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ON ZETA AND DIRICHLET BETA FUNCTION FAMILIES AS GENERATORS OF GENERALIZED MATHIEU SERIES, PROVIDING APPROXIMATION AND BOUNDS

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Abstract. Integral representations for a generalized Mathieu series and its companions are used to undertake analysis leading to novel insights for Zeta and Dirichlet Beta function families. The bounds are procured using sharp bounds of Zeta and Dirichlet family bounds to procure approximating and bounds utilising integral representation of generalized Mathieu series results using in particular Hardy-type upper bounds.
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1. Introduction

The series, known in the literature as the Mathieu series,

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{\left(n^{2}+r^{2}\right)^{2}},\quad r>0, \tag{1.1}$$

has been extensively studied in the past since its introduction by Mathieu [28] in 1890, where it arose in connection with work on elasticity of solid bodies. The reader is directed to the references and the books [4], [5] and [33] for further illustration of various representations and bounds. The various applications of areas involve the solution of the biharmonic equation in a rectangular two dimensional domain using

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the so called *superposition method* and the interested reader is referred to the work of Meleshko ([29],[30],[31]) for excellent coverage and further references. A Literature search in MathScinet with 'Mathieu series' results in over 800 hits demnostrates that the area continues to attract many avenues of research and application. See also some of the recent activity such as in [5, 19, 32, 36].

One of the main questions addressed in relation to the series is obtaining sharp bounds.

Building on some results from [37], Alzer, Brenner and Ruehr [1] showed that the best constants a and b in

$$\frac{1}{x^2+a} < S(x) < \frac{1}{x^2+b}, \quad x \neq 0 \tag{1.2}$$

are $a = \frac{1}{2\zeta(3)}$ and $b = \frac{1}{6}$ where $\zeta(\cdot)$ denotes the Riemann zeta function defined by

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}. \tag{1.3}$$

An integral representation for $S(r)$ as given in (1.1) was presented in [18] and [20] as

$$S(r) = \frac{1}{r} \int_0^{\infty} \frac{x}{e^x - 1} \sin(rx)\, dx. \tag{1.4}$$

Guo [22] utilised (1.4) to obtain bounds on $S(r)$.

Guo in [22] posed the interesting problem as to whether there is an integral representation of the generalized Mathieu series

$$S_\mu(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^{1+\mu}}, \quad r > 0,\ \mu > 0. \tag{1.5}$$

The challenge by Guo [22] to obtain an integral representation for $S_\mu(r)$ as defined in (1.5), was successfully answered by Cerone and Lenard [15] in which the following two theorems were proved.

Theorem 1.1. *The generalized Mathieu series $S_\mu(r)$ defined by (1.5) may be represented in the integral form*

$$S_\mu(r) = C_\mu(r) \int_0^{\infty} \frac{x^{\mu+\frac{1}{2}}}{e^x - 1} J_{\mu-\frac{1}{2}}(rx)\, dx, \quad \mu > 0, \tag{1.6}$$

where

$$C_\mu(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\, \Gamma(\mu+1)} \tag{1.7}$$

and $J_\nu(z)$ is the ν^{th} order Bessel function of the first kind.

The emphasis as in [15], became the derivation of bounds for the generalized Mathieu series $S_\mu(r)$. The first approach utilized sharp bounds for the Bessel function $|J_\nu(z)|$. To this end, in an article by Landau [25], the best possible uniform bounds were obtained for Bessel function.

The following results were obtained in [15] using a weighted Čebyšev functional approach. See also [10] where the approach was utilized for a greater variety of special functions. Further relating reference papers, [7], [12],and books, [14], [23], [38] that have contributed to the work.

Theorem 1.2. *For $\mu > 0$ and $r > 0$ the generalized Mathieu series $S_\mu(r)$ satisfies*

$$\left| S_\mu(r) - \frac{\pi^2}{12\mu\left(r^2+\frac{1}{4}\right)^\mu} \right| \tag{1.8}$$

$$\leq \kappa \left[\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(2\mu-\frac{1}{2}\right)}{2^{2\mu-1}\Gamma^2(\mu+1)} \int_0^{\frac{\pi}{2}} \frac{\cos^{2\mu-1}\phi}{\left[\left(\frac{1}{4}\right)^2 + r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}} d\phi - \frac{1}{2\mu^2\left(r^2+\frac{1}{4}\right)^{2\mu}} \right]^{\frac{1}{2}}$$

$$\leq \kappa \left[\frac{\Gamma\left(2\mu-\frac{1}{2}\right)\Gamma\left(\mu+\frac{1}{2}\right)}{2^{2\mu}\Gamma^3(\mu+1)} \cdot \frac{1}{r^{4\mu-1}} - \frac{1}{2\mu^2\left(r^2+\frac{1}{4}\right)^{2\mu}} \right]^{\frac{1}{2}},$$

where

$$\kappa = \left[\pi^2\left(1-\frac{\pi^2}{72}\right) - 7\zeta(3) \right]^{\frac{1}{2}} = 0.3198468959\ldots. \tag{1.9}$$

