *Liu* and *Wang* (see [10]) in Euclidean 3-space and Minkowski 3-space.

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Another interesting example is *Bertrand* curves. A Bertrand curve is a curve in the Euclidean space such that its principal normal is the principal normal of the second curve ([3],[18]). The study of this kind of curves has been extended to many other ambient spaces. In [12], Pears studied this problem for curves in the n-dimensional Euclidean space $\mathbb{E}^n$, $n > 3$, and showed that a Bertrand curve in $\mathbb{E}^n$ must belong to a three-dimensional subspace $\mathbb{E}^3 \subset \mathbb{E}^n$. This result is restated by Matsuda and Yorozu [11]. They proved that there was not any special Bertrand curves in $\mathbb{E}^n$ $(n > 3)$ and defined a new kind, which is called $(1,3)$-type Bertrand curves in 4-dimensional Euclidean space. Bertrand curves and their characterizations were studied by many researchers in Minkowski 3-space and Minkowski space-time (see [1], [2], [6], [7], [14], [15]) as well as in Euclidean space. In addition, $(1,3)$-type Bertrand curves were studied in semi-Euclidean 4-space with index 2 ([16],).

Following [17], in this paper, we consider Cartan null Bertrand curves in Minkowski 3-space. Since the principal normal vector of a null curve is a spacelike vector, the Bertrand mate curve of a null curve can be a null curve, a timelike curve and a spacelike curve with spacelike principal normal. The case where the Bertrand mate curve is a null curve, were studied in [2]. Thus, we give the necessary and sufficient conditions for other cases to be Bertrand curves and we also give the related examples.

2. Preliminaries

The Minkowski space $\mathbb{E}^3_1$ is the Euclidean 3-space $\mathbb{E}^3$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3_1$. Recall that a vector $v \in \mathbb{E}^3_1 \backslash \{0\}$ can be *spacelike* if $g(v,v) > 0$, *timelike* if $g(v,v) < 0$ and *null (lightlike)* if $g(v,v) = 0$ and $v \neq 0$. In particular, the vector $v = 0$ is a spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$, and two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}^3_1$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null ([9]). Spacelike curve in $\mathbb{E}^3_1$ is called *pseudo null curve* if its principal normal vector N is null [4]. A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$. Also null curve is called null Cartan curve if it is parameterized by pseudo-arc function. A spacelike or a timelike curve $\alpha(s)$ has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$ ([4]).

Let $\{T, N, B\}$ be the moving Frenet frame along a curve α in $\mathbb{E}^3_1$, consisting of the tangent, the principal normal and the binormal vector fields respectively. Depending on the causal character of α, the Frenet equations have the following forms.

Case I. If α is a non-null curve, the Frenet equations are given by ([9]):

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2 k_1 & 0 \\ -\epsilon_1 k_1 & 0 & \epsilon_3 k_2 \\ 0 & -\epsilon_2 k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1}$$

where k_1 and k_2 are the first and the second curvature of the curve respectively. Moreover, the following conditions hold:

$$g(T,T) = \epsilon_1 = \pm 1, g(N,N) = \epsilon_2 = \pm 1, g(B,B) = \epsilon_3 = \pm 1$$

and

$$g(T,N) = g(T,B) = g(N,B) = 0.$$

Case II. If α is a null Cartan curve, the Cartan equations are given by ([4])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_2 & 0 & -k_1 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.2}$$

where the first curvature $k_1 = 0$ if α is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$$g(T,T) = g(B,B) = g(T,N) = g(N,B) = 0, g(N,N) = g(T,B) = 1.$$

3. Cartan Null Bertrand curves in Minkowski 3-space

In this section, we consider the Cartan null Bertrand curves in $\mathbb{E}_1^3$. We get the necessary and sufficient conditions for the Cartan null curves to be Bertrand curves in $\mathbb{E}_1^3$ and we also give the related examples.

