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# A FIXED POINT APPROACH TO STABILITY OF A CUBIC FUNCTIONAL EQUATION IN 2-BANACH SPACES

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**Abstract.** In this paper, we prove a new type of stability and hyperstability results for the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$$

in 2-Banach spaces using fixed point approach.
**Keywords**: stability, hyperstability, fixed point, 2-Banach spaces, cubic functional equation.

## 1. Introduction

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and the set of real numbers by $\mathbb{R}$. By $\mathbb{N}_m$, $m \in \mathbb{N}$, we will denote the set of all natural numbers greater than or equal to $m$. Let $\mathbb{R}_+ = [0, \infty)$ be the set of nonnegative real numbers. We write $B^A$ to mean the family of all functions mapping from a nonempty set $A$ into a nonempty set $B$ and we use the notation $E_0$ for the set $E\backslash\{0\}$.

We need to recall some basic facts concerning 2-normed spaces and some preliminary results [6]).

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**Definition 1.1.** let $X$ be a real linear space with $dimX > 1$ and $\|\cdot,\cdot\| : X \times X \longrightarrow \mathbb{R}_+$ be a function satisfying the following properties:

1. $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly dependent,
2. $\|x, y\| = \|y, x\|$,
3. $\|\lambda x, y\| = |\lambda|\|x, y\|$,
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$,

for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$. Then the function $\|\cdot,\cdot\|$ is called *a 2-norm* on $X$ and the pair $(X, \|\cdot,\cdot\|)$ is called a *linear 2-normed space*. Sometimes the condition (4) called the *triangle inequality*.

**Definition 1.2.** A sequence $\{x_k\}$ in a 2-normed space $X$ is called a *convergent sequence* if there is an $x \in X$ such that

$$\lim_{k\to\infty} \|x_k - x, y\| = 0,$$

for all $y \in X$. If $\{x_k\}$ converges to $x$, write $x_k \longrightarrow x$ with $k \longrightarrow \infty$ and call $x$ the limit of $\{x_k\}$. In this case, we also write $\lim_{k\to\infty} x_k = x$.

**Definition 1.3.** A sequence $\{x_k\}$ in a 2-normed space $X$ is said to be a *Cauchy sequence* with respect to the 2-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l, y\| = 0,$$

for all $y \in X$. If every Cauchy sequence in $X$ converges to some $x \in X$, then $X$ is said to be *complete* with respect to the 2-norm. Any complete 2-normed space is said to be a *2-Banach space*.

Now, we state the following results as lemma (see [12] for the details).

**Lemma 1.1.** *Let $X$ be a 2-normed space. Then,*

1. $\big|\|x, z\| - \|y, z\|\big| \leq \|x - y, z\|$ *for all* $x, y, z \in X$,
2. *if* $\|x, z\| = 0$ *for all* $z \in X$, *then* $x = 0$,
3. *for a convergent sequence* $x_n$ *in* $X$,

$$\lim_{n\longrightarrow\infty} \|x_n, z\| = \left\|\lim_{n\longrightarrow\infty} x_n, z\right\|$$

*for all* $z \in X$.

The method of the proof of the main result corresponds to some observations in [4]. The problem of the stability of functional equations was first raised by Ulam [17]. This included the following question concerning the stability of group homomorphisms.

*Let $(G_1, *_1)$ be a group and let $(G_2, *_2)$ be a metric group with a metric $d(.,.)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality*

$$d\big(h(x *_1 y), h(x) *_2 h(y)\big) < \delta$$

*for all $x, y \in G_1$, then there exist a homomorphism $H : G_1 \to G_2$ with*

$$d\big(h(x), H(x)\big) < \varepsilon$$

*for all $x \in G_1$?*

If the answer is affirmative, we say that the equation of homomorphism

$$h(x *_1 y) = h(x) *_2 H(y)$$

is stable.

In 1941, Hyers [8] provided the first partial answer to Ulam's question and defined the result of stability where $G_1$ and $G_2$ are Banach spaces.

Later, Aoki [1] considered the problem of stability with unbounded Cauchy differences. Rassias [14] used a direct method to prove a generalization of Hyers result. The following theorem is the most classical result concerning the Hyers–Ulam stability of the Cauchy equation

$$T(x + y) = T(x) + T(y). \tag{1.1}$$

**Theorem 1.1.** *Let $E_1$ be a normed space, $E_2$ be a Banach space, and $f : E_1 \to E_2$ be a function. If $f$ satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \theta\big(\|x\|^p + \|y\|^p\big) \tag{1.2}$$

*for some $\theta \ge 0$, for some $p \in \mathbb{R}$ with $p \ne 1$, and for all $x, y \in E_1 - \{0_{E_1}\}$, then there exists a unique additive function $T : E_1 \to E_2$ such that*

$$\|f(x) - T(x)\| \le \frac{2\theta}{|2 - 2^p|}\|x\|^p \tag{1.3}$$

*for each $x \in E_1 - \{0_{E_1}\}$.*

It is due to Aoki [1] (for $0 < p < 1$; see also [13]), Gajda [7] for $p > 1$ and Rassias [14] for $p < 0$. Also, Brzdęk [2] showed that estimation (1.3) is optimal for $p \ge 0$ in

the general case. Recently, Brzdęk [3] showed that Theorem 1.1 can be significantly improved; namely, in the case $p < 0$, each $f : E_1 \to E_2$ satisfying (1.2) must actually be additive, and the assumption of completeness of $E_2$ is not necessary.
The cubic function $f(x) = cx^3$ satisfies the functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.4}$$

In 2002, Jun and Kim [9] established the general solution and the Hyers–Ulam stability of the cubic functional equation (1.4) for mappings $f : X \to Y$, where $X$ is a real normed space and $Y$ is a Banach space. Recently, interesting results concerning the cubic functional equation (1.4) have been obtained, for example, in [10] and [11].

In 2018, Brzdęk and Ciepliński [4] proved a new fixed point theorem in 2-Banach spaces and showed its applications to the Ulam stability of some single-variable equations and the most important functional equation in several variables. In addition, Brzdęk and El-hady [5] provided an extension of an earlier stability result that has been motivated by a problem of Rassias for the functions taking values in 2- Banach spaces. After that, Sayar and Bergam proved stability results for the quadratic functional equation in 2-Banach spaces [16].

In our paper, we discuss some stability and hyperstability results for the cubic functional equation (1.4) in 2-Banach spaces.

Now, we present the fixed point theorem concerning 2-Banach spaces given in [4]. First, we need the following hypotheses:

(H1) $E$ is a nonempty set, $\big(Y, \|\cdot,\cdot\|\big)$ is a 2-Banach space, $Y_0$ is a subset of $Y$ containing two linearly independent vectors, $j \in \mathbb{N}$, $f_i : E \to E$, $g_i : Y_0 \to Y_0$, and $L_i : E \times Y_0 \to \mathbb{R}_+$ for $i = 1, ..., j$;

(H2) $\mathcal{T} : Y^E \to Y^E$ is an operator satisfying the inequality

$$\big\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), y\big\| \le \sum_{i=1}^{j} L_i(x,y)\Big\|\xi\big(f_i(x)\big) - \mu\big(f_i(x)\big), g_i(y)\Big\|, \ \xi, \mu \in Y^E, x \in E, y \in Y_0; \tag{1.5}$$

(H3) $\Lambda : \mathbb{R}_+^{E\times Y_0} \to \mathbb{R}_+^{E\times Y_0}$ is an operator defined by

$$\Lambda\delta(x,y) := \sum_{i=1}^{j} L_i(x,y)\delta\big(f_i(x), g_i(y)\big), \quad \delta \in \mathbb{R}_+^{E\times Y_0}, \ x \in E, y \in Y_0. \tag{1.6}$$

**Theorem 1.2.** *[4] Let hypotheses (H1)-(H3) hold and functions $\varepsilon : E \times Y_0 \to R_+$ and $\varphi : E \to Y$ fulfill the following two conditions:*

$$\Big\|\mathcal{T}\varphi(x) - \varphi(x), y\Big\| \le \varepsilon(x,y) \quad x \in E, y \in Y_0, \tag{1.7}$$

$$\varepsilon^*(x,y) := \sum_{n=0}^{\infty} \big(\Lambda^n \varepsilon\big)(x,y) < \infty \quad x \in E, y \in Y_0. \tag{1.8}$$

*Then, there exists a unique fixed point $\psi$ of $\mathcal{T}$ for which*

$$\big\|\varphi(x) - \psi(x), y\big\| \leq \varepsilon^*(x,y) \quad x \in E, y \in Y_0. \tag{1.9}$$

*Moreover,*

$$\psi(x) := \lim_{n\to\infty} (\mathcal{T}^n \varphi)(x) \quad x \in E. \tag{1.10}$$

## 2. Main results

In this section, we begin with a discussion of the stability of the cubic equation (1.4) in 2-Banach spaces using Theorem 1.2. Then, we will present some sufficient conditions to prove that this equation is hyperstable, namely, the approximated solution of that equation is an exact solution of it.

In what follows $E$ is a normed space, $(Y, \|\cdot,\cdot\|)$ is a real 2-Banach space and $Y_0$ is a subset of $Y$ containing two linearly independent vectors.

**Theorem 2.1.** *Let $h_1, h_2 : E_0 \times Y_0 \to \mathbb{R}_+$ be two functions, such that*

$$\mathcal{U} := \{n \in \mathbb{N} \colon \alpha_n < 1\} \neq \phi \tag{2.1}$$

*where*

$$\alpha_n := 2\lambda_1(3n-1)\lambda_2(3n-1) + 2\lambda_1(1-n)\lambda_2(1-n) + 12\lambda_1(n)\lambda_2(n) + \lambda_1(4n-1)\lambda_2(4n-1)$$

$$\lambda_i(n) := \inf\{t \in \mathbb{R}_+ \colon h_i(nx,z) \leq t\, h_i(x,z), \;\; x \in E_0, z \in Y_0\} \tag{2.2}$$

*for all $n \in \mathbb{N}$, where $i = 1,2$. Assume that $f : X \to Y$ satisfies the inequality*

$$\|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x), z\| \leq h_1(x,z)h_2(y,z) \tag{2.3}$$

*for all $x, y \in E_0, z \in Y_0$, such that $x+y \neq 0, x-y \neq 0, 2x+y \neq 0$ and $2x-y \neq 0$. Then there exists a unique cubic function $F : E \to Y$, such that*

$$\big\|f(x) - F(x), z\big\| \leq \lambda_0 h_1(x,z)h_2(x,z) \tag{2.4}$$

*for all $x, z \in X_0$, where*

$$\lambda_0 := \inf_{n\in\mathcal{U}} \left\{ \frac{\lambda_1(n)\lambda_2(2n-1)}{1-\alpha_n} \right\}.$$

*Proof.* Replacing $x$ with $mx$ and $y$ with $(2m-1)x$, where $x \in X_0$ and $m \in \mathbb{N}$, in inequality (2.3), we get

$$\begin{aligned}\big\|2f\big((3m-1)x\big) + 2f\big((1-m)x\big) + 12f(mx) - f\big((4m-1)x\big) - f(x), z\big\| \\ \leq h_1(mx,z)h_2((2m-1)x,z)\end{aligned}$$

for all $x \in E_0, z \in Y_0$. For each $m \in \mathbb{N}$, we define the operator $\mathcal{T}_m : Y^{E_0} \to Y^{E_0}$ by

$$\mathcal{T}_m\xi(x) := 2\xi\big((3m-1)x\big)+2\xi\big((1-m)x\big)+12\xi(mx)-\xi\big((4m-1)x\big),\ \xi \in Y^{E_0}, x \in E_0. \tag{2.5}$$

Further put

$$\varepsilon_m(x,z) := h_1(mx,z)h_2((2m-1)x,z),\ x \in E_0, z \in Y_0, \tag{2.6}$$

and observe that

$$\varepsilon_m(x,z) := h_1(mx,z)h_2((2m-1)x,z) \leq \lambda_1(m)\lambda_2(2m-1)h_1(x,z)h_2(x,z), \tag{2.7}$$

for all $x \in E_0, z \in Y_0, m \in \mathbb{N}$. Then the inequality (2.5) takes the form

$$\big\|\mathcal{T}_m f(x) - f(x), z\big\| \leq \varepsilon_m(x,z), \quad x \in E_0, z \in Y_0. \tag{2.8}$$

Furthermore, for every $x \in E_0, z \in Y_0$, $\xi, \mu \in Y^{E_0}$, we obtain

$$\begin{aligned}\Big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x), z\Big\| &= \Big\|2\xi\big((3m-1)x\big) + 2\xi\big((1-m)x\big) + 12\xi(mx)\\ &\qquad -\xi\big((4m-1)x\big) - 2\mu\big((3m-1)x\big)\\ &\qquad -2\mu\big((1-m)x\big) - 12\mu(mx) + \mu\big((4m-1)x\big), z\Big\|\\ &\leq 2\Big\|(\xi-\mu)\big((3m-1)x\big), z\Big\| + 2\Big\|(\xi-\mu)\big((1-m)x\big), z\Big\|\\ &\qquad +12\Big\|(\xi-\mu)(mx), z\Big\| + \Big\|(\xi-\mu)\big((4m-1)x\big), z\Big\|.\end{aligned}$$

So, (H2) is valid for $\mathcal{T}_m$.
This brings us to define the operator $\Lambda_m : \mathbb{R}_+^{E_0\times Y_0} \to \mathbb{R}_+^{E_0\times Y_0}$ by

$$\Lambda_m\delta(x,z) := 2\delta\big((3m-1)x,z\big) + 2\delta\big((1-m)x,z\big) + 12\delta(mx,z) + \delta\big((4m-1)x,z\big),$$

for all $\delta \in \mathbb{R}_+^{E_0\times Y_0}, x \in E_0, z \in Y_0, m \in \mathbb{N}$. The above operator has the form described in (H3) with $f_1(x) = (3m-1)x$, $f_2(x) = (1-m)x$, $f_3(x) = mx$, $f_4(x) = (4m-1)x$, $g_1(z) = g_2(z) = z$ and $L_1(x) = L_2(x) = 2$, $L_3(x) = 12$ and $L_4(x) = 1$ for all $x \in X_0$. By induction on $n \in \mathbb{N}_0$, we will show

$$(\Lambda_m^n\varepsilon_m)(x,z) \leq \lambda_1(m)\lambda_2(2m-1)\alpha_m^n h_1(x,z)h_2(x,z) \tag{2.9}$$

where

$$\alpha_m = 2\lambda_1(3m-1)\lambda_2(3m-1)+2\lambda_1(1-m)\lambda_2(1-m)+12\lambda_1(m)\lambda_2(m)+\lambda_1(4m-1)\lambda_2(4m-1).$$