Corollary 1.1. *The Mathieu series $S(r)$, satisfies the following bounds*

$$\left| \sum_{n=1}^{\infty} \frac{2n}{\left(n^2+r^2\right)^2} - \frac{\pi^2}{12\left(r^2+\frac{1}{4}\right)} \right| \leq 2\sqrt{2}\cdot\kappa \left\{ \frac{2}{1+(4r)^2} - \frac{1}{\left[1+(2r)^2\right]^2} \right\}^{\frac{1}{2}} \tag{1.10}$$

where κ is as given by (1.9).

As explained in Pogany *et al.*[32], motivated by [15], a family of *Mathieu* **a**-*series* were introduced by Pogany *et al.*[35] together with their integral representations, various approaches and results were used to procure bounds.

The *alternating generalized Mathieu series*, companion to $S_\mu(r)$,was introduced by Pogany *et al.* [36] and is represented by

$$\tilde{S}_\mu(r) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot 2n}{\left(n^2+r^2\right)^{1+\mu}}, \qquad r>0,\ \mu>0. \tag{1.11}$$

which can be also expressed in the following integral form

$$\tilde{S}_\mu(r) = C_\mu(r) \int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x+1} J_{\mu-\frac{1}{2}}(rx)\,dx, \qquad \mu > 0, \tag{1.12}$$

where $C_\mu(r)$ is as given in (1.7).

In paper [16], bounds were obtained for the alternating generalized Mathieu series $\tilde{S}_\mu(r)$, the odd $\phi_\mu(r)$ and even $\psi_\mu(r)$ generalized Mathieu series. This was accomplished by using their integral representations via *Čebyšev Functional* bounds which is presented in Subsection 2.1. The methodology produces both the approximation and bounds for the companion series of the *generalized Mathieu series.* It is further demonstrated that the relationship between the Zeta function, the alternating Zeta function and the odd Zeta function family is recaptured by allowing $r \to 0$.

In the paper [11] the emphasis was to extend the methodology for the Zeta family to the Dirichlet Beta L- function family through the generalized Mathieu series which is presented in section 2. The work in [16] emphasised the extension to a generalized Mathieu series $S_\mu(r)$ by the Zeta function, $\zeta(\cdot)$ as generator.The work is based on the generator as the sum of the reciprocal powers of odd positive numbers, $\lambda(s)$ .The Dirichlet Beta function $\beta(s) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{(2n-1)^s}$ has the honour as the lead of this family.

The generalized Mathieu series are based on two parameters r and μ, as exemplified by (1.5) and (1.6) in addition to various generators.The Dirichlet L-series have played a great deal of attention in number theory. These are also relevant to lattice sums which may be represented by lower dimensional lattice sums.The classic example of this was first given by Lorenz [26] as well in [39] , given by,

$$\sum_{m,n\neq 0,0}^{\infty} \frac{1}{(m^2+n^2)^s} = 4\zeta(s)\cdot\beta(s). \tag{1.13}$$

The reader is encouraged to refer to [39] , [6] and [17] for interest and further references.

Section 3 demonstrates a result of Hardy [24] which enables sharp upper bounds of a well known function which, it is seemed not to have been utilized in the literature.The upper bounds of Zeta and Beta families and, double bounds for Eta and Dirichlet Beta functions are also shown. Section 4 demonstrates application of Hardy-type upper bounds for generalized Mathieu series and compares with results of *Čebyšev Functional* bounds. Some further applications of obtaining upper bounds with Hardy -type results are also shown in section 5.

2. Some Results on Bounding the Generalized Mathieu Series via the Čebyšev Functional

The current section presents a key methodology to procure approximation and bounds for the integral representations of the Zeta and Dirichlet Beta function companions, as generators of generalized Mathieu series.