Definition 3.1. A Cartan null curve $\alpha : I \to \mathbb{E}_1^3$ with $\kappa_1(s) \neq 0$ is a Bertrand curve if there is a curve $\alpha^* : I^* \to \mathbb{E}_1^3$ such that the principal normal vectors of $\alpha(s)$ and $\alpha^*(s^*)$ at $s \in I$, $s^* \in I^*$ are equal. In this case, $\alpha^*(s^*)$ is the Bertrand mate of $\alpha(s)$.

Let $\beta : I \to \mathbb{E}_1^3$ be a Cartan null Bertrand curve in $\mathbb{E}_1^3$ with the Frenet frame $\{T, N, B\}$ and the curvatures κ_1, κ_2, and $\beta^* : I \to \mathbb{E}_1^3$ be a Bertrand mate curve of β with the Frenet frame $\{T^*, N^*, B^*\}$ and the curvatures κ_1^*, κ_2^*.

Theorem 3.1. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a Cartan null curve parametrized by pseudo arc parameter with curvatures $\kappa_1 \neq 0, \kappa_2$. Then the curve β is a Bertrand curve with Bertrand mate β^* if and only if one of the following conditions holds:*
(i) there exists constant real numbers λ and h satisfying

$$h < 0, \quad 1 + \lambda\kappa_2 = -h\lambda\kappa_1, \quad \kappa_2 - h\kappa_1 \neq 0, \quad \kappa_2 + h\kappa_1 \neq 0. \tag{3.1}$$

In this case, β^* is a timelike curve in $\mathbb{E}_1^3$.

(ii) there exists constant real numbers λ and h satisfying

$$h > 0, \quad 1 + \lambda\kappa_2 = -h\lambda\kappa_1, \quad \kappa_2 - h\kappa_1 \neq 0, \quad \kappa_2 + h\kappa_1 \neq 0. \tag{3.2}$$

In this case, β^* is a spacelike curve with spacelike principal normal in $\mathbb{E}_1^3$.

Proof. Assume that β is a Cartan null Bertrand curve parametrized by pseudo arc parameter s with $\kappa_1 \neq 0, \kappa_2$ and the curve β^* is the Bertrand mate curve of the curve β parametrized by with arc-length or pseudo arc s^*.

(i) Let β^* be a timelike curve. Then, we can write the curve β^* as

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda(s)N(s) \tag{3.3}$$

for all $s \in I$ where $\lambda(s)$ is $C^\infty-$function on I. Differentiating (3.3) with respect to s and using (2.1),(2.2), we get

$$T^* f' = (1 + \lambda\kappa_2)T + \lambda' N - \lambda\kappa_1 B. \tag{3.4}$$

By taking the scalar product of (3.4) with N, we have

$$\lambda' = 0. \tag{3.5}$$

Substituting (3.5) in (3.4), we find

$$T^* f' = (1 + \lambda\kappa_2)T - \lambda\kappa_1 B. \tag{3.6}$$

By taking the scalar product of (3.6) with itself, we obtain

$$(f')^2 = 2\lambda\kappa_1(1 + \lambda\kappa_2). \tag{3.7}$$

If we denote

$$\delta = \frac{1 + \lambda\kappa_2}{f'} \quad \text{and} \quad \gamma = \frac{-\lambda\kappa_1}{f'}, \tag{3.8}$$

we get

$$T^* = \delta T + \gamma B. \tag{3.9}$$

Differentiating (3.9) with respect to s and using (2.1),(2.2), we find

$$f'\kappa_1^* N^* = \delta' T + (\delta\kappa_1 - \gamma\kappa_2) N + \gamma' B. \tag{3.10}$$

By taking the scalar product of (3.10) with itself, we get

$$\delta' = 0 \quad \text{and} \quad \gamma' = 0. \tag{3.11}$$

Since $\gamma \neq 0$, we have $1 + \lambda\kappa_2 = -h\lambda\kappa_1$ where $h = \delta/\gamma$. Substituting (3.11) in (3.10) , we find

$$f'\kappa_1^* N^* = (\delta\kappa_1 - \gamma\kappa_2) N \tag{3.12}$$

By taking the scalar product of (3.12) with itself, using (3.7) and (3.8), we have

