

SOME VECTOR FIELDS ON THE TANGENT BUNDLE WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\bar{\nabla}$. In this paper, we examine some special vector fields, such as incompressible vector fields, harmonic vector fields, concurrent vector fields, conformal vector fields and projective vector fields on TM with respect to the semi-symmetric metric connection $\bar{\nabla}$ and obtain some properties related to them.

Key words: Complete lift metric, semi-symmetric metric connection, tangent bundle, vector fields.

1. Introduction

Friedmann and Schouten introduced the notion of a semi-symmetric linear connection on a differentiable manifold [1]. Semi-symmetric metric connections play an important role in the study of Riemannian manifolds. In [2], Hayden introduced the idea of a metric connection with torsion on a Riemannian manifold. Using Hayden's idea, Yano [6] studied a semi-symmetric metric connection on a Riemannian manifold. He proved that a Riemannian manifold endowed with the semi-symmetric metric connection has vanishing curvature tensor if and only if the Riemannian manifold is conformally flat. After that, the generalization of this result for vanishing Ricci tensor of the semi-symmetric metric connection was shown by Imai in [3, 4].

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The geometry of tangent bundle TM is based on the fundamental paper of Sasaki [5] published in 1958. He used a given Riemannian metric g on a differentiable manifold M to construct a metric \tilde{g} on the tangent bundle TM of M . Today this metric is called the *Sasaki metric*. The well-known Riemannian or pseudo-Riemannian metrics on TM are constructed from the Riemannian metric g given on M by classical lifts, such as

1. The complete lift metric or the metric II ;
2. The metric $I + II$;
3. The Sasaki metric or the metric $I + III$;

4. The metric $II + III$; where $I = g_{ij}dx^i dx^j$, $II = 2g_{ij}dx^i \delta y^j$, $III = g_{ij}\delta y^i \delta y^j$ are all quadratic differential forms defined globally on the tangent bundle TM over M [8].

In our paper [9], we originally define a semi-symmetric metric connection on the tangent bundle equipped with complete lift metric. We compute all forms of the curvature tensors of the semi-symmetric metric connection and study their properties. Also, we have investigated conditions for the tangent bundle with this connection and the complete lift metric to be locally conformally flat. The goal of the present paper is to characterize some vector fields such as incompressible, harmonic, concurrent, conformal, projective with respect to the semi-symmetric metric connection on the tangent bundle over a Riemannian manifold.

2. Preliminaries

Let M be an n -dimensional differentiable manifold and TM be its tangent bundle with the natural projection $\pi : TM \rightarrow M$. Coordinate systems in M are denoted by (U, x^h) , where U is the coordinate neighborhood and (x^h) , $h = 1, \dots, n$ are the coordinate functions. Let $(y^{\bar{h}}) = (x^{\bar{h}})$, $\bar{h} = n+1, \dots, 2n$ be the Cartesian coordinates in each tangent space $T_p M$ at $p \in M$ with respect to natural basis $\{\frac{\partial}{\partial x^{\bar{h}}} |_p\}$, where p is an arbitrary point in U with local coordinates (x^h) . Then we can introduce the local coordinates $(x^h, y^{\bar{h}})$ on the open set $\pi^{-1}(U) \subset TM$. Here, the coordinate system of $(x^h, y^{\bar{h}}) = (x^h, x^{\bar{h}})$ is called induced coordinates on $\pi^{-1}(U)$ from (U, x^h) . In the paper, we use Einstein's convention on repeated indices.

Let $X = X^h \frac{\partial}{\partial x^h}$ be the local expression in U of a vector field X on M . Let ∇ be a (torsion-free) linear connection on M . The vertical lift ${}^V X$, the horizontal lift ${}^H X$ and the complete lift ${}^C X$ of X are given respectively by

$${}^V X = X^h \partial_{\bar{h}},$$

$${}^H X = X^h \partial_h - y^s \Gamma_{sk}^h X^k \partial_{\bar{h}}$$

and

$${}^C X = X^h \partial_h + y^s \partial_s X^h \partial_{\bar{h}}$$

with respect to the induced coordinates, where $\partial_h = \frac{\partial}{\partial x^h}$, $\partial_{\bar{h}} = \frac{\partial}{\partial y^h}$ and Γ_{jk}^h are the components of the connection ∇ .

Suppose that a (p, q) tensor field S on M , $q > 1$, is given. We then define a $(p, q - 1)$ tensor field γS on TM by

$$\gamma S = (y^s S_{s i_2 \dots i_q}^{j_1 \dots j_p}) \partial_{\bar{j}_1} \otimes \dots \otimes \partial_{\bar{j}_p} \otimes dx^{i_2} \otimes \dots \otimes dx^{i_q}$$

with respect to the induced coordinates (x^i, y^i) [8]. The tensor field γS determines a global tensor field on TM . We easily see that for any $(1, 1)$ tensor field P , γP has components

$$(\gamma P) = \begin{pmatrix} 0 \\ y^j P_j^i \end{pmatrix}$$

and γP is a vertical vector field on TM .