The weighted Čebyšev functional defined by

$$T(f,g;p) := \mathcal{M}(fg;p) - \mathcal{M}(f;p)\,\mathcal{M}(g;p)\,, \tag{2.1}$$

where $\mathcal{M}$ is the weighted integral mean

$$\mathcal{M}(h;p) := \frac{\int_a^b p(x)\,h(x)\,dx}{P}, \tag{2.2}$$

where P is given by,

$$P = \int_a^b p(x)\,dx\ . \tag{2.3}$$

has been extensively investigated in the literature with the view of determining its bounds. The unweighted Čebyšev functional $T(f,g;1)$,was bounded by Grüss in [21] by the product of the difference of the functions and their function bounds.

Cerone and Dragomir [13] showed that the best K, in the following lemma, in the sense of providing the sharpest bound for the Euclidean or $2-$norm, results when $K = \mathcal{M}(f;p)$.

Lemma 2.1. *The sharpest bound for the Čebyšev functional involving the Euclidean norm is given by*

$$\begin{aligned} & P\cdot|T(f,g;p)| \\ & \le \inf_K \left[\int_a^b p(t)\,(f(t)-K)^2\,dt\right]^{\frac{1}{2}} \left[\int_a^b p(t)\,(g(t)-\mathcal{M}(g;p))^2\,dt\right]^{\frac{1}{2}} \\ & = \left[\int_a^b p(t)\,f^2(t)\,dt - \mathcal{M}^2(f;p)\right]^{\frac{1}{2}} \left[\int_a^b p(t)\,g^2(t)\,dt - \mathcal{M}^2(g;p)\right]^{\frac{1}{2}}. \end{aligned} \tag{2.4}$$

The following technical lemma involving the Euler beta function $B(x,y)$ is represented in terms of the gamma function by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.5}$$

The bounds of the Lemma below provides a courser bounds for Čebyšev Functional bounds of the Generalized Mathieu series of Zeta and Beta family results.

Lemma 2.2. *The following result holds (see [16] for the proof)*

$$\frac{1}{2}\cdot\frac{B(\frac{1}{2},\mu)}{[\alpha^2+r^2]^{2\mu-\frac{1}{2}}} \le \int_0^{\frac{\pi}{2}} \frac{\cos^{2\mu-1}\phi}{[\alpha^2+r^2\cos^2\phi]^{2\mu-\frac{1}{2}}}d\phi \le \frac{1}{2}\cdot\frac{B(\frac{1}{2},\mu)}{\alpha^{4\mu-1}}, \tag{2.6}$$

It is noted that equality follows in (2.6) when $r=0$.

2.1. Bounds for $\tilde{S}_\mu(r)$,Odd and Even Generalized Mathieu Series via the Čebyšev Functional

Bounds on the Čebyšev functional (2.1) may be looked upon as estimating the distance of the weighted mean of the product of two functions from the product of the weighted means of the two functions. This proves to be quite useful since the individual means are invariably easier to evaluate.

Theorem 2.1. *(see [16] for the proof).For $\mu > 0$ and $r > 0$, the alternating generalized Mathieu series $\tilde{S}_\mu(r)$ satisfies the following bounds,*

$$\left|\tilde{S}_\mu(r) - \frac{\pi^2}{24\mu\left(r^2+\frac{1}{4}\right)^\mu}\right| \tag{2.7}$$

$$\leq \tilde{\kappa}\left[\frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma\left(2\mu-\frac{1}{2}\right)}{2^{2\mu-1}\Gamma^2(\mu+1)}\int_0^{\frac{\pi}{2}}\frac{\cos^{2\mu-1}\phi}{\left[\left(\frac{1}{4}\right)^2+r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}}d\phi - \frac{1}{2\mu^2\left(r^2+\frac{1}{4}\right)^{2\mu}}\right]^{\frac{1}{2}}$$

$$\leq \tilde{\kappa}\left[\frac{1}{2^{3\mu-1}\mu^2(\mu-\frac{1}{2})B(\mu,\mu-\frac{1}{2})} - \frac{1}{2\mu^2\left(r^2+\frac{1}{4}\right)^{2\mu}}\right]^{\frac{1}{2}},$$

where $\tilde{\kappa}$ is as given by

$$\tilde{\kappa} = \left[\frac{\pi^3}{4} - 8\cdot G - 2\cdot\left(\frac{\pi^2}{24}\right)^2\right]^{\frac{1}{2}} = 0.29260623049\ldots. \tag{2.8}$$