$$(f')^2 (\kappa_1^*)^2 = -\frac{(\kappa_2 - h\kappa_1)^2}{2h} \tag{3.13}$$

where $\kappa_2 - h\kappa_1 \neq 0$ and $h < 0$. If we put $v = \frac{\delta\kappa_1 - \gamma\kappa_2}{f'\kappa_1^*}$, we get

$$N^* = vN. \tag{3.14}$$

Differentiating (3.14) with respect to s and using (2.1),(2.2), we find

$$f'\kappa_2^* B^* = v\kappa_2 T - v\kappa_1 B - f'\kappa_1^* T^* \tag{3.15}$$

where $v' = 0$. Rewriting (3.15) by using (3.6) , we get

$$f'\kappa_2^* B^* = P(s)\,T + Q(s)\,B$$

where

$$\begin{aligned} P(s) &= \frac{\lambda\kappa_1 (\kappa_2 - h\kappa_1)(\kappa_2 + h\kappa_1)}{2(f')^2\kappa_1^*}, \\ Q(s) &= \frac{-\lambda\kappa_1 (\kappa_2 - h\kappa_1)(\kappa_2 + h\kappa_1)}{2h(f')^2\kappa_1^*} \end{aligned}$$

which implies that $\kappa_2 + h\kappa_1 \neq 0$.

Conversely, assume that β is a Cartan null curve parametrized by pseudo arc parameter s with $\kappa_1 \neq 0, \kappa_2$ and the conditions of (3.1) holds for constant real numbers λ and h. Then, we can define a curve β^* as

$$\beta^*(s^*) = \beta(s) + \lambda N(s). \tag{3.16}$$

Differentiating (3.16) with respect to s and using (2.2), we find

$$\frac{d\beta^*}{ds} = -\lambda\kappa_1 \{hT + B\} \tag{3.17}$$

which leads to that

$$f' = \sqrt{\left| g\left(\frac{d\beta^*}{ds}, \frac{d\beta^*}{ds}\right)\right|} = m_1\lambda\kappa_1\sqrt{-2h}$$

where $m_1 = \pm 1$ such that $m_1\lambda\kappa_1 > 0$. Rewriting (3.17), we obtain

$$T^* = \frac{-m_1}{\sqrt{-2h}}\{hT + B\}, \quad g(T^*, T^*) = -1. \tag{3.18}$$

Differentiating (3.18) with respect to s and using (2.2), we get

$$\frac{dT^*}{ds^*} = \frac{m_1 (\kappa_2 - h\kappa_1)}{f'\sqrt{-2h}} N$$

which causes that

$$\kappa_1^* = \left\| \frac{dT^*}{ds^*} \right\| = \frac{m_2 (\kappa_2 - h\kappa_1)}{f' \sqrt{-2h}} \tag{3.19}$$

where $m_2 = \pm 1$ such that $m_2 (\kappa_2 - h\kappa_1) > 0$. Now, we can find N^* as

$$N^* = m_1 m_2 N, \quad g(N^*, N^*) = 1. \tag{3.20}$$

Differentiating (3.20) with respect to s, using (3.18) and (3.19), we get

$$\frac{dN^*}{ds^*} - \kappa_1^* T^* = \frac{m_1 m_2 (\kappa_2 + h\kappa_1)}{2hf'} \{hT - B\}$$

which bring about that

$$\kappa_2^* = \frac{m_3 (\kappa_2 + h\kappa_1)}{f' \sqrt{-2h}},$$

where $m_3 = \pm 1$ such that $m_3 (\kappa_2 + h\kappa_1) > 0$. Lastly, we define B^* as

$$B^* = \frac{m_1 m_2 m_3 \sqrt{-2h}}{2} \left\{ T - \frac{1}{h} B \right\}, \quad g(B^*, B^*) = 1.$$

Then β^* is a timelike curve and the Bertrand mate curve of β. Thus β is a Bertrand curve.