With the connection ∇ , the set of the $2n$ linearly independent vector fields on each induced coordinate neighbourhood $\pi^{-1}(U)$ of TM which are the following forms:

$$\begin{aligned} E_j &= \partial_j - y^s \Gamma_{sj}^h \partial_{\bar{h}}, \\ E_{\bar{j}} &= \partial_{\bar{j}}. \end{aligned}$$

is a frame field [8]. We call it the adapted frame and it will be written by $\{E_\beta\} = \{E_j, E_{\bar{j}}\}$. With respect to adapted frame $\{E_\beta\}$, the vertical lift ${}^V X$, the horizontal lift ${}^H X$ and the complete lift ${}^C X$ of X are respectively expressed by [8]

$$(2.1) \quad \begin{aligned} {}^V X &= X^j E_{\bar{j}}, \\ {}^H X &= X^j E_j, \\ {}^C X &= X^j E_j + y^s \nabla_s X^j E_{\bar{j}}. \end{aligned}$$

The complete lift metric ${}^C g$ on the tangent bundle TM over a (pseudo-)Riemannian manifold (M, g) is defined as follows:

$$\begin{aligned} {}^C g({}^H X, {}^H Y) &= 0, \\ {}^C g({}^H X, {}^V Y) &= {}^C g({}^V X, {}^H Y) = g(X, Y), \\ {}^C g({}^V X, {}^V Y) &= 0 \end{aligned}$$

for all vector fields X and Y on M [8]. Note that ${}^C g$ is a pseudo-Riemannian metric on TM . The covariant and contravariant components of the complete lift metric ${}^C g$ on TM are respectively given in the adapted local frame by

$${}^C g_{\alpha\beta} = \begin{pmatrix} 0 & g_{ij} \\ g_{ij} & 0 \end{pmatrix}$$

and

$${}^C g^{\alpha\beta} = \begin{pmatrix} 0 & g^{ij} \\ g^{ij} & 0 \end{pmatrix}.$$

The semi-symmetric metric connection $\bar{\nabla}$ on TM with respect to the complete lift metric ${}^C g$ is given as follows.

Proposition 2.1. [9] *The semi-symmetric metric connection $\bar{\nabla}$ on the tangent bundle TM with the complete lift metric ${}^C g$ over a (pseudo-)Riemannian manifold (M, g) is given by*

$$(2.2) \quad \begin{cases} \bar{\nabla}_{E_i} E_j = \Gamma_{ij}^k E_k + \{y^s R_{sij}{}^k + y_j \delta_i^k - y^k g_{ij}\} E_{\bar{k}}, \\ \bar{\nabla}_{E_i} E_{\bar{j}} = \Gamma_{ij}^k E_{\bar{k}}, \\ \bar{\nabla}_{E_{\bar{i}}} E_j = 0, \bar{\nabla}_{E_{\bar{i}}} E_{\bar{j}} = 0 \end{cases}$$

with respect to the adapted frame $\{E_\beta\}$, where Γ_{ij}^h and $R_{hji}{}^s$ respectively denote components of the Levi-Civita connection ∇ and the Riemannian curvature tensor field R of the pseudo-Riemannian metric g on M .

3. Some Vector Fields on TM with respect to Semi-symmetric Metric Connection

In this section, we firstly search the properties of being harmonic and incompressible of the lifting vector fields. After that we will find the general forms of concurrent, conformal, projective vector fields with respect to the semi-symmetric metric connection on the tangent bundle TM and give some important results related to them.

3.1. Lifting vector fields being incompressible (divergence-free) and harmonic

Firstly, we shall give the definition of an incompressible vector field on TM with respect to the semi-symmetric metric connection.

Definition 3.1. Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\bar{\nabla}$. A vector field $\tilde{V} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ on TM is called incompressible vector field with respect to the semi-symmetric metric connection if \tilde{V} satisfies the following condition

$$\text{trace}(\bar{\nabla}\tilde{V}) = \bar{\nabla}_\alpha \tilde{V}^\alpha = 0.$$

Proposition 3.1. *Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\bar{\nabla}$. Then, for any vector field V on M ,*

i) The vertical lift ${}^V V$ is an incompressible vector field on TM with respect to the semi-symmetric metric connection $\bar{\nabla}$;

ii) The horizontal lift ${}^H V$ or the complete lift ${}^C V$ is an incompressible vector field on TM with respect to the semi-symmetric metric connection $\bar{\nabla}$ if and only if the vector field V is incompressible on M with respect to the Levi-Civita connection ∇ .

Proof. Using (2.1) and (2.2), we calculate

$$\begin{aligned}
 \text{trace}(\bar{\nabla}{}^V V) &= \bar{\nabla}_\alpha{}^V V^\alpha = \bar{\nabla}_{\bar{h}} v^h = 0 \\
 \text{trace}(\bar{\nabla}{}^H V) &= \bar{\nabla}_\alpha{}^H V^\alpha = \bar{\nabla}_h v^h \\
 &= (\partial_h - y^s \Gamma_{sh}^m \partial_{\bar{m}}) v^h + \bar{\Gamma}_{hm}^h v^m \\
 &= \nabla_h v^h = \text{trace}(\nabla V) \\
 \text{trace}(\bar{\nabla}{}^C V) &= \bar{\nabla}_\alpha{}^C V^\alpha = \bar{\nabla}_h v^h + \bar{\nabla}_{\bar{h}} v^{\bar{h}} \\
 &= (\partial_h - y^s \Gamma_{sh}^m \partial_{\bar{m}}) v^h + \bar{\Gamma}_{hm}^h v^m + \partial_{\bar{h}} (y^s \nabla_s v^h) \\
 &= 2\nabla_h v^h = 2\text{trace}(\nabla V)
 \end{aligned}$$

from which, it is easy to see that the results (i) and (ii). \square

Definition 3.2. Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\bar{\nabla}$. A vector field $\tilde{V} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ on TM is called a harmonic vector field with respect to the semi-symmetric metric connection $\bar{\nabla}$ if \tilde{V} satisfies the following condition

$$\left(\bar{\nabla}_i \tilde{V}^\epsilon \right)^C g_{\epsilon j} - \left(\bar{\nabla}_j \tilde{V}^\epsilon \right)^C g_{\epsilon i} = 0,$$

where ${}^C g_{ij}$ are the components of the complete lift metric ${}^C g$ on TM .