Using the generalized Mathieu series, $S_\mu(r)$ as given in (1.1) and (1.6)-(1.7) together with the *alternating* generalized Mathieu series $\tilde{S}_\mu(r)$ as given in (1.11)-(1.12) we introduce the *odd generalized Mathieu series*, $\phi_\mu(r)$ and the *even generalized Mathieu series*, $\psi_\mu(r)$. These are given by [16]

$$\phi_\mu(r) := \frac{S_\mu(r)+\tilde{S}_\mu(r)}{2} = \sum_{n=1}^\infty \frac{2\cdot(2n-1)}{\left((2n-1)^2+r^2\right)^{1+\mu}} \tag{2.9}$$

$$= C_\mu(r)\cdot 2\int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x-e^{-x}}J_{\mu-\frac{1}{2}}(rx)\,dx, \quad r,\mu>0,$$

and

$$\psi_\mu(r) := \frac{S_\mu(r)-\tilde{S}_\mu(r)}{2} = \sum_{n=1}^\infty \frac{2\cdot(2n)}{\left((2n)^2+r^2\right)^{1+\mu}}$$

$$= C_\mu(r)\cdot 2\int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^{2x}-1}J_{\mu-\frac{1}{2}}(rx)\,dx, \qquad r,\mu>0,$$

where $C_\mu(r)$ is positive as defined in (1.7).

Remark 2.1. It may be noticed that if we have *identities* for *any two* of the generalized Mathieu type series $S_\mu(r)$, $\tilde{S}_\mu(r)$, $\phi_\mu(r)$, $\psi_\mu(r)$ then we may deduce the other two. In particular $S_\mu(r) = \frac{\phi_\mu(r) + \psi_\mu(r)}{2}$ and $\tilde{S}_\mu(r) = \frac{\phi_\mu(r) - \psi_\mu(r)}{2}$.This however, is **not** the case with regards to inequalities or bounds since, recourse to the triangle inequality would result in a coarser bound. We may further notice that their integral representation may be given by

$$2C_\mu(r)\int_0^\infty H(x)\cdot x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)\,dx, \qquad r,\mu>0 \tag{2.10}$$

where $C_\mu(r)$ is positive as defined in (1.7)and $H(x)$ is one of the following

$$H_M(x)=\frac{x}{e^x-1},\ H_A(x)=\frac{x}{e^x+1},\ H_O(x)=\frac{x}{e^x-e^{-x}},\ H_E(x)=\frac{x}{e^{2x}-1}, \tag{2.11}$$

where the subscripts relate to the generalized Mathieu , alternating Mathieu, odd Mathieu and even Mathieu series integral representations, respectively.

Remark 2.2. It should be emphasized that the $H_\cdot(\cdot)$ in (2.11) represent the weights associated with the integral representation of the generalized Mathieu and its companions. They satisfy the following conditions

$$\begin{aligned} &H_A(x)<H_E(x)<H_O(x)<H_M(x) \quad ,x<\ln(2)\\ &H_E(x)<H_A(x)<H_O(x)<H_M(x) \quad ,x>\ln(2). \end{aligned} \tag{2.12}$$

In ([16])the odd and even generalized Mathieu series bounds were obtained via a Čebyšev functional approach. If we allow the subscripts of O and E to represent the cases related to $\phi_\mu(r)$ (odd) and $\psi_\mu(r)$ (even).

We note from (2.9) that

$$\frac{\phi_\mu(r)}{2C_\mu(r)}=\int_0^\infty H_O(x)\cdot x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)\,dx, \qquad r,\mu>0 \tag{2.13}$$

where from (2.11)

$$H_O(x)=\frac{x}{e^x-e^{-x}}. \tag{2.14}$$

The following theorem is a corection of the result in [16] . The $(\frac{1}{2})^2$ within the integral was $(1)^2$.