(ii) Let β^* be a spacelike curve with spacelike principal normal in $\mathbb{E}_1^3$. Then the proof can be done similarly to (i). □

In the following results, the relationships between the Frenet vectors and curvature functions of Cartan Null Bertrand Curve and its Bertrand Mate curve are given

Corollary 3.1. *Let $\beta : I \to \mathbb{E}_1^3$ be a Cartan null Bertrand curve with the Frenet frame $\{T, N, B\}$ and the curvatures κ_1, κ_2, and $\beta^* : I \to \mathbb{E}_1^3$ be a spacelike Bertrand mate curve with spacelike principal normal of β with the Frenet frame $\{T^*, N^*, B^*\}$ and the curvatures κ_1^*, κ_2^*. Then the curvatures of β and β^*satisfy the relations*

$$\begin{aligned} \kappa_1^* &= \frac{\lambda(\kappa_2 - h)}{(f')^2}, \\ \kappa_2^* &= \frac{1}{(f')^3} \sqrt{-2 \left(h\lambda(\lambda\kappa_2 - h\lambda) - \kappa_2 (f')^2 \right) \left(\lambda(\lambda\kappa_2 - h\lambda) + (f')^2 \right)} \end{aligned}$$

and the corresponding frames of β and $\beta^*$are related by

$$\begin{aligned} T^* &= \left(\frac{h\lambda}{f'} \right) T - \left(\frac{\lambda}{f'} \right) B, \\ N^* &= N, \end{aligned}$$

$$B^* = \left(\frac{h\lambda(\lambda\kappa_2 - h\lambda) - \kappa_2 (f')^2}{\sqrt{-2\left(h\lambda(\lambda\kappa_2 - h\lambda) - \kappa_2 (f')^2\right)\left(\lambda(\lambda\kappa_2 - h\lambda) + (f')^2\right)}}\right) T + \left(\frac{\lambda(\lambda\kappa_2 - h\lambda) + (f')^2}{\sqrt{-2\left(h\lambda(\lambda\kappa_2 - h\lambda) - \kappa_2 (f')^2\right)\left(\lambda(\lambda\kappa_2 - h\lambda) + (f')^2\right)}}\right) B$$

where $(f')^2 = 2\lambda^2 h$ *and* $1 + \lambda\kappa_2 = -h\lambda$, $h > 0$, $\lambda \neq 0$.

Corollary 3.2. *Let $\beta : I \to \mathbb{E}_1^3$ be a Cartan null Bertrand curve with the Frenet frame $\{T, N, B\}$ and the curvatures κ_1, κ_2, and $\beta^* : I \to \mathbb{E}_1^3$ be a timelike Bertrand mate curve of β with the Frenet frame $\{T^*, N^*, B^*\}$ and the curvatures κ_1^*, κ_2^*. Then the curvatures of β and β^* satisfy the relations*

$$\kappa_1^* = \frac{\lambda(\kappa_2 - h)}{(f')^2},$$
$$\kappa_2^* = \frac{1}{(f')^3}\sqrt{2\left(h\lambda(\lambda\kappa_2 - h\lambda) + \kappa_2 (f')^2\right)\left(\lambda(\lambda\kappa_2 - h\lambda) - (f')^2\right)}$$

and the corresponding frames of β and β^* are related by

$$T^* = \left(\frac{-h\lambda}{f'}\right) T - \left(\frac{\lambda}{f'}\right) B,$$
$$N^* = N,$$
$$B^* = \left(\frac{h\lambda(\lambda\kappa_2 - h\lambda) + \kappa_2 (f')^2}{\sqrt{2\left(h\lambda(\lambda\kappa_2 - h\lambda) + \kappa_2 (f')^2\right)\left(\lambda(\lambda\kappa_2 - h\lambda) - (f')^2\right)}}\right) T + \left(\frac{\lambda(\lambda\kappa_2 - h\lambda) - (f')^2}{\sqrt{2\left(h\lambda(\lambda\kappa_2 - h\lambda) + \kappa_2 (f')^2\right)\left(\lambda(\lambda\kappa_2 - h\lambda) - (f')^2\right)}}\right) B$$

where $(f')^2 = -2\lambda^2 h$ *and* $1 + \lambda\kappa_2 = -h\lambda$, $h < 0$, $\lambda \neq 0$.

Remark 3.1. It can easily be proved that a Cartan null curve has no pseudo null Bertrand mate in $\mathbb{E}_1^3$.

Example 3.1. Let us consider a Cartan null curve in $\mathbb{E}_1^3$ parametrized by

$$\beta(s) = (\sinh s, \cosh s, s)$$

with

$$\begin{aligned} T(s) &= (\cosh s, \sinh s, 1), \\ N(s) &= (\sinh s, \cosh s, 0), \\ B(s) &= \left(-\tfrac{\cosh s}{2}, -\tfrac{\sinh s}{2}, \tfrac{1}{2}\right) \\ \kappa_1(s) &= 1 \quad \text{and} \quad \kappa_2(s) = 1/2. \end{aligned}$$

If we take $h = -3/2$ and $\lambda = 1$ in (i) of theorem 3.1, then we get the curve β^* as follows:

$$\beta^*(s) = \beta(s) + N(s) = (2\sinh s, 2\cosh s, s)$$

By straight calculations, we get

$$\begin{aligned} T^*(s) &= \left(\tfrac{2\cosh s}{\sqrt{3}}, \tfrac{2\sinh s}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right), \\ N^*(s) &= (\sinh s, \cosh s, 0), \\ B^*(s) &= \left(-\tfrac{\cosh s}{\sqrt{3}}, -\tfrac{\sinh s}{\sqrt{3}}, -\tfrac{2}{\sqrt{3}}\right), \\ \kappa_1^*(s) &= 2/3, \quad \kappa_2^*(s) = 1/3. \end{aligned}$$

It can be easily seen that the curve β^* is a timelike Bertrand mate curve of the curve β.

Example 3.2. For the same Cartan null curve β in Example 1, if we take $h = 3/2$ and $\lambda = -1/2$ in (ii) of theorem 3.1, then we get the curve β^* as follows:

$$\beta^*(s) = \beta(s) - \frac{1}{2}N(s) = \left(\frac{\sinh s}{2}, \frac{\cosh s}{2}, s\right)$$

By straight calculations, we get

$$\begin{aligned} T^*(s) &= \left(\tfrac{\cosh s}{\sqrt{3}}, \tfrac{\sinh s}{\sqrt{3}}, \tfrac{2}{\sqrt{3}}\right), \\ N^*(s) &= (\sinh s, \cosh s, 0), \\ B^*(s) &= \left(-\tfrac{2\cosh s}{\sqrt{3}}, -\tfrac{2\sinh s}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\right), \\ \kappa_1^*(s) &= 2/3, \quad \kappa_2^*(s) = 4/3. \end{aligned}$$

It can be easily seen that the curve β^* is a spacelike Bertrand mate curve of the curve β.

In the graphic below, the curves given in Example 3.1 and Example 3.2 are illustrated together.

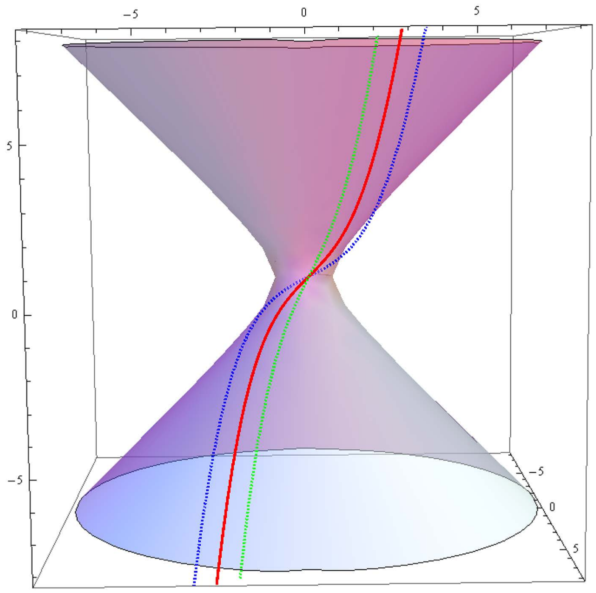

FIG. 3.1: Cartan null Bertrand curve β (red) and its spacelike (blue) and timelike (green) Bertrand mates curves in $\mathbb{E}_1^3$

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