The following lemma comes immediate from standard calculations.

Lemma 3.1. Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\bar{\nabla}$. Then

i) For the vertical lift ${}^V V$, we get

$$(3.1) \quad \left(\bar{\nabla}_\alpha{}^V V^\epsilon \right)^C g_{\epsilon\beta} - \left(\bar{\nabla}_\beta{}^V V^\epsilon \right)^C g_{\epsilon\alpha} = \begin{pmatrix} \nabla_i v_j - \nabla_j v_i & 0 \\ 0 & 0 \end{pmatrix};$$

ii) For the horizontal lift ${}^H X$, we get

$$(3.2) \quad \begin{aligned}
 &\left(\bar{\nabla}_\alpha{}^H V^\epsilon \right)^C g_{\epsilon\beta} - \left(\bar{\nabla}_\beta{}^H V^\epsilon \right)^C g_{\epsilon\alpha} \\
 &= \begin{pmatrix} y^s [R_{siaj} - R_{sjai} + g_{si} g_{ja} - g_{sj} g_{ia}] v^a & \nabla_i v_j \\ -\nabla_j v_i & 0 \end{pmatrix};
 \end{aligned}$$

iii) For the complete lift ${}^C V$, we get

$$(3.3) \quad \begin{aligned} & (\bar{\nabla}_\alpha {}^C V^\epsilon) {}^C g_{\epsilon\beta} - (\bar{\nabla}_\beta {}^C V^\epsilon) {}^C g_{\epsilon\alpha} \\ &= \begin{pmatrix} y^s [\nabla_s (\nabla_i v_j - \nabla_j v_i) + (g_{si} g_{ja} - g_{sj} g_{ia}) v^a] & \nabla_i v_j - \nabla_j v_i \\ \nabla_i v_j - \nabla_j v_i & 0 \end{pmatrix}. \end{aligned}$$

A manifold whose curvature tensor is of the form

$$R_{ijkl} = \kappa(g_{il} g_{jk} - g_{jl} g_{ik})$$

is called a manifold of constant curvature [7]. Here κ is the sectional curvature of the manifold.

From (3.2) and the above definition, we write

$$(3.4) \quad \begin{aligned} R_{siaj} &= \kappa(g_{sj} g_{ia} - g_{ij} g_{sa}) \\ R_{sjai} &= \kappa(g_{si} g_{ja} - g_{ji} g_{sa}) \\ \Rightarrow R_{siaj} - R_{sjai} &= \kappa(g_{sj} g_{ia} - g_{si} g_{ja}). \end{aligned}$$

When we use the above equation (3.4) on the equation (3.2) and take $\kappa = 1$, we obtain

$$\begin{aligned} & y^s [R_{siaj} - R_{sjai} + g_{si} g_{ja} - g_{sj} g_{ia}] v^a \\ &= y^s [\kappa(g_{sj} g_{ia} - g_{si} g_{ja}) + g_{si} g_{ja} - g_{sj} g_{ia}] v^a \\ &= y^s [(g_{sj} g_{ia} - g_{si} g_{ja}) + g_{si} g_{ja} - g_{sj} g_{ia}] v^a \\ &= 0. \end{aligned}$$

Hence, as a corollary of Lemma 3.1, we obtain

Proposition 3.2. *Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\bar{\nabla}$. Then, for any vector field V on M ,*

i) The vertical lift ${}^V V$ is a harmonic vector field on TM with respect to the semi-symmetric metric connection $\bar{\nabla}$ if and only if the vector field V is a harmonic vector field with respect to the Levi-Civita connection ∇ ;

ii) The complete lift ${}^C V$ is a harmonic vector field on TM with respect to the semi-symmetric metric connection $\bar{\nabla}$ if and only if the vector field V is a harmonic vector field with respect to the Levi-Civita connection ∇ and $g_{si} g_{ja} - g_{sj} g_{ia} = 0$;

iii) The horizontal lift ${}^H V$ is a harmonic vector field on TM with respect to the semi-symmetric metric connection $\bar{\nabla}$ if and only if the vector field V is parallel with respect to the Levi-Civita connection ∇ and M has constant sectional curvature 1.

3.2. Concurrent vector fields

Definition 3.3. A vector field $\tilde{V} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ on TM is called a concurrent vector field with respect to the semi-symmetric metric connection $\bar{\nabla}$ if it satisfies

$$(3.5) \quad \bar{\nabla}_\beta \tilde{V}^\epsilon = \bar{\nabla}_{E_\beta} \tilde{V}^\epsilon = \tilde{k} \delta_\beta^\epsilon,$$

where \tilde{k} is a function on TM and δ_β^ϵ is the Kronecker symbol.