Theorem 2.2. *(see [16] for the proof). For $\mu>0$ and $r>0$ the **odd** generalized Mathieu series $\phi_\mu(r)$ satisfies the following relationship, namely,*

$$\left|\phi_\mu(r)-\frac{\pi^2}{4\mu\left(r^2+1\right)^\mu}\right| \tag{2.15}$$

$$\leq \kappa_O\left[\frac{4\Gamma\left(2\mu-\frac{1}{2}\right)}{2^{2\mu-1}\sqrt{\pi}\Gamma^2(\mu+1)}\int_0^{\frac{\pi}{2}}\frac{\cos^{2\mu-1}\phi}{\left[\,(\frac{1}{2})^2+r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}}d\phi-\frac{4}{\mu^2\left(1^2+r^2\right)^{2\mu}}\right]^{\frac{1}{2}}$$

$$\leq \kappa_O \left[\frac{2^{2\mu+3}}{\mu^2(\mu-\frac{1}{2}) \cdot B\left(\mu, \mu-\frac{1}{2}\right)} - \frac{4}{\mu^2\left(1^2+r^2\right)^{2\mu}} \right]^{\frac{1}{2}},$$

where,

$$\kappa_O = \left[\frac{\pi^2}{8}(1-\frac{\pi^2}{8}) + \frac{7}{8}\zeta(3) \right]^{\frac{1}{2}}$$

and $B(x,y)$ is the Euler beta function given by (2.5).

We note from (2.9) that

$$\frac{\psi_\mu(r)}{2C_\mu(r)} = \int_0^\infty H_E(x) \cdot x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\, dx, \qquad r, \mu > 0 \tag{2.16}$$

where from (2.11)

$$H_E(x) = \frac{x}{e^{2x}-1}. \tag{2.17}$$

Theorem 2.3. *(see [16] for the proof). For $\mu > 0$ and $r > 0$ the **even** generalized Mathieu series $\psi_\mu(r)$ satisfies the following relationship*

$$\left| \psi_\mu(r) - \frac{\pi^2}{6\mu\left(r^2+2^2\right)^\mu} \right| \tag{2.18}$$

$$\leq \kappa_E \left[\frac{4\Gamma\left(2\mu-\frac{1}{2}\right)}{2^{2\mu-1}\sqrt{\pi}\Gamma^2(\mu+1)} \int_0^{\frac{\pi}{2}} \frac{\cos^{2\mu-1}\phi}{\left[(1)^2+r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}} d\phi - \frac{8}{\mu^2\left(1^2+r^2\right)^{2\mu}} \right]^{\frac{1}{2}}$$

$$\leq \kappa_E \left[\frac{1}{4^{\mu-1}\mu^2(\mu-\frac{1}{2}) \cdot B\left(\mu, \mu-\frac{1}{2}\right)} - \frac{8}{\mu^2\left(1^2+r^2\right)^{2\mu}} \right]^{\frac{1}{2}},$$

where,

$$\kappa_E = \left[\frac{\pi^2}{24}(1-\frac{\pi^2}{12}) \right]^{\frac{1}{2}} \tag{2.19}$$

and $B(x,y)$ is the Euler beta function given by (2.5).

The remainder of the results in this Section were developed in [16] to complete the interplay between the generators as depicted by the Zeta family and the generalized Mathieu series expressions.

The following lemma demonstrates the relationship for the generalised Mathieu series and its companions.

Lemma 2.3. *(see [16] for the proof) The companion generalised Mathieu series may be expressed in terms of the generalised Mathieu series, namely,*

$$\tilde{S}_{\mu}(r) = S_{\mu}(r) - 4^{-\mu} S_{\mu}\left(\frac{r}{2}\right)$$

$$\phi_{\mu}(r) = S_{\mu}(r) - 2^{-2\mu-1} S_{\mu}\left(\frac{r}{2}\right) \tag{2.20}$$

$$\psi_{\mu}(r) = 2^{-2\mu-1} S_{\mu}\left(\frac{r}{2}\right).$$

Remark 2.3. It is important to emphasize, as mentioned earlier, that obtaining bounds for the companions in terms of those of the generalised Mathieu series would produce inferior bounds *from using* the triangle inequality required for the first two results in (2.20).

Theorem 2.4. *The following relationships holds,*

$$S_{\mu}(r) = \left\{ \begin{array}{c} 2\phi_{\mu}(r) - \tilde{S}_{\mu}(r) \\ 2\psi(r) + \tilde{S}_{\mu}(r) \end{array} \right\}. \tag{2.21}$$

Proof. The first relationship (2.21) follows easily from (2.20) by subtracting the first equation from twice the second. The second result is obtained by noting that $2\psi_{\mu}(r) = 4^{-\mu} S_{\mu}\left(\frac{r}{2}\right)$ in the third equation and substitution in the first of (2.20).