From (2.6) and (2.7), we obtain that the inequality (2.9) holds for $n = 0$. Next, we assume that (2.9) holds for $n = k$, where $k \in \mathbb{N}$ and we have

$$(\Lambda_m^{k+1}\varepsilon_m)(x,z) = \Lambda_m\left((\Lambda_m^k\varepsilon_m)(x,z)\right)$$

$$
\begin{aligned}
&= 2(\Lambda_m^k\varepsilon_m)\big((3m-1)x,z\big) + 2(\Lambda_m^k\varepsilon_m)\big((1-m)x,z\big) \\
&\quad +12(\Lambda_m^k\varepsilon_m)(mx,z) + (\Lambda_m^k\varepsilon_m)\big((4m-1)x,z\big) \\
&\leq \Big(2\lambda_1(m)\lambda_2(2m-1)\alpha_m^k h_1((3m-1)x,z)h_2((3m-1)x,z) \\
&\quad +2\lambda_1(m)\lambda_2(2m-1)\alpha_m^k h_1((1-m)x,z)h_2((1-m)x,z) \\
&\quad +12\lambda_1(m)\lambda_2(2m-1)\alpha_m^k h_1(mx,z)h_2(mx,z) \\
&\quad +\lambda_1(m)\lambda_2(2m-1)\alpha_m^k h_1((4m-1)x,z)h_2((4m-1)x,z)\Big) \\
&= \lambda_1(m)\lambda_2(2m-1)\alpha_m^{k+1}h_1(x,z)h_2(x,z)
\end{aligned}
$$

for all $x \in E_0, z \in Y_0$, $m \in \mathcal{U}$. This shows that (2.9) holds for $n = k+1$. Now, we can conclude that the inequality (2.9) holds for all $n \in \mathbb{N}_0$. By (2.9), we obtain

$$
\begin{aligned}
\varepsilon_m^*(x,z) &= \textstyle\sum_{n=0}^{\infty}(\Lambda_m^n\varepsilon_m)(x,z) \\
&\leq \textstyle\sum_{n=0}^{\infty}\lambda_1(m)\lambda_2(2m-1)\alpha_m^n h_1(x,z)h_2(x,z) \\
&= \tfrac{\lambda_1(m)\lambda_2(2m-1)h_1(x,z)h_2(x,z)}{(1-\alpha_m)} < \infty
\end{aligned}
$$

for all $x \in E_0, z \in Y_0$ and all $m \in \mathcal{U}$. Therefore, according to Theorem 1.2 with $\varphi = f$, the limit

$$F_m(x) := \lim_{n\to\infty}\big(\mathcal{T}_m^n f\big)(x)$$

exists for each $x \in E_0$ and $m \in \mathcal{U}$, and

$$\big\|f(x) - F_m(x), z\big\| \leq \frac{\lambda_1(m)\lambda_2(2m-1)h_1(x,z)h_2(x,z)}{(1-\alpha_m)}, \tag{2.10}$$

for all $x, \in E_0, z \in Y_0$ $m \in \mathcal{U}$. To prove that $F_m$ satisfies the functional equation (1.4), just prove the following inequality

$$
\begin{aligned}
&\Big\|(\mathcal{T}_m^n f)\big(2x+y\big) + (\mathcal{T}_m^n f)\big(2x-y\big) - 2(\mathcal{T}_m^n f)\big(x+y\big) - 2(\mathcal{T}_m^n f)\big(x-y\big) - 12(\mathcal{T}_m^n f)(x), z\Big\| \\
&\qquad \leq \alpha_m^n h_1(x,z)h_2(y,z) \qquad (2.11)
\end{aligned}
$$

for every $x, y \in E_0, z \in Y_0$ such that $x+y \neq 0, x-y \neq 0, 2x+y \neq 0$ and $2x-y \neq 0$, $n \in \mathbb{N}_0$, and $m \in \mathcal{U}$. Since the case $n = 0$ is just (2.3), take $k \in \mathbb{N}$ and assume that (2.11) holds for $n = k$. Then, for each $x, y \in E_0, z \in Y_0$ and $m \in \mathcal{U}$, we get

$$
\begin{aligned}
&\Big\|(\mathcal{T}_m^{k+1}f)\big(2x+y\big) + (\mathcal{T}_m^{k+1}f)\big(2x-y\big) - 2(\mathcal{T}_m^{k+1}f)\big(x+y\big) \\
&\qquad -2(\mathcal{T}_m^{k+1}f)\big(x-y\big) - 12(\mathcal{T}_m^{k+1}f)(x), z\Big\| \\
&= \Big\|2\mathcal{T}_m^k f\left((3m-1)(2x+y)\right) + 2\mathcal{T}_m^k f\left((1-m)(2x+y)\right) + 12\mathcal{T}_m^k f\left(m(2x+y)\right) \\
&\quad -\mathcal{T}_m^k f\left((4m-1)(2x+y)\right) + 2\mathcal{T}_m^k f\left((3m-1)(2x-y)\right) + 2\mathcal{T}_m^k f\left((1-m)(2x-y)\right) \\
&\quad +12\mathcal{T}_m^k f\left(m(2x-y)\right) - \mathcal{T}_m^k f\left((4m-1)(2x-y)\right) - 2\Big(2\mathcal{T}_m^k f\left((3m-1)\left(x+y\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+2\mathcal{T}_m^k f\left((1-m)(x+y)\right)+12\mathcal{T}_m^k f\left(m(x+y)\right)-\mathcal{T}_m^k f\left((4m-1)(x+y)\right)\Big)\\
&-2\Big(2\mathcal{T}_m^k f\left((3m-1)(x-y)\right)+2\mathcal{T}_m^k f\left((1-m)(x-y)\right)+12\mathcal{T}_m^k f\left(m(x-y)\right)\\
&-\mathcal{T}_m^k f\left((4m-1)(x-y)\right)\Big)-12\Big(2\mathcal{T}_m^k f\left((3m-1)x\right)+2\mathcal{T}_m^k f\left((1-m)x\right)\\
&+12\mathcal{T}_m^k f\left(mx\right)-\mathcal{T}_m^k f\left((4m-1)x\right)\Big),z\Big\|\\
\leq\ & 2\Big\|\mathcal{T}_m^k f\left((3m-1)(2x+y)\right)+\mathcal{T}_m^k f\left((3m-1)(2x-y)\right)-2\mathcal{T}_m^k ff\left((3m-1)(x+y)\right)\\
&-2\mathcal{T}_m^k f\left((3m-1)(x-y)\right)-12\mathcal{T}_m^k f\left((3m-1)\left(x\right)\right),z\Big\|\\
&+2\Big\|\mathcal{T}_m^k f\left((1-m)(2x+y)\right)+\mathcal{T}_m^k f\left((1-m)(2x-y)\right)-2\mathcal{T}_m^k f\left((1-m)(x+y)\right)\\
&-2\mathcal{T}_m^k f\left((1-m)(x-y)\right)-12\mathcal{T}_m^k f\left((1-m)x\right),z\Big\|\\
&+12\Big\|\mathcal{T}_m^k f\left((m)(2x+y)\right)+\mathcal{T}_m^k f\left(m(2x-y)\right)-2\mathcal{T}_m^k f\left(m(x+y)\right)\\
&-2\mathcal{T}_m^k f\left(m(x-y)\right)-12\mathcal{T}_m^k f\left(mx\right),z\Big\|\\
&+\Big\|\mathcal{T}_m^k f\left((4m-1)(2x+y)\right)+\mathcal{T}_m^k f\left((4m-1)(2x-y)\right)-2\mathcal{T}_m^k f\left((4m-1)(x+y)\right)\\
&-2\mathcal{T}_m^k f\left((4m-1)(x-y)\right)-12\mathcal{T}_m^k f\left((4m-1)x\right),z\Big\|\\
\leq\ & 2\alpha_m^k h_1\big((3m-1)x,z\big)h_2\big((3m-1)y,z\big)+2\alpha_m^k h_1\big((1-m)x,z\big)h_2\big((1-m)y,z\big)\\
&+12\alpha_m^k h_1\big(mx,z\big)h_2\big(my,z\big)+\alpha_m^k h_1\big((4m-1)x,z\big)h_2\big((4m-1)y,z\big)\\
=\ &\alpha_m^{k+1}h_1(x,z)h_2(y,z).
\end{aligned}
$$

Thus, by induction on $n \in \mathbb{N}_0$, we have shown that (2.11) holds for all $x, y \in E_0, z \in Y_0$, such that $x+y \neq 0, x-y \neq 0, 2x+y \neq 0$ and $2x-y \neq 0$, and $m \in \mathcal{U}$. Letting $n \to \infty$ in (2.11), we obtain the equality

$$F_m(2x+y)+F_m(2x-y)=2F_m(x+y)+2F_m(x-y)+12F_m(x), \tag{2.12}$$

for all $x, y \in E_0$, such that $x+y \neq 0, x-y \neq 0, 2x+y \neq 0$ and $2x-y \neq 0$, $m \in \mathcal{U}$. This implies that $F_m : E \to Y$, defined in this way, is a solution of the equation

$$F(x)=2F\big((3m-1)x\big)+2F\big((1-m)x\big)+12F(mx)-F\big((4m-1)x\big), \tag{2.13}$$

for all $x \in E_0, m \in \mathcal{U}$. Next, we will prove that each cubic function $F : E \to Y$ satisfying the inequality

$$\big\|f(x)-F(x),z\big\| \leq L\, h_1(x,z)h_2(x,z), \quad x \in E_0, z \in Y_0 \tag{2.14}$$

with some $L > 0$, is equal to $F_m$ for each $m \in \mathcal{U}$. To this end, we fix $m_0 \in \mathcal{U}$ and $F : E \to Y$ satisfying (2.14). From (2.10), for each $x \in E_0$, we get

$$\big\|F(x)-F_{m_0}(x),z\big\| \leq \big\|F(x)-f(x),z\big\|+\big\|f(x)-F_{m_0}(x),z\big\|$$

$$\le L\ h_1(x,z)h_2(x,z) + \varepsilon^*_{m_0}(x,z)$$

$$\le L_0\ h_1(x,z)h_2(x,z)\sum_{n=0}^{\infty}\alpha^n_{m_0}, \tag{2.15}$$

where $L_0 := (1-\alpha_{m_0})L + \lambda_1(m_0)\lambda_2(2m_0-1) > 0$ and we exclude the case that $h_1(x,z)\equiv 0$ or $h_2(x,z)\equiv 0$ which is trivial. Observe that $F$ and $F_{m_0}$ are solutions to equation (2.13) for all $m\in\mathcal{U}$. Next, we show that, for each $j\in\mathbb{N}_0$, we have

$$\big\|F(x)-F_{m_0}(x),z\big\| \le L_0\ h_1(x,z)h_2(x,z)\sum_{n=j}^{\infty}\alpha^n_{m_0},\ \ x,z\in E_0. \tag{2.16}$$

The case $j=0$ is exactly (2.15). We fix $k\in\mathbb{N}$ and assume that (2.16) holds for $j=k$. Then, in view of (2.15), for each $x,z\in E_0$, we get

$$\begin{aligned}
\big\|F(x)-F_{m_0}(x),z\big\| &= \big\|2F\big((3m_0-1)x\big)+2F\big((1-m_0)x\big)+12F(m_0x)\\
&\quad -F\big((4m_0-1)x\big)-2F_{m_0}\big((3m_0-1)x\big)-2F_{m_0}\big((1-m_0)x\big)\\
&\quad -12F_{m_0}(m_0x)+F_{m_0}\big((4m_0-1)x\big),z\big\|\\
&\le 2\big\|F\big((3m_0-1)x\big)-F_{m_0}\big((3m_0-1)x\big),z\big\|\\
&\quad +2\big\|F\big((1-m_0)x\big)-F_{m_0}\big((1-m_0)x\big),z\big\|\\
&\quad +12\big\|F(m_0x)-F_{m_0}(m_0x),z\big\|\\
&\quad +\big\|F\big((4m_0-1)x\big)-F_{m_0}\big((4m_0-1)x\big),z\big\|\\
&\le 2L_0\ h_1\big((3m_0-1)x,z\big)h_2\big((3m_0-1)x,z\big)\textstyle\sum_{n=k}^{\infty}\alpha^n_{m_0}\\
&\quad +2L_0\ h_1\big((1-m_0)x,z\big)h_2\big((1-m_0)x,z\big)\textstyle\sum_{n=k}^{\infty}\alpha^n_{m_0}\\
&\quad +12L_0\ h_1\big(m_0x,z\big)h_2\big(m_0x,z\big)\textstyle\sum_{n=k}^{\infty}\alpha^n_{m_0}\\
&\quad +L_0h_1\big((4m_0-1)x,z\big)h_2\big((4m_0-1)x,z\big)\textstyle\sum_{n=k}^{\infty}\alpha^n_{m_0}\\
&= L_0\ \Big(2h_1\big((3m_0-1)x,z\big)h_2\big((3m_0-1)x,z\big)\\
&\quad +2h_1\big((1-m_0)x,z\big)h_2\big((1-m_0)x,z\big)+12h_1\big(m_0x,z\big)h_2\big(m_0x,z\big)\\
&\quad +h_1\big((4m_0-1)x,z\big)h_2\big((4m_0-1)x,z\big)\Big)\textstyle\sum_{n=k}^{\infty}\alpha^n_{m_0}\\
&\le L_0\ \alpha_{m_0}h_1(x,z)h_2(x,z)\textstyle\sum_{n=k}^{\infty}\alpha^n_{m_0}\\
&= L_0\ h_1(x,z)h_2(x,z)\textstyle\sum_{n=k+1}^{\infty}\alpha^n_{m_0}.
\end{aligned}$$

This shows that (2.16) holds for $j=k+1$. Now we can conclude that the inequality (2.16) holds for all $j\in\mathbb{N}_0$. Now, letting $j\to\infty$ in (2.16), we get

$$F = F_{m_0}. \tag{2.17}$$

Thus, we have also proved that $F_m = F_{m_0}$ for each $m\in\mathcal{U}$, which (in view of (2.10)) yields

$$\big\|f(x)-F_{m_0}(x),z\big\| \le \frac{\lambda_1(m)\lambda_2(2m-1)h_1(x,z)h_2(x,z)}{1-\alpha_m},\ \ x,\in E_0, z\in Y_0\ m\in\mathcal{U}.$$

This implies (2.4) with $F = F_{m_0}$ and (2.17) confirms the uniqueness of $F$. $\square$

## 3. Applications

According to above theorem, we can obtain the following corollary for the hyperstability results of the cubic equation (1.4) in 2-Banach spaces.

**Corollary 3.1.** *Let $h_1, h_2$ and $\mathcal{U}$ be as in Theorem 2.1. Assume that*

$$\lim_{n\to\infty} \lambda_1(n)\lambda_2(2n-1) = 0, \tag{3.1}$$

*Then every $f : E \to Y$ satisfying (2.3) is a solution of (1.4) on $E_0$.*

*Proof.* Suppose that $f : E \to Y$ satisfies (2.3). Then, by Theorem 2.1, there exists a mapping $F : E \to Y$ satisfies (1.4) and

$$\|f(x) - F(x), z\| \leq \lambda_0 h_1(x, z)h_2(x, z) \tag{3.2}$$

for all $x \in E_0, z \in Y_0$, where

$$\lambda_0 := \inf_{n\in\mathcal{U}} \left\{ \frac{\lambda_1(n)\lambda_2(2n-1)}{1-\alpha_n} \right\}.$$

Since, in view of (3.1), $\lambda_0 = 0$. This means that $f(x) = F(x)$ for all $x \in E_0$, whence

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x),$$

for all $x, y \in E_0$ such that $x + y \neq 0, x - y \neq 0, 2x + y \neq 0$ and $2x - y \neq 0$, which implies that $f$ satisfies the functional equation (1.4) on $E_0$. $\square$

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