Proposition 3.3. Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\bar{\nabla}$. The vector field \tilde{V} on TM is concurrent with respect to semi-symmetric metric connection $\bar{\nabla}$ if and only if the vector field \tilde{V} has the form

$$\tilde{V} = \left(\begin{array}{c} v^h \\ \frac{1}{n} [\text{trace}(\nabla V)] y^h \end{array} \right)$$

and the following condition is satisfied

$$\frac{1}{n} [\nabla_j (\text{trace}(\nabla V)) y^h] + (y^s R_{sja}^h + y_a \delta_j^h - y^h g_{ja}) v^a = 0.$$

Proof. With respect to the adapted frame, firstly putting $\epsilon = h, \beta = \bar{j}$ in (3.5), it follows that

$$\begin{aligned} \bar{\nabla}_{\bar{j}} v^h &= E_{\bar{j}} v^h + \bar{\Gamma}_{\bar{j}a}^h v^a + \bar{\Gamma}_{\bar{j}\bar{a}}^h v^{\bar{a}} = \tilde{k} \delta_{\bar{j}}^h \\ &\Rightarrow \partial_{\bar{j}} v^h = 0 \\ &\Rightarrow v^h = v^h(x^h). \end{aligned}$$

Similarly putting $\epsilon = h, \beta = j$ and $\epsilon = \bar{h}, \beta = \bar{j}$, we respectively get

$$\begin{aligned} \bar{\nabla}_j v^h &= E_j v^h + \bar{\Gamma}_{ja}^h v^a + \bar{\Gamma}_{j\bar{a}}^h v^{\bar{a}} = \tilde{k} \delta_j^h \\ &\Rightarrow \partial_j v^h + \bar{\Gamma}_{ja}^h v^a = \tilde{k} \delta_j^h \\ &\Rightarrow \nabla_j v^h = \tilde{k} \delta_j^h \quad (h \rightarrow j) \\ &\Rightarrow \frac{1}{n} \nabla_j v^j = \tilde{k} \end{aligned}$$

and

$$\begin{aligned} \bar{\nabla}_{\bar{j}} v^{\bar{h}} &= E_{\bar{j}} v^{\bar{h}} + \bar{\Gamma}_{\bar{j}a}^{\bar{h}} v^a + \bar{\Gamma}_{\bar{j}\bar{a}}^{\bar{h}} v^{\bar{a}} = \tilde{k} \delta_{\bar{j}}^{\bar{h}} \\ &\Rightarrow \partial_{\bar{j}} v^{\bar{h}} = \frac{1}{n} \nabla_j v^j \cdot \delta_{\bar{j}}^{\bar{h}} \\ &\Rightarrow \partial_{\bar{j}} v^{\bar{h}} = \frac{1}{n} [\text{trace}(\nabla V) \delta_{\bar{j}}^{\bar{h}}] \\ &\Rightarrow \partial_{\bar{j}} v^{\bar{h}} = \frac{1}{n} [\text{trace}(\nabla V) (\partial_{\bar{j}} y^h)] \end{aligned}$$

$$\begin{aligned} \Rightarrow \partial_{\bar{j}} v^{\bar{h}} &= \partial_{\bar{j}} \left[\frac{1}{n} \text{trace}(\nabla V) y^h \right] \\ \Rightarrow v^{\bar{h}} &= \frac{1}{n} [\text{trace}(\nabla V)] y^h. \end{aligned}$$

Finally putting $\epsilon = \bar{h}, \beta = j$, we find

$$\begin{aligned} \bar{\nabla}_j v^{\bar{h}} &= E_j v^{\bar{h}} + \bar{\Gamma}_{ja}^{\bar{h}} v^a + \bar{\Gamma}_{j\bar{a}}^{\bar{h}} v^{\bar{a}} = \tilde{k} \delta_j^{\bar{h}} \\ \Rightarrow E_j \left[\frac{1}{n} [\text{trace}(\nabla V)] y^h \right] &+ (y^s R_{sja}^h + y_a \delta_j^h - y^h g_{ja}) v^a \\ &+ \Gamma_{ja}^h \left[\frac{1}{n} [\text{trace}(\nabla V)] y^a \right] = 0 \\ \Rightarrow (\partial_j - y^s \Gamma_{sj}^m \partial_{\bar{m}}) \left[\frac{1}{n} [\text{trace}(\nabla V)] y^h \right] &+ (y^s R_{sja}^h + y_a \delta_j^h - y^h g_{ja}) v^a + y^a \Gamma_{ja}^h \left[\frac{1}{n} [\text{trace}(\nabla V)] \right] = 0 \\ \Rightarrow \frac{1}{n} [\partial_j (\text{trace}(\nabla V)) y^h] - y^s \Gamma_{sj}^h \left[\frac{1}{n} [\text{trace}(\nabla V)] \right] &+ (y^s R_{sja}^h + y_a \delta_j^h - y^h g_{ja}) v^a + y^a \Gamma_{ja}^h \left[\frac{1}{n} [\text{trace}(\nabla V)] \right] = 0 \\ \Rightarrow \frac{1}{n} [\partial_j (\text{trace}(\nabla V)) y^h] + (y^s R_{sja}^h + y_a \delta_j^h - y^h g_{ja}) v^a &= 0 \\ \Rightarrow \frac{1}{n} [\bar{\nabla}_j (\text{trace}(\nabla V)) y^h] + (y^s R_{sja}^h + y_a \delta_j^h - y^h g_{ja}) v^a &= 0. \end{aligned}$$

□

3.3. Conformal vector fields

Let \tilde{V} be a vector field on TM with components $(v^h, v^{\bar{h}})$ with respect to the adapted frame $\{E_\beta\}$. Then \tilde{V} is a fibre-preserving vector field on TM if and only if v^h depends only on the variables (x^h) .

Definition 3.4. A vector field $\tilde{V} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ on TM is called a fibre-preserving conformal vector field with respect to the semi-symmetric metric connection $\bar{\nabla}$ if it satisfies

$$L_{\tilde{V}}^C g_{\alpha\beta} = (\bar{\nabla}_\alpha \tilde{V}^\epsilon)^C g_{\epsilon\beta} + (\bar{\nabla}_\beta \tilde{V}^\epsilon)^C g_{\epsilon\alpha} = 2\tilde{\Omega}^C g_{\alpha\beta}.$$

Putting $(\alpha, \beta) = (i, \bar{j}), (\bar{i}, j)$ and (i, j) , from the above equation, it can be written the following system

$$(3.6) \quad \left\{ \begin{array}{l} i) (\nabla_i v^h) g_{hj} + (E_{\bar{j}} v^{\bar{h}}) g_{hi} = 2\tilde{\Omega} g_{ij}, \\ ii) (E_{\bar{i}} v^{\bar{h}}) g_{hj} + (\nabla_j v^h) g_{hi} = 2\tilde{\Omega} g_{ij}, \\ iii) \left[E_i v^{\bar{h}} + (y^s R_{sia}^h + y_a \delta_i^h - y^h g_{ia}) v^a + \Gamma_{ia}^h v^{\bar{a}} \right] g_{hj} \\ \quad + \left[E_j v^{\bar{h}} + (y^s R_{sja}^h + y_a \delta_j^h - y^h g_{ja}) v^a + \Gamma_{ja}^h v^{\bar{a}} \right] g_{hi} = 0. \end{array} \right.$$

Proposition 3.4. The scalar function $\tilde{\Omega}$ on TM depends only on the variables (x^h) with respect to the induced coordinates (x^h, y^h) .

Proof. Applying $E_{\bar{k}}$ to the both sides of the equation (ii) in (3.6), we have

$$g_{hj} E_{\bar{k}} E_{\bar{i}} v^{\bar{h}} = 2E_{\bar{k}}(\bar{\Omega}) g_{ij}$$

from which we get

$$E_{\bar{k}}(\bar{\Omega}) g_{ij} = E_{\bar{i}}(\bar{\Omega}) g_{kj}.$$

It follows that

$$(n - 1)E_{\bar{k}}(\bar{\Omega}) = 0.$$

This shows that the scalar function $\tilde{\Omega}$ on TM depends only on the variables (x^h) with respect to the induced coordinates (x^h, y^h) . Thus we can regard $\tilde{\Omega}$ as a function on M and in the following we write ρ instead of $\tilde{\Omega}$. \square

From (3.6) and Proposition 3.4, $E_{\bar{i}}(v^{\bar{h}})$ depends only the variables (x^h) , thus we can put

$$(3.7) \quad v^{\bar{h}} = y^a A_a^h + B^h,$$

where A_a^h and B^h are certain functions which depend only on the variable (x^h) . Furthermore, we can easily show that A_a^h and B^h are the components of a $(1, 1)$ tensor field and a contravariant vector field on M , respectively.

Any vector field V on a (pseudo-)Riemannian manifold (M, g) is a Killing vector field if $L_V g_{ij} = \nabla_i v_j + \nabla_j v_i = 0$.

Proposition 3.5. *If we put*

$$B = B^h \frac{\partial}{\partial x^h},$$

then the vector field B on M is a Killing vector field with respect to the Levi-Civita connection ∇ .

Proof. Substituting (3.7) into the equation (iii) in (3.6) we have

$$(3.8) \quad \nabla_i B_j + \nabla_j B_i = 0$$

and

$$(3.9) \quad v^a (R_{siaj} + R_{sjai} + g_{sa} g_{ij} - g_{ia} g_{sj} + g_{sa} g_{ji} - g_{ja} g_{si}) + \nabla_i A_{sj} + \nabla_j A_{si} = 0$$

where $B_i = g_{im} B^m$ and $A_{sj} = g_{hj} A_s^h$. Hence by (3.8), it follows

$$L_B g_{ij} = \nabla_i B_j + \nabla_j B_i = 0.$$

This shows B is a Killing vector field on M with respect to the Levi-Civita connection ∇ . \square

Substituting (3.7) into the equation (ii) in (3.6), we have

$$\begin{aligned}
 (3.10) \quad & E_{\bar{i}}(v^{\bar{h}}) g_{hj} + (\nabla_j v^h) g_{hi} = 2\rho g_{ij} \\
 \Rightarrow & \partial_{\bar{i}}(y^s A_s^h + B^h) g_{hj} + (\nabla_j v^h) g_{hi} = 2\rho g_{ij} \\
 \Rightarrow & A_i^h g_{hj} + (\nabla_j v^h) g_{hi} = 2\rho g_{ij} \\
 \Rightarrow & g_{hj} A_i^h = 2\rho g_{ij} - g_{hi} (\nabla_j v^h).
 \end{aligned}$$

Let ∇ be a linear connection on M . A vector field V on M is said to be a projective vector field if there exists a 1-form θ such that

$$(L_V \nabla)(X, Y) = \theta(X)Y + \theta(Y)X$$

for any vector fields X and Y on M . In this case θ is called the associated 1-form of V . It can locally be expressed in the following form

$$L_V \Gamma_{ij}^h = \theta_i \delta_j^h + \theta_j \delta_i^h.$$

Proposition 3.6. *The vector field V with components (v^h) is a projective vector field on M with respect to the Levi-Civita connection ∇ , if $\delta_a^h g_{ij} - g_{ia} \delta_j^h + \delta_a^h g_{ji} - g_{ja} \delta_i^h = 0$.*

Proof. Applying the covariant derivative ∇_k to the both sides of (3.10), we obtain

$$\begin{aligned}
 (3.11) \quad g_{hj} \nabla_k A_i^h &= \nabla_k [2\rho g_{ij} - g_{hi} (\nabla_j v^h)] \\
 &= 2(\nabla_k \rho) g_{ij} - g_{hi} \nabla_k \nabla_j v^h \\
 &= 2\rho_k g_{ij} - g_{hi} (L_V \Gamma_{kj}^h - R_{akj}^h v^a) \\
 \nabla_k A_{ij} &= 2\rho_k g_{ij} - L_V \Gamma_{kj}^h g_{hi} - R_{akij} v^a.
 \end{aligned}$$

Substituting (3.11) into (3.9), we have

$$v^a (R_{siaj} + R_{sjai} + g_{sa} g_{ij} - g_{ia} g_{sj} + g_{sa} g_{ji} - g_{ja} g_{si}) + \nabla_i A_{sj} + \nabla_j A_{si} = 0$$

$$\begin{aligned}
 & v^a (R_{siaj} + R_{sjai} + g_{sa} g_{ij} - g_{ia} g_{sj} + g_{sa} g_{ji} - g_{ja} g_{si}) \\
 & + 2\rho_i g_{sj} - L_V \Gamma_{ij}^h g_{hs} - R_{aisj} v^a + 2\rho_j g_{si} - L_V \Gamma_{ji}^h g_{hs} - R_{ajsi} v^a = 0
 \end{aligned}$$

$$v^a (g_{sa} g_{ij} - g_{ia} g_{sj} + g_{sa} g_{ji} - g_{ja} g_{si}) + 2(\rho_i g_{sj} + \rho_j g_{si}) = 2L_V \Gamma_{ij}^h g_{hs}$$

$$L_V \Gamma_{ij}^h = \rho_i \delta_j^h + \rho_j \delta_i^h + \frac{1}{2} v^a (\delta_a^h g_{ij} - g_{ia} \delta_j^h + \delta_a^h g_{ji} - g_{ja} \delta_i^h),$$

where $\rho_i = \nabla_i \rho$. Hence, V is a projective vector field on M with respect to the Levi-Civita connection ∇ . \square

Now we consider the converse problem, that is, let M admit a projective vector field $V = v^h \frac{\partial}{\partial x^h}$ with respect to the Levi-Civita connection ∇ . Then we have the following proposition.

Proposition 3.7. *The vector field \tilde{V} on TM defined by*

$$\tilde{V} = v^h E_h + (y^s A_s^h + B^h) E_{\bar{h}}$$

is a fibre-preserving conformal vector field on TM with respect to the semi-symmetric metric connection $\bar{\nabla}$, where $A_i^h = g^{ha} A_{ai}$, $A_{ij} = 2\rho g_{ij} - \nabla_j v_i$, and $g_{ji} B^j = B_i$, $2p_i g_{sj} - L_V \Gamma_{ij}^h g_{hs} + (g_{sm} g_{ij} - g_{im} g_{sj}) = 0$.

Proof. If B_h , v^h and A_i^h are given so that they satisfy the above assumptions, we see that $\tilde{V} = v^h E_h + (y^s A_s^h + B^h) E_{\bar{h}}$ is a fibre-preserving conformal vector field on TM with respect to the semi-symmetric metric connection $\bar{\nabla}$. We omit standard calculations. \square

3.4. Projective vector fields

In this section, we study fibre-preserving projective vector fields on TM with respect to the semi-symmetric metric connection $\bar{\nabla}$. We shall first state following lemma which is needed later on.

Lemma 3.2. *The Lie derivations of the adapted frame with respect to the fibre-preserving vector field $\tilde{V} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ are given as follows*

$$L_{\tilde{V}} E_h = -(\partial_h v^a) E_a + \left\{ y^b v^c R_{hcb}^a - v^{\bar{b}} \Gamma_{bh}^a - (E_h v^{\bar{a}}) \right\} E_{\bar{a}},$$

$$L_{\tilde{V}} E_{\bar{h}} = \left\{ v^b \Gamma_{bh}^a - (E_{\bar{h}} v^{\bar{a}}) \right\} E_{\bar{a}}.$$

The general form of fibre-preserving vector fields on TM with respect to the semi-symmetric metric connection $\bar{\nabla}$ are given by

Theorem 3.1. *Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\bar{\nabla}$. Then a vector field \tilde{V} is a fibre-preserving projective vector field with associated 1-form $\bar{\theta}$ on TM with respect to the semi-symmetric metric connection $\bar{\nabla}$ if and only if the vector field \tilde{V} has the following form*

$$\tilde{V} = {}^H V + {}^V B + \gamma A,$$

where the vector fields $V = (v^h)$, $B = (B^h)$, the $(1, 1)$ -tensor field $A = (A_i^h)$ and the associated 1-form $\bar{\theta}$ satisfy the following conditions

$$\begin{aligned} (i) \bar{\theta} &= \theta_i dx^i, \\ (ii) \nabla_i \theta_j &= (n-1)(L_V g_{ij}), \\ (iii) \nabla_j A_i^h &= \theta_j \delta_i^h - v^c R_{cji}^h, \\ (iv) \nabla_i \nabla_j v^h + R_{aij}^h v^a &= \theta_i \delta_j^h + \theta_j \delta_i^h, \\ (v) \nabla_i \nabla_j B^k + R_{hij}^k B^h + B^h g_{hj} \delta_i^k - B^k g_{ij} &= 0, \\ (vi) L_V \Gamma_{ij}^h &= \theta_i \delta_j^h + \theta_j \delta_i^h. \end{aligned}$$

Proof. A vector field $\tilde{V} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ on TM is a fibre-preserving projective vector field with respect to the semi-symmetric metric connection $\bar{\nabla}$ if and only if there exists a 1-form $\tilde{\theta}$ with components $(\tilde{\theta}_i, \tilde{\theta}_{\bar{i}})$ on TM such that

$$\begin{aligned} (L_{\tilde{X}} \bar{\nabla})(\tilde{Y}, \tilde{Z}) &= L_{\tilde{X}}(\bar{\nabla}_{\tilde{Y}} \tilde{Z}) - \bar{\nabla}_{\tilde{Y}}(L_{\tilde{X}} \tilde{Z}) - \bar{\nabla}_{(L_{\tilde{X}} \tilde{Y})} \tilde{Z} \\ &= \tilde{\theta}(\tilde{Y}) \tilde{Z} + \tilde{\theta}(\tilde{Z}) \tilde{Y} \end{aligned}$$

for any vector fields \tilde{Y} and \tilde{Z} on TM . We compute the following system

$$\begin{aligned} (3.12) \quad (L_{\tilde{V}} \bar{\nabla})(E_{\bar{i}}, E_{\bar{j}}) &= L_{\tilde{V}}(\bar{\nabla}_{E_{\bar{i}}} E_{\bar{j}}) - \bar{\nabla}_{E_{\bar{i}}}(L_{\tilde{V}} E_{\bar{j}}) - \bar{\nabla}_{(L_{\tilde{V}} E_{\bar{i}})} E_{\bar{j}} \\ &= \tilde{\theta}(E_{\bar{i}}) E_{\bar{j}} + \tilde{\theta}(E_{\bar{j}}) E_{\bar{i}}, \end{aligned}$$

$$\begin{aligned} (3.13) \quad (L_{\tilde{V}} \bar{\nabla})(E_{\bar{i}}, E_j) &= L_{\tilde{V}}(\bar{\nabla}_{E_{\bar{i}}} E_j) - \bar{\nabla}_{E_{\bar{i}}}(L_{\tilde{V}} E_j) - \bar{\nabla}_{(L_{\tilde{V}} E_{\bar{i}})} E_j \\ &= \tilde{\theta}(E_{\bar{i}}) E_j + \tilde{\theta}(E_j) E_{\bar{i}}, \end{aligned}$$

$$\begin{aligned} (3.14) \quad (L_{\tilde{V}} \bar{\nabla})(E_i, E_j) &= L_{\tilde{V}}(\bar{\nabla}_{E_i} E_j) - \bar{\nabla}_{E_i}(L_{\tilde{V}} E_j) - \bar{\nabla}_{(L_{\tilde{V}} E_i)} E_j \\ &= \tilde{\theta}(E_i) E_j + \tilde{\theta}(E_j) E_i. \end{aligned}$$

From (3.12), by virtue of (2.2) and Lemma 3.2 we obtain

$$(3.15) \quad \left\{ \partial_{\bar{i}}(\partial_{\bar{j}} v^{\bar{a}}) \right\} E_{\bar{a}} = \tilde{\theta}_{\bar{i}} E_{\bar{j}} + \tilde{\theta}_{\bar{j}} E_{\bar{i}}.$$

Similarly, from (3.13) we get

$$(3.16) \quad \left\{ -v^c R_{jci}^a + (E_{\bar{i}} v^{\bar{b}}) \Gamma_{bj}^a + E_{\bar{i}}(E_j v^{\bar{a}}) \right\} E_{\bar{a}} = \tilde{\theta}_{\bar{i}} E_j + \tilde{\theta}_j E_{\bar{i}}$$

from which, we have

$$(3.17) \quad \tilde{\theta}_{\bar{i}} = 0.$$

Due to $\tilde{\theta}_{\bar{i}} = 0$, (3.15) to

$$\partial_{\bar{i}}(\partial_{\bar{j}} v^{\bar{a}}) = 0,$$

and we obtain

$$(3.18) \quad v^{\bar{a}} = y^s A_s^a + B^a,$$

where A_s^a and B^a are certain functions which depend only on the variables (x^h) and the coordinate transformation rule implies that A is a $(1,1)$ -tensor field with components (A_s^a) and B is a vector field with components (B^a) . Hence, the fibre-preserving projective vector field \tilde{V} on TM can be expressed in the following form

$$(3.19) \quad \begin{aligned} \tilde{V} &= v^h E_h + v^{\bar{h}} E_{\bar{h}} = v^h E_h + \{y^s A_s^a + B^a\} E_{\bar{h}} \\ &= {}^H V + {}^V B + \gamma A. \end{aligned}$$

Substituting (3.18) into (3.16), we obtain

$$(3.20) \quad R_{aji}{}^h v^a + \nabla_j A_i^h = \delta_i^h \theta_j.$$

Substituting (3.18) and (3.20) into (3.14), we have

$$(3.21) \quad \begin{aligned} &\{\nabla_i \nabla_j v^h + R_{aij}{}^h v^a\} E_h + \{\nabla_i \nabla_j B^k + R_{hij}{}^k B^h + B^h g_{hj} \delta_i^k \\ &- B^k g_{ij} + y^s (\nabla_i \nabla_j A_s^k + A_s^h R_{hij}{}^k - R_{sij}{}^a A_a^k + v^h \nabla_h R_{sij}{}^k \\ &- v^h \nabla_i R_{jhs}{}^k + \nabla_j v^h R_{sih}{}^k + \nabla_i v^h R_{sjh}{}^k + \nabla_j v^a g_{sa} \delta_i^k \\ &- \nabla_j v^a \delta_s^k g_{ia} + \nabla_i v^a g_{sj} \delta_a^k - \nabla_i v^a \delta_s^k g_{aj} + A_s^h g_{hj} \delta_i^k - g_{sj} A_i^k)\} E_{\bar{h}} \\ &= \tilde{\theta}_i E_j + \tilde{\theta}_j E_i. \end{aligned}$$

From (3.21), we have

$$(3.22) \quad \nabla_i \nabla_j v^h + R_{aij}{}^h v^a = \tilde{\theta}_i \delta_j^h + \tilde{\theta}_j \delta_i^h,$$

$$(3.23) \quad \nabla_i \nabla_j B^k + R_{hij}{}^k B^h + B^h g_{hj} \delta_i^k - B^k g_{ij} = 0,$$

$$(3.24) \quad \begin{aligned} &\nabla_i \nabla_j A_s^k + A_s^h R_{hij}{}^k - R_{sij}{}^a A_a^k + v^h \nabla_h R_{sij}{}^k \\ &- v^h \nabla_i R_{jhs}{}^k + \nabla_j v^h R_{sih}{}^k + \nabla_i v^h R_{sjh}{}^k \\ &+ \nabla_j v^a g_{sa} \delta_i^k - \nabla_j v^a \delta_s^k g_{ia} + \nabla_i v^a g_{sj} \delta_a^k \\ &- \nabla_i v^a \delta_s^k g_{aj} + A_s^h g_{hj} \delta_i^k - g_{sj} A_i^k \\ &= 0. \end{aligned}$$

The equation (3.22) shows that the induced vector field $V = v^h \frac{\partial}{\partial x^h}$ is a projective vector field with respect to the Levi-Civita Connection ∇ . Hence we obtain

$$(3.25) \quad L_V R_{ij} = -(n-1) \nabla_i \theta_j.$$

Contracting k and s in (3.24) and using (3.20) and (3.25), we get

$$\nabla_i \theta_j = (n-1) (L_V g)_{ij}.$$

In the case, (3.24) is reduced to

$$\begin{aligned} & A_s^h R_{hij}^k - R_{sij}^a A_a^k + v^h \nabla_h R_{sij}^k + \nabla_j v^h R_{sij}^k \\ & + \nabla_i v^h R_{shj}^k + \nabla_j v_s \delta_i^k + \nabla_i v^k g_{sj} + A_{sj} \delta_i^k - g_{sj} A_i^k \\ & = 0. \end{aligned}$$

Conversely, if B^h, v^h, θ_h and A_i^h are given so that they satisfy (i)-(vi), reserving the above steps, we see that $\tilde{X} =^H V +^V B + \gamma A$ is a fibre-preserving projective vector field on TM with respect to the semi-symmetric metric connection $\bar{\nabla}$. Hence, the proof is complete. \square

Let \tilde{V} be a fibre-preserving vector field on TM with components $(v^h, v^{\bar{h}})$. It is well-known that every fibre-preserving vector field \tilde{V} on TM induces a vector field V on M with components (v^h) . The below result follows immediately from Theorem 3.1 and from its Proof.

Corollary 3.1. *Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\bar{\nabla}$. Every fibre-preserving projective vector field \tilde{V} is of the form (3.19) and it naturally induces a projective vector field V on M .*

Let \tilde{V} be a vector field on TM with components $(v^h, v^{\bar{h}})$ with respect to the adapted frame $\{E_\beta\}$. Then \tilde{V} is a vertical vector field on TM if and only if $v^h = 0$. In the present case, the vector field \tilde{V} in Theorem 3.1 reduces to $\tilde{V} =^V B + \gamma A$. Hence, from the Theorem 3.1, we obtain the following conclusion.

Corollary 3.2. *Let M be a (pseudo-)Riemannian manifold and TM be its tangent bundle with the semi-symmetric metric connection $\bar{\nabla}$. If TM admits a vertical projective vector field \tilde{V} , then the vector field \tilde{V} is defined by*

$$\tilde{V} =^V B + \gamma A,$$

where the vector field $B = (B^h)$, the (1,1)-tensor field $A = (A_i^h)$ and the associated 1-form $\tilde{\theta}$ satisfy the following conditions

$$\begin{aligned} & (i) \bar{\theta} = \theta_i dx^i, \\ & (ii) \nabla_j A_i^h = \theta_j \delta_i^h, \\ & (iii) \nabla_i \theta_j = 0, \\ & (iv) \nabla_i \nabla_j B^k + R_{hij}^k B^h + B^h g_{hj} \delta_i^k - B^k g_{ij} = 0, \\ & (v) A_s^h R_{hij}^k - R_{sij}^a A_a^k + A_{sj} \delta_i^k - g_{sj} A_i^k = 0. \end{aligned}$$

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