The first equation in (2.21) recaptures, on allowing $r- > 0$, the well known result involving the Zeta function, $\zeta(x)$

$$\zeta(x) = 2\lambda(x) - \eta(x) \tag{2.22}$$

where $\lambda(x)$ is the odd zeta, $\eta(x)$ is the alternating zeta ,and $x = 2\mu + 1$. This demonstrates that (2.21) is an extention of the Zeta expression (2.22) through the variable r of Mathieu type functions. A similar resoning gives $\zeta(x) = 2\psi(x) + \eta(x)$. $\square$

2.2. Dirichlet Beta and L-Function Generalized Mathieu Series Bounds

The Dirichlet beta function or Dirichlet L-function is given by [19]

$$\beta(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^x}, \quad x > 0 \tag{2.23}$$

where $\beta(2) = G$, Catalan's constant.See [8] and [9] in which sharp double bounds were obtained.

It is readily observed that $\beta(x)$ is the alternating version of $\lambda(x)$, however, it cannot be directly related to $\zeta(x)$. It is also related to $\eta(x)$ in that only the odd terms are summed.

The beta function may be evaluated explicitly at positive odd integer values of x, namely,

$$\beta(2n+1) = (-1)^n \frac{E_{2n}}{2(2n)!}\left(\frac{\pi}{2}\right)^{2n+1}, \tag{2.24}$$

where E_n are the Euler numbers generated by

$$\operatorname{sech}(x) = \frac{2e^x}{e^{2x}+1} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

The Dirichlet beta function may be analytically continued over the whole complex plane by the functional equation

$$\beta(1-z) = \left(\frac{2}{\pi}\right)^z \sin\left(\frac{\pi z}{2}\right)\Gamma(z)\,\beta(z).$$

The function $\beta(z)$ is defined everywhere in the complex plane and has no singularities, unlike the Riemann zeta function, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, which has a simple pole at $s = 1$.

The Dirichlet beta function and the zeta function have important applications in a number of branches of mathematics, and in particular in Analytic number theory. See for example [3], [17].

Further, $\beta(x)$ has an alternative integral representation [19, p. 56]. Namely,

$$\beta(x) = \frac{1}{2\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{\cosh(t)} dt, \quad x > 0.$$

That is,

$$\beta(x) = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t + e^{-t}} dt, \quad x > 0. \tag{2.25}$$

The function $\beta(x)$ is also connected to prime number theory [19] which may perhaps be best summarised by

$$\beta(x) = \prod_{\substack{p \text{ prime} \\ p \equiv 1 \bmod 4}} \left(1 - p^{-x}\right)^{-1} \cdot \prod_{\substack{p \text{ prime} \\ p \equiv 3 \bmod 4}} \left(1 + p^{-x}\right)^{-1} = \prod_{\substack{p \text{ odd} \\ \text{prime}}} \left(1 - (-1)^{\frac{p-1}{2}} p^{-x}\right)^{-1},$$

where the rearrangement of factors is permitted because of absolute convergence.

The main thrust of the article is to investigate the Dirichlet Beta function via generalised Mathieu series approach, which may be looked upon as an alternating odd generalized Mathieu series, $\tilde{\phi}_\mu(r)$, namely

$$\tilde{\phi}_\mu(r) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 2(2n-1)}{\left((2n-1)^2 + r^2\right)^{1+\mu}} \tag{2.26}$$

$$= C_{\mu}(r) \cdot 2 \int_0^{\infty} \frac{x^{\mu+\frac{1}{2}}}{e^x + e^{-x}} J_{\mu-\frac{1}{2}}(rx)\, dx, \quad r, \mu > 0.$$

This is, in part, inspired by the alternating odd zeta function, $\beta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s}$ which has explicit closed form solution in terms of Euler polynomials for $s = 2m+1$ whereas $\zeta(2m)$ for $m \in \mathbb{N}$, is explicitly given in terms of Bernoulli polynomials.This is so, since using a limiting argument $\tilde{\phi}_{\mu}(0) = 2\beta(2\mu+1)$.

Theorem 2.5. *(See [11] for the proof). For $\mu > 0$ and $r > 0$ the alternating odd generalized Mathieu series $\tilde{\phi}_{\mu}(r)$ satisfies the following relationship, namely,*

$$\left| \tilde{\phi}_{\mu}(r) - \frac{2 \cdot G}{\mu\,(r^2+1)^{\mu}} \right| \tag{2.27}$$

$$\leq \kappa_{\tilde{O}} \left[\frac{4\Gamma\left(2\mu - \frac{1}{2}\right)}{2^{2\mu-1}\sqrt{\pