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# SOME BOUNDS FOR THE COMPLEX ČEBYŠEV FUNCTIONAL OF ABSOLUTELY CONTINUOUS FUNCTIONS

**Silvestru Sever Dragomir**

**Mathematics, College of Engineering & Science**
**Victoria University, PO Box 14428, Melbourne City, Victoria, Australia**

**Abstract.** In this paper we provide several bounds for the modulus of the *complex Čebyšev functional*

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\int_a^b g(t)\,dt$$

under various assumptions for the integrable functions $f,\ g : [a,b] \to \mathbb{C}$. We show amongst others that, if $f$ and $g$ are absolutely continuous on $[a,b]$ with $f' \in L_p[a,b]$, $g' \in L_q[a,b]$, $p,\ q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\max\{|C(f,g)|, |C(|f|,g)|, |C(f,|g|)|, |C(|f|,|g|)|\}$$

$$\leq \left[C\left(\ell, F_{|f'|^p}\right)\right]^{1/p}\left[C\left(\ell, F_{|g'|^q}\right)\right]^{1/q},$$

where $F_{|h|} : [a,b] \to [0,\infty)$ is defined by $F_{|h|}(t) := \int_a^t |h(t)|\,dt$ and $\ell : [a,b] \to [a,b]$, $\ell(t) = t$ is the identity function on the interval $[a,b]$. Applications for the trapezoid inequality are also provided.

**Key words:** complex Čebyšev functional, trapezoid inequality, inequalities for sums, series and integrals.

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Corresponding Author: Sever S. Dragomir, Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, Victoria, Australia | E-mail: sever.dragomir@vu.edu.au

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## 1. Introduction

For Lebesgue integrable functions $f$, $g : [a, b] \to \mathbb{C}$ we consider the *complex Čebyšev functional*

$$C(f, g) := \frac{1}{b-a} \int_a^b f(t) g(t) \, dt - \frac{1}{b-a} \int_a^b f(t) \, dt \frac{1}{b-a} \int_a^b g(t) \, dt.$$

For two integrable real-valued functions $f$, $g : [a, b] \to \mathbb{R}$, in order to compare the integral mean of the product with the product of the integral means, in 1934, G. Grüss [14] showed that

$$|C(f, g)| \leq \frac{1}{4} (M - m)(N - n), \tag{1.1}$$

provided $m$, $M$, $n$, $N$ are real numbers with the property that

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on} \quad [a, b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one. For other results, see [4], [3], [16], [6] and [7].

In order to extend this inequality for complex-valued functions we need the following preparations.

For $\phi$, $\Phi \in \mathbb{C}$ and $[a, b]$ an interval of real numbers, define the sets of complex-valued functions (see [6], [8] and [13])

$$\bar{U}_{[a,b]}(\phi, \Phi) := \left\{ g : [a, b] \to \mathbb{C} | \ \operatorname{Re}\left[ (\Phi - g(t)) \left( \overline{g(t)} - \overline{\phi} \right) \right] \geq 0 \text{ for a.e. } t \in [a, b] \right\}$$

and

$$\bar{\Delta}_{[a,b]}(\phi, \Phi) := \left\{ g : [a, b] \to \mathbb{C} | \ \left| g(t) - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi| \text{ for a.e. } t \in [a, b] \right\}.$$

For any $\phi$, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that $\bar{U}_{[a,b]}(\phi, \Phi)$ and $\bar{\Delta}_{[a,b]}(\phi, \Phi)$ are nonempty, convex and closed sets and

$$\bar{U}_{[a,b]}(\phi, \Phi) = \bar{\Delta}_{[a,b]}(\phi, \Phi). \tag{1.3}$$

We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi|$$

if and only if

$$\operatorname{Re}\left[ (\Phi - z) \left( \bar{z} - \overline{\phi} \right) \right] \geq 0.$$

This follows by the equality

$$\frac{1}{4} |\Phi - \phi|^2 - \left| z - \frac{\phi + \Phi}{2} \right|^2 = \operatorname{Re}\left[ (\Phi - z) \left( \bar{z} - \overline{\phi} \right) \right]$$

that holds for any $z \in \mathbb{C}$.

The equality (1.3) is thus a simple consequence of this fact.

For any $\phi$, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$,we also have that

$$\bar{U}_{[a,b]}(\phi,\Phi) = \{g : [a,b] \to \mathbb{C} \mid (\operatorname{Re}\Phi - \operatorname{Re} g(t))(\operatorname{Re} g(t) - \operatorname{Re}\phi) + (\operatorname{Im}\Phi - \operatorname{Im} g(t))(\operatorname{Im} g(t) - \operatorname{Im}\phi) \geq 0 \text{ for a.e. } t \in [a,b]\}. \tag{1.4}$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\bar{S}_{[a,b]}(\phi,\Phi) := \{g : [a,b] \to \mathbb{C} \mid \operatorname{Re}(\Phi) \geq \operatorname{Re} g(t) \geq \operatorname{Re}(\phi) \text{ and } \operatorname{Im}(\Phi) \geq \operatorname{Im} g(t) \geq \operatorname{Im}(\phi) \text{ for a.e. } t \in [a,b]\}. \tag{1.5}$$

One can easily observe that $\bar{S}_{[a,b]}(\phi,\Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\phi,\Phi) \subseteq \bar{U}_{[a,b]}(\phi,\Phi). \tag{1.6}$$

This fact provides also numerous example of complex functions belonging to the class $\bar{\Delta}_{[a,b]}(\phi,\Phi)$.

In [6] we obtained the following complex version of Grüss' inequality:

$$|C(f,\overline{g})| \leq \frac{1}{4}|\Phi - \phi||\Psi - \psi| \tag{1.7}$$

provided $f \in \bar{\Delta}_{[a,b]}(\phi,\Phi)$ and $g \in \bar{\Delta}_{[a,b]}(\psi,\Psi)$, where $\overline{g}$ denotes the complex conjugate function of $g$.

We denote *the variance* of the complex-valued function $f : [a,b] \to \mathbb{C}$ by $D(f)$ and defined as

$$D(f) = \left[C\left(f,\bar{f}\right)\right]^{1/2} = \left[\frac{1}{b-a}\int_a^b |f(t)|^2\,dt - \left|\frac{1}{b-a}\int_a^b f(t)\,dt\right|^2\right]^{1/2},$$

where $\bar{f}$ denotes the complex conjugate function of $f$.

If we apply the inequality (1.7) for $g = f$, then we get

$$D(f) \leq \frac{1}{2}|\Phi - \phi|. \tag{1.8}$$

We observe that, if $g \in \bar{\Delta}_{[a,b]}(\psi,\Psi)$, then $\left|g(t) - \frac{\psi+\Psi}{2}\right| \leq \frac{1}{2}|\Psi - \psi|$ for a.e. $t \in [a,b]$ that is equivalent to $\left|\overline{g(t)} - \frac{\overline{\psi}+\overline{\Psi}}{2}\right| \leq \frac{1}{2}\left|\overline{\Psi} - \overline{\psi}\right|$ meaning that $\overline{g} \in \bar{\Delta}_{[a,b]}\left(\overline{\psi},\overline{\Psi}\right)$ and by 1.7, for $\overline{g}$ instead of $g$ we also have

$$|C(f,g)| \leq \frac{1}{4}|\Phi - \phi||\Psi - \psi| \tag{1.9}$$

provided $f \in \bar{\Delta}_{[a,b]}(\phi,\Phi)$ and $g \in \bar{\Delta}_{[a,b]}(\psi,\Psi)$.

We can also consider the following quantity associated with a complex-valued function $f : [a,b] \to \mathbb{C}$,

$$E(f) := |C(f,f)|^{1/2} = \left| \frac{1}{b-a} \int_a^b f^2(t)\,dt - \left( \frac{1}{b-a} \int_a^b f(t)\,dt \right)^2 \right|^{1/2}.$$

By using (1.9) we also have

$$E(f) \le \frac{1}{2} |\Phi - \phi|. \tag{1.10}$$

For an integrable function $f : [a,b] \to \mathbb{C}$, consider the *mean deviation* of $f$ defined by

$$R(f) := \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| dt.$$

The following result holds (see [11] or the more extensive preprint version [10]).

**Theorem 1.1.** *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|C(f,g)| \le \frac{1}{2} \bigvee_a^b (f)\, R(g) \le \frac{1}{2} \bigvee_a^b (f)\, D(g), \tag{1.11}$$

*where $\bigvee_a^b (f)$ denotes the total variation of $f$ on the interval $[a,b]$. The constant $\frac{1}{2}$ is best possible in* (1.11).

**Corollary 1.1.** *If $f,\ g : [a,b] \to \mathbb{C}$ are of bounded variation on $[a,b]$, then*

$$|C(f,g)| \le \frac{1}{2} \bigvee_a^b (f)\, R(g) \le \frac{1}{2} \bigvee_a^b (f)\, D(g) \le \frac{1}{4} \bigvee_a^b (f) \bigvee_a^b (g). \tag{1.12}$$

*The constant $\frac{1}{4}$ is best possible in* (1.12).

*We also have*

$$D(f) \le \frac{1}{2} \bigvee_a^b (f), \tag{1.13}$$

*and the constant $\frac{1}{2}$ is best possible in* (1.13).

Utilising the above results we can state, for a function of bounded variation $f:[a,b]\to\mathbb{C}$, that

$$E^{2}(f)\leq\frac{1}{2}\bigvee_{a}^{b}(f)\,R(f)\leq\frac{1}{2}\bigvee_{a}^{b}(f)\,D(f)\leq\frac{1}{4}\left[\bigvee_{a}^{b}(f)\right]^{2}. \tag{1.14}$$

In the recent paper [12] we obtained the following result that extends to complex functions the inequalities obtained in [1]

**Theorem 1.2.** *Let $f,\ g:[a,b]\to\mathbb{C}$ be measurable on $[a,b]$. Then*

$$|C(f,g)|\leq\begin{cases}\inf\limits_{\gamma\in\mathbb{C}}\|g-\gamma\|_{\infty}R(f) & \text{if } g\in L_{\infty}[a,b] \text{ and } f\in L[a,b],\\[2ex] \frac{1}{(b-a)^{1/q}}\inf\limits_{\gamma\in\mathbb{C}}\|g-\gamma\|_{q}R_{p}(f), & g\in L_{q}[a,b],\ f\in L_{p}[a,b],\\ \text{and } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, & \\[2ex] \frac{1}{b-a}\inf\limits_{\gamma\in\mathbb{C}}\|g-\gamma\|_{1}R_{\infty}(f) & \text{if } g\in L[a,b] \text{ and } f\in L_{\infty}[a,b].\end{cases} \tag{1.15}$$

An important corollary of this result is:

**Corollary 1.2.** *Assume that $g:[a,b]\to\mathbb{C}$ is measurable on $[a,b]$ and $g\in\bar{\Delta}_{[a,b]}(\psi,\Psi)$ for some distinct complex numbers $\psi$, $\Psi$. Then*

$$|C(f,g)|\leq\frac{1}{2}|\Psi-\psi|\,R(f) \tag{1.16}$$

*if $f\in L[a,b]$.*

*In particular, we have*

$$D^{2}(g)\leq\frac{1}{2}|\Psi-\psi|\,R(g). \tag{1.17}$$

This generalizes the following result obtained by Cheng and Sun [5] by a more complicated technique

$$|C(f,g)|\leq\frac{1}{2}(M-m)\,R(f), \tag{1.18}$$

provided $m\leq g\leq M$ for a.e. $x\in[a,b]$. The constant $\frac{1}{2}$ is best in (1.18) as shown by Cerone and Dragomir in [2] where a general version for Lebesgue integral and measurable spaces was also given.

Motivated by the above results, in this paper we establish other bounds for the absolute value of the Čebyšev functional when the complex-valued functions are absolutely continuous. Applications for the trapezoid type inequalities are also provided.

## 2. Main Results

For an absolutely continuous function $f:[a,b]\to\mathbb{C}$ we define the function $F_{|f'|}:[a,b]\to[0,\infty)$, $F_{|f'|}(x):=\int_a^x|f'(t)|\,dt$. We observe that $F_{|f'|}$ is monotonic nondecreasing and absolutely continuous on $[a,b]$ and $F'_{|f'|}(x)=|f'(x)|$ for a.e. $x\in[a,b]$. We also have the bounds

$$0\le F_{|f'|}(x)\le\|f'\|_{[a,b],1}\ \text{ for any } x\in[a,b]$$

where $\|\cdot\|_{[a,b],1}$ is the *Lebesgue norm*

$$\|h\|_{[a,b],1}:=\int_a^b|h(t)|\,dt,\ \text{ if } h\in L_1[a,b].$$

We have the following inequality for the complex Čebyšev functional that extends naturally the real case:

**Theorem 2.1.** *Assume that $g:[a,b]\to\mathbb{C}$ is absolutely continuous on $[a,b]$.*

*(i) If $f:[a,b]\to\mathbb{C}$ is absolutely continuous on $[a,b]$, then*

$$\max\{|C(f,g)|,|C(|f|,g)|,|C(f,|g|)|,|C(|f|,|g|)|\}\le C\left(F_{|f'|},F_{|g'|}\right);\tag{2.1}$$

*(ii) If $f:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[a,b]$, i.e.*

$$|f(t)-f(s)|\le L|t-s|\ \textit{ for any } t,\ s\in[a,b]$$

*then*

$$\max\{|C(f,g)|,|C(|f|,g)|,|C(f,|g|)|,|C(|f|,|g|)|\}\le LC\left(\ell,F_{|g'|}\right),\tag{2.2}$$

*where $\ell:[a,b]\to[a,b]$, $\ell(t)=t$ is the identity function on the interval $[a,b]$;*

*(iii) If $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

$$\max\{|C(f,g)|,|C(|f|,g)|,|C(f,|g|)|,|C(|f|,|g|)|\}\le C\left(f,F_{|g'|}\right).\tag{2.3}$$

*Proof.* As in the real case, we have *Korkine's identity*

$$C(f,g):=\frac{1}{2(b-a)^2}\int_a^b\int_a^b(f(t)-f(s))(g(t)-g(s))\,dtds,$$

that can be proved directly by doing the calculations in the right hand side.

By the properties of modulus, we have

$$|(f(t)-f(s))(g(t)-g(s))|=|f(t)-f(s)|\,|g(t)-g(s)|$$

$$\ge\begin{cases}|(|f(t)|-|f(s)|)(g(t)-g(s))|,\\|(f(t)-f(s))(|g(t)|-|g(s)|)|,\\|(|f(t)|-|f(s)|)(|g(t)|-|g(s)|)|\end{cases}$$

for any $t,\ s\in[a,b]$.

Using the properties of the integral versus the modulus, we also have

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b |f(t)-f(s)|\,|g(t)-g(s)|\,dtds \tag{2.4}$$

$$\geq \frac{1}{2(b-a)^2}\begin{cases}\int_a^b\int_a^b |(f(t)-f(s))(g(t)-g(s))|\,dtds,\\ \int_a^b\int_a^b |(|f(t)|-|f(s)|)(g(t)-g(s))|\,dtds,\\ \int_a^b\int_a^b |(f(t)-f(s))(|g(t)|-|g(s)|)|\,dtds,\\ \int_a^b\int_a^b |(|f(t)|-|f(s)|)(|g(t)|-|g(s)|)|\,dtds,\end{cases}$$

$$\geq \frac{1}{2(b-a)^2}\begin{cases}\left|\int_a^b\int_a^b (f(t)-f(s))(g(t)-g(s))\,dtds\right|,\\ \left|\int_a^b\int_a^b (|f(t)|-|f(s)|)(g(t)-g(s))\,dtds\right|,\\ \left|\int_a^b\int_a^b (f(t)-f(s))(|g(t)|-|g(s)|)\,dtds\right|,\\ \left|\int_a^b\int_a^b (|f(t)|-|f(s)|)(|g(t)|-|g(s)|)\,dtds\right|,\end{cases} = \begin{cases}|C(f,g)|,\\ |C(|f|,g)|,\\ |C(f,|g|)|,\\ |C(|f|,|g|)|.\end{cases}$$

(i) Now, since $f,\ g:[a,b]\to\mathbb{C}$ are absolutely continuous on $[a,b]$, then for any $t,\ s\in[a,b]$

$$|f(t)-f(s)|\,|g(t)-g(s)| = \left|\int_s^t f'(u)\,du\right|\left|\int_s^t g'(u)\,du\right|$$

$$\leq \left|\int_s^t |f'(u)|\,du\right|\left|\int_s^t |g'(u)|\,du\right| = \left|F_{|f'|}(t)-F_{|f'|}(s)\right|\left|F_{|g'|}(t)-F_{|g'|}(s)\right|$$

$$= \left(F_{|f'|}(t)-F_{|f'|}(s)\right)\left(F_{|g'|}(t)-F_{|g'|}(s)\right)$$

since both functions $F_{|f'|}$ and $F_{|g'|}$ are monotonic nondecreasing on $[a,b]$.

Then

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b |f(t)-f(s)|\,|g(t)-g(s)|\,dtds \tag{2.5}$$

$$\leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(F_{|f'|}(t)-F_{|f'|}(s)\right)\left(F_{|g'|}(t)-F_{|g'|}(s)\right)dtds = C\left(F_{|f'|},F_{|g'|}\right).$$

If we use (2.4) and (2.5), then we get (2.1).

(ii) If $f:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ and $g:[a,b]\to\mathbb{C}$ is absolutely continuous on $[a,b]$, then

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b |f(t)-f(s)|\,|g(t)-g(s)|\,dtds \tag{2.6}$$

$$\leq \frac{1}{2(b-a)^2}L\int_a^b\int_a^b |t-s|\left|F_{|g'|}(t)-F_{|g'|}(s)\right|dtds$$

$$= \frac{1}{2(b-a)^2} L \int_a^b \int_a^b (t-s) \left( F_{|g'|}(t) - F_{|g'|}(s) \right) dtds = LC\left(\ell, F_{|g'|}\right).$$

If we use (2.4) and (2.6), then we get (2.2).

(iii) If $f : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ is absolutely continuous on $[a,b]$, then

$$\frac{1}{2(b-a)^2} \int_a^b \int_a^b |f(t) - f(s)| \, |g(t) - g(s)| \, dtds \tag{2.7}$$

$$\leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b |f(t) - f(s)| \left| F_{|g'|}(t) - F_{|g'|}(s) \right| dtds$$

$$= \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f(t) - f(s)) \left( F_{|g'|}(t) - F_{|g'|}(s) \right) dtds = C\left(f, F_{|g'|}\right).$$

If we use (2.4) and (2.7), then we get (2.3). □

For an absolutely continuous function $f : [a,b] \to \mathbb{C}$ we define the function $F_{|f'|^p} : [a,b] \to [0,\infty)$, $p \geq 1$ by $F_{|f'|^p}(x) := \int_a^x |f'(t)|^p \, dt$, where $|f'|^p$ is integrable on $[a,b]$. We observe that $F_{|f'|^p}$ is monotonic nondecreasing and absolutely continuous on $[a,b]$ and $F'_{|f'|^p}(x) = |f'(x)|^p$ for a.e. $x \in [a,b]$. We also have the bounds

$$0 \leq F_{|f'|^p}(x) \leq \|f'\|_{[a,b],p}^p \text{ for any } x \in [a,b]$$

where $\|\cdot\|_{[a,b],p}$ is the *Lebesgue norm*

$$\|h\|_{[a,b],p} := \left( \int_a^b |h(t)|^p \, dt \right)^{1/p}, \text{ if } h \in L_p[a,b].$$

We have the following result:

**Theorem 2.2.** *Assume that $f$, $g : [a,b] \to \mathbb{C}$ are absolutely continuous on $[a,b]$. If $p$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $f' \in L_p[a,b]$, $g' \in L_q[a,b]$, then*

$$\max\{|C(f,g)|, |C(|f|,g)|, |C(f,|g|)|, |C(|f|,|g|)|\} \tag{2.8}$$

$$\leq \left[C\left(\ell, F_{|f'|^p}\right)\right]^{1/p} \left[C\left(\ell, F_{|g'|^q}\right)\right]^{1/q}.$$

*In particular, if $f'$, $g' \in L_2[a,b]$, then*

$$\max\left\{|C(f,g)|^2, |C(|f|,g)|^2, |C(f,|g|)|^2, |C(|f|,|g|)|^2\right\} \tag{2.9}$$

$$\leq C\left(\ell, F_{|f'|^2}\right) C\left(\ell, F_{|g'|^2}\right).$$

*Proof.* Since $f,\ g:[a,b]\to\mathbb{C}$ are absolutely continuous on $[a,b]$, then for any $t$, $s\in[a,b]$ and by applying Hölder's integral inequality we have for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$|f(t)-f(s)|\,|g(t)-g(s)|=\left|\int_s^t f'(u)\,du\right|\left|\int_s^t g'(u)\,du\right|$$

$$\leq|t-s|^{1/q}\left|\int_s^t|f'(u)|^p\,du\right|^{1/p}|t-s|^{1/p}\left|\int_s^t|g'(u)|^q\,du\right|^{1/q}.$$

Then by Hölder's integral inequality we have for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b|f(t)-f(s)|\,|g(t)-g(s)|\,dtds \tag{2.10}$$

$$\leq\frac{1}{2(b-a)^2}\int_a^b\int_a^b|t-s|^{1/p}\left|\int_s^t|f'(u)|^p\,du\right|^{1/p}|t-s|^{1/q}\left|\int_s^t|g'(u)|^q\,du\right|^{1/q}dtds$$

$$\leq\frac{1}{2(b-a)^2}\left[\int_a^b\int_a^b\left(|t-s|^{1/p}\left|\int_s^t|f'(u)|^p\,du\right|^{1/p}\right)^p dtds\right]^{1/p}$$

$$\times\left[\int_a^b\int_a^b\left(|t-s|^{1/q}\left|\int_s^t|g'(u)|^q\,du\right|^{1/q}\right)^q dtds\right]^{1/q}$$

$$=\frac{1}{2(b-a)^2}\left[\int_a^b\int_a^b|t-s|\left|\int_s^t|f'(u)|^p\,du\right|dtds\right]^{1/p}$$

$$\times\left[\int_a^b\int_a^b|t-s|\left|\int_s^t|g'(u)|^q\,du\right|dtds\right]^{1/q}$$

$$=\left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b|t-s|\left|\int_s^t|f'(u)|^p\,du\right|dtds\right]^{1/p}$$

$$\times\left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b|t-s|\left|\int_s^t|g'(u)|^q\,du\right|dtds\right]^{1/q}.$$

Now, observe that

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b|t-s|\left|\int_s^t|f'(u)|^p\,du\right|dtds$$

$$=\frac{1}{2(b-a)^2}\int_a^b\int_a^b|t-s|\left|F_{|f'|^p}(t)-F_{|f'|^p}(s)\right|dtds$$

$$= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (t-s)\left(F_{|f'|^p}(t)-F_{|f'|^p}(s)\right)dtds = C\left(\ell, F_{|f'|^p}\right)$$

since $F_{|f|^p}$ is monotonic nondecreasing on $[a,b]$.

In a similar way, we have

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b |t-s|\left|\int_s^t |g'(u)|^q du\right| dtds = C\left(\ell, F_{|g'|^q}\right).$$

By using (2.10) we deduce

(2.11)
$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b |f(t)-f(s)|\,|g(t)-g(s)|\,dtds \le \left[C\left(\ell, F_{|f'|^p}\right)\right]^{1/p}\left[C\left(\ell, F_{|g'|^q}\right)\right]^{1/q},$$

that is of interest in itself.

By using (2.4) and (2.11) we deduce the desired result (2.8). □

**Corollary 2.1.** *Assume that $f,\ g : [a,b] \to \mathbb{C}$ are absolutely continuous on $[a,b]$. If $p,\ q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $f' \in L_p[a,b]$, $g' \in L_q[a,b]$, then*

$$\max\{|C(f,g)|, |C(|f|,g)|, |C(f,|g|)|, |C(|f|,|g|)|\} \tag{2.12}$$

$$\le \frac{1}{2(b-a)}\left[\int_a^b (t-a)(b-t)|f'(t)|^p dt\right]^{1/p}\left[\int_a^b (t-a)(b-t)|g'(t)|^q dt\right]^{1/q}.$$

*In particular, if $f',\ g' \in L_2[a,b]$, then*

$$\max\left\{|C(f,g)|^2, |C(|f|,g)|^2, |C(f,|g|)|^2, |C(|f|,|g|)|^2\right\} \tag{2.13}$$

$$\le \frac{1}{4(b-a)^2}\int_a^b (t-a)(b-t)|f'(t)|^2 dt\int_a^b (t-a)(b-t)|g'(t)|^2 dt.$$

*Proof.* If $h : [a,b] \to \mathbb{C}$ is a function of bounded variation, since the function $u(t) := (t-a)(b-t)$ is continuous, then the Stieltjes integral $\int_a^b (t-a)(b-t)\,dh(t)$ exists and integrating by parts, we have

$$\int_a^b (t-a)(b-t)\,dh(t) = 2\int_a^b \left(t-\frac{a+b}{2}\right)h(t)\,dt = 2(b-a)\,C(\ell,h)$$

giving the identity of interest for complex valued functions, see also [3] for the real case,

$$C(\ell,h) = \frac{1}{2(b-a)}\int_a^b (t-a)(b-t)\,dh(t).$$

By (2.8) we then obtain

$$\max\{|C(f,g)|, |C(|f|,g)|, |C(f,|g|)|, |C(|f|,|g|)|\}$$

$$\leq \left[\frac{1}{2(b-a)}\int_a^b (t-a)(b-t)\left|f'(t)\right|^p dt\right]^{1/p}\left[\frac{1}{2(b-a)}\int_a^b (t-a)(b-t)\left|g'(t)\right|^q dt\right]^{1/q}$$

$$= \frac{1}{2(b-a)}\left[\int_a^b (t-a)(b-t)\left|f'(t)\right|^p dt\right]^{1/p}\left[\int_a^b (t-a)(b-t)\left|g'(t)\right|^q dt\right]^{1/q},$$

which proves (2.12). □

**Remark 2.1.** The inequality

$$\left|C(f,g)\right|^2 \leq \frac{1}{4(b-a)^2}\int_a^b (t-a)(b-t)\left|f'(t)\right|^2 dt \int_a^b (t-a)(b-t)\left|g'(t)\right|^2 dt$$

was proved for real-valued functions in [3].

## 3. Some Examples

If we use Grüss' inequality (1.1) for the functions $F_{|f'|}$ and $F_{|g'|}$, we have

$$\left|C\left(F_{|f'|},F_{|g'|}\right)\right| \leq \frac{1}{4}\left\|f'\right\|_{[a,b],1}\left\|g'\right\|_{[a,b],1} \tag{3.1}$$

for any $f,\, g:[a,b]\to\mathbb{C}$ absolutely continuous functions on $[a,b]$.

Using the inequality (2.1), we deduce

$$\max\left\{\left|C(f,g)\right|,\left|C(|f|,g)\right|,\left|C(f,|g|)\right|,\left|C(|f|,|g|)\right|\right\} \leq \frac{1}{4}\left\|f'\right\|_{[a,b],1}\left\|g'\right\|_{[a,b],1} \tag{3.2}$$

for any $f,\, g:[a,b]\to\mathbb{C}$ absolutely continuous functions on $[a,b]$.

If we use the inequality (1.12) for the functions $F_{|f'|}$ and $F_{|g'|}$ we have

$$\left|C\left(F_{|f'|},F_{|g'|}\right)\right| \leq \frac{1}{2}\left\|f'\right\|_{[a,b],1} R\left(F_{|g'|}\right) \tag{3.3}$$

$$\leq \frac{1}{2}\left\|f'\right\|_{[a,b],1} D\left(F_{|g'|}\right) \leq \frac{1}{4}\left\|f'\right\|_{[a,b],1}\left\|g'\right\|_{[a,b],1},$$

where

$$R\left(F_{|g'|}\right) = \frac{1}{b-a}\int_a^b \left|F_{|g'|}(t) - \frac{1}{b-a}\int_a^b F_{|g'|}(s)\,ds\right| dt$$

and

$$D\left(F_{|g'|}\right) = \frac{1}{b-a}\int_a^b F_{|g'|}^2(t)\,dt - \left(\frac{1}{b-a}\int_a^b F_{|g'|}(t)\,dt\right)^2.$$

Using the inequality (2.1), we deduce

$$\max\left\{\left|C(f,g)\right|,\left|C(|f|,g)\right|,\left|C(f,|g|)\right|,\left|C(|f|,|g|)\right|\right\} \tag{3.4}$$

$$\leq \frac{1}{2}\left\|f'\right\|_{[a,b],1} R\left(F_{|g'|}\right) \leq \frac{1}{2}\left\|f'\right\|_{[a,b],1} D\left(F_{|g'|}\right) \leq \frac{1}{4}\left\|f'\right\|_{[a,b],1}\left\|g'\right\|_{[a,b],1},$$

for any $f,\, g : [a,b] \to \mathbb{C}$ absolutely continuous functions on $[a,b]$.

The inequality (3.4) is a refinement of (3.2).

In 1970, A. M. Ostrowski [17] proved amongst others the following result that is somehow a mixture of the Čebyšev and Grüss results

$$|C(f,g)| \leq \frac{1}{8}(b-a)(M-m)\left\|g'\right\|_{\infty}, \tag{3.5}$$

provided $f$ is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_{\infty}[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In [9] we obtained the following refinement of (3.5).

**Theorem 3.1.** *Let $f : [a,b] \to \mathbb{R}$ be measurable and such that there exist the constants $m,\, M \in \mathbb{R}$ with*

$$-\infty < m \leq f(x) \leq M < \infty \text{ for a.e. } x \text{ on } [a,b]. \tag{3.6}$$

*If $g : [a,b] \to \mathbb{C}$ is absolutely continuous on $[a,b]$ with $g' \in L_{\infty}[a,b]$ then we have the inequality*

$$|C(f,g)| \leq \frac{1}{2}\left\|g'\right\|_{\infty} \frac{\left(\frac{1}{b-a}\int_a^b f(x)\,dx - m\right)\left(M - \frac{1}{b-a}\int_a^b f(x)\,dx\right)}{M-m}(b-a) \tag{3.7}$$

$$\leq \frac{1}{8}(b-a)(M-m)\left\|g'\right\|_{\infty}.$$

*The constants $\frac{1}{2}$ and $\frac{1}{8}$ are sharp in the above sense.*

If we use the inequality (1.12) for the functions $F_{|f'|}$ and $F_{|g'|}$ we have

$$\left|C\left(F_{|f'|}, F_{|g'|}\right)\right| \leq \frac{1}{2}\left\|g'\right\|_{\infty}\left(1 - \frac{1}{(b-a)\left\|f'\right\|_{[a,b],1}}\int_a^b F_{|f'|}(x)\,dx\right)\int_a^b F_{|f'|}(x)\,dx \tag{3.8}$$

$$\leq \frac{1}{8}(b-a)\left\|f'\right\|_{[a,b],1}\left\|g'\right\|_{\infty}$$

for any $f,\, g : [a,b] \to \mathbb{C}$ absolutely continuous functions on $[a,b]$ and $g' \in L_{\infty}[a,b]$.

Using the inequality (2.1), we deduce

$$\max\left\{|C(f,g)|, |C(|f|,g)|, |C(f,|g|)|, |C(|f|,|g|)|\right\} \tag{3.9}$$

$$\leq \frac{1}{2}\left\|g'\right\|_{\infty}\left(1 - \frac{1}{(b-a)\left\|f'\right\|_{[a,b],1}}\int_a^b F_{|f'|}(x)\,dx\right)\int_a^b F_{|f'|}(x)\,dx$$

$$\leq \frac{1}{8}(b-a)\left\|f'\right\|_{[a,b],1}\left\|g'\right\|_{\infty},$$

for any $f,\ g : [a,b] \to \mathbb{C}$ absolutely continuous functions on $[a,b]$ and $g' \in L_\infty [a,b]$.

Now, we observe that for $f = \ell$, where $\ell$ is the identity mapping of the interval $[a,b]$, namely $\ell(t) = t$, $t \in [a,b]$, we have

$$R(\ell) = \frac{1}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{1}{4}(b-a).$$

Then we have by (1.18) that

$$\left| C\left(\ell, F_{|g'|}\right) \right| \le \frac{1}{8}(b-a) \|g'\|_{[a,b],1}. \tag{3.10}$$

If $f : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ is absolutely continuous on $[a,b]$, then by (2.2) we have

$$\max\{|C(f,g)|, |C(|f|,g)|, |C(f,|g|)|, |C(|f|,|g|)|\} \le \frac{1}{8}(b-a) L \|g'\|_{[a,b],1}. \tag{3.11}$$

Assume that $f,\ g : [a,b] \to \mathbb{C}$ are absolutely continuous on $[a,b]$. If $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $f' \in L_p[a,b]$, $g' \in L_q[a,b]$, then by (3.5) we have

$$\left| C\left(\ell, F_{|f'|^p}\right) \right| \le \frac{1}{8}(b-a) \|f'\|_{[a,b],p}^p, \tag{3.12}$$

and

$$\left| C\left(\ell, F_{|g'|^q}\right) \right| \le \frac{1}{8}(b-a) \|g'\|_{[a,b],q}^q. \tag{3.13}$$

By using (2.8) we then get

$$\begin{aligned} &\max\{|C(f,g)|, |C(|f|,g)|, |C(f,|g|)|, |C(|f|,|g|)|\} \\ &\qquad \le \frac{1}{8}(b-a) \|f'\|_{[a,b],p} \|g'\|_{[a,b],q} \end{aligned} \tag{3.14}$$

provided $f,\ g : [a,b] \to \mathbb{C}$ are absolutely continuous on $[a,b]$, $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $f' \in L_p[a,b]$, $g' \in L_q[a,b]$.

## 4. Applications for Trapezoid Inequality

Let $h : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then we have the following well known *trapezoid equality* in terms of the first derivative

$$\frac{h(a) + h(b)}{2} - \frac{1}{b-a} \int_a^b h(t)\,dt = \frac{1}{b-a} \int_a^b \left( t - \frac{a+b}{2} \right) (h'(t) - \delta)\,dt \tag{4.1}$$

for any $\delta \in \mathbb{C}$. This is obvious integrating by parts in the right hand side of the equality.

Consider $f = h'$ and $g = \ell - \frac{a+b}{2}$. Then

$$F_{|f'|}(t) = \int_a^t |h''(s)|\, ds \text{ and } F_{|g'|}(t) = \int_a^t ds = t - a$$

and if we use the inequality (3.4) we have

$$\frac{1}{b-a}\left|\int_a^b \left(t - \frac{a+b}{2}\right) h'(t)\, dt\right| \leq \frac{1}{2}\left\|h''\right\|_{[a,b],1} R(\ell - a) = \frac{1}{8}(b-a)\left\|h''\right\|_{[a,b],1}. \tag{4.2}$$

Therefore, by (4.1) for $\delta = 0$ we obtain the inequality

$$\left|\frac{h(a) + h(b)}{2} - \frac{1}{b-a}\int_a^b h(t)\, dt\right| \leq \frac{1}{8}(b-a)\left\|h''\right\|_{[a,b],1}, \tag{4.3}$$

provided $h'$ is absolutely continuous on $[a, b]$.

If we use the inequality (3.9) for $f = h'$ and $g = \ell - \frac{a+b}{2}$ we get

$$\begin{aligned} &\frac{1}{b-a}\left|\int_a^b \left(t - \frac{a+b}{2}\right) h'(t)\, dt\right| \\ &\leq \frac{1}{2}\left(1 - \frac{1}{(b-a)\left\|h''\right\|_{[a,b],1}}\int_a^b \left(\int_a^x |h''(s)|\, ds\right) dx\right) \\ &\quad\times \int_a^b \left(\int_a^x |h''(s)|\, ds\right) dx \leq \frac{1}{8}(b-a)\left\|h''\right\|_{[a,b],1}. \end{aligned} \tag{4.4}$$

By (4.1) for $\delta = 0$ and (4.4) we obtain the inequality

$$\begin{aligned} &\left|\frac{h(a) + h(b)}{2} - \frac{1}{b-a}\int_a^b h(t)\, dt\right| \\ &\leq \frac{1}{2}\left(1 - \frac{1}{(b-a)\left\|h''\right\|_{[a,b],1}}\int_a^b \left(\int_a^x |h''(s)|\, ds\right) dx\right) \\ &\quad\times \int_a^b \left(\int_a^x |h''(s)|\, ds\right) dx \leq \frac{1}{8}(b-a)\left\|h''\right\|_{[a,b],1}, \end{aligned} \tag{4.5}$$

which is an improvement of (4.3).

Using the inequality (3.4) for $f = \ell - \frac{a+b}{2}$ and $g = h'$ we get

$$\frac{1}{b-a}\left|\int_a^b \left(t - \frac{a+b}{2}\right) h'(t)\, dt\right| \tag{4.6}$$

$$\leq \frac{1}{2}\int_a^b \left|\int_a^x |h''(s)|\,ds - \frac{1}{b-a}\int_a^b \left(\int_a^t |h''(s)|\,ds\right)dt\right| dx,$$

and by (4.1) for $\delta = 0$ we get

$$\left|\frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(t)\,dt\right| \tag{4.7}$$

$$\leq \frac{1}{2}\int_a^b \left|\int_a^x |h''(s)|\,ds - \frac{1}{b-a}\int_a^b \left(\int_a^t |h''(s)|\,ds\right)dt\right| dx,$$

provided that $h'$ is absolutely continuous on $[a,b]$.

Now, if we use the inequality (2.12) for $f = h'$ and $g = \ell - \frac{a+b}{2}$ we get

$$\frac{1}{b-a}\left|\int_a^b \left(t - \frac{a+b}{2}\right) h'(t)\,dt\right| \tag{4.8}$$

$$\leq \frac{1}{2(b-a)}\left[\int_a^b (t-a)(b-t)|h''(t)|^p\,dt\right]^{1/p}\left[\int_a^b (t-a)(b-t)\,dt\right]^{1/q}$$

$$= \frac{1}{2\cdot 6^{1/q}}(b-a)^{3/q-1}\left[\int_a^b (t-a)(b-t)|h''(t)|^p\,dt\right]^{1/p},$$

where $p$, $q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$.

By (4.1) for $\delta = 0$ we get

$$\left|\frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(t)\,dt\right| \tag{4.9}$$

$$\leq \frac{1}{2\cdot 6^{1/q}}(b-a)^{3/q-1}\left[\int_a^b (t-a)(b-t)|h''(t)|^p\,dt\right]^{1/p},$$

provided $h'' \in L_p[a,b]$.

In particular, we have

$$\left|\frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(t)\,dt\right| \tag{4.10}$$

$$\leq \frac{1}{2}\sqrt{\frac{b-a}{6}}\left[\int_a^b (t-a)(b-t)|h''(t)|^2\,dt\right]^{1/2},$$

provided $h'' \in L_2[a,b]$.

The following identity of trapezoid type in terms of the second derivative is also well known:

$$\frac{h(a)+h(b)}{2}-\frac{1}{b-a}\int_a^b h(t)\,dt=\frac{1}{2(b-a)}\int_a^b (t-a)(b-t)\,h''(t)\,dt, \tag{4.11}$$

provided the first derivative $h'$ is absolutely continuous on $[a,b]$.

Consider $f=h''$ and assume that, for some constants $p$, $P$ we have $p\le h''(t)\le P$ for a.e. $t\in[a,b]$. If $g=\frac{1}{2}(\ell-a)(b-\ell)$, then $g'=\frac{a+b}{2}-\ell$,

$$\|g'\|_{[a,b],\infty}=\frac{1}{2}(b-a),$$

and by the inequality (3.7) we then have

$$\left|\frac{1}{2(b-a)}\int_a^b (t-a)(b-t)\,h''(t)\,dt-\frac{1}{2(b-a)^2}\int_a^b (t-a)(b-t)\,dt\int_a^b h''(t)\,dt\right|$$

$$\le\frac{1}{4}(b-a)^2\frac{\left(\frac{h'(b)-h'(a)}{b-a}-p\right)\left(P-\frac{h'(b)-h'(a)}{b-a}\right)}{P-p}\le\frac{1}{16}(b-a)^2(P-p),$$

which is equivalent to

$$\left|\frac{h(a)+h(b)}{2}-\frac{1}{b-a}\int_a^b h(t)\,dt-\frac{1}{12}(b-a)\left[h'(b)-h'(a)\right]\right|$$

$$\le\frac{1}{4}(b-a)^2\frac{\left(\frac{h'(b)-h'(a)}{b-a}-p\right)\left(P-\frac{h'(b)-h'(a)}{b-a}\right)}{P-p}$$

$$\le\frac{1}{16}(b-a)^2(P-p).$$

For $g=\frac{1}{2}(\ell-a)(b-\ell)$ we have

$$\|g'\|_{[a,b],q}=\left(\int_a^b\left|t-\frac{a+b}{2}\right|^q dt\right)^{1/q}=\frac{(b-a)^{1+1/q}}{2(q+1)^{1/q}}$$

and by taking $f=h''$ in (3.14) we get

$$\left|\frac{h(a)+h(b)}{2}-\frac{1}{b-a}\int_a^b h(t)\,dt-\frac{1}{12}(b-a)\left[h'(b)-h'(a)\right]\right| \tag{4.12}$$

$$\le\frac{1}{16(q+1)^{1/q}}(b-a)^{2+1/q}\|h'''\|_{[a,b],p},$$

where the second derivative $h''$ is absolutely continuous on $[a,b]$.

Similar bounds may be obtained by utilising the inequality

$$|C(f,g)| \leq \frac{1}{2(b-a)} \left[ \int_a^b (t-a)(b-t) |f'(t)|^p dt \right]^{1/p} \times \left[ \int_a^b (t-a)(b-t) |g'(t)|^q dt \right]^{1/q} \tag{4.13}$$

for $g = \frac{1}{2}(\ell - a)(b - \ell)$ and $f = h''$ provided the second derivative $h''$ is absolutely continuous on $[a,b]$. The details are omitted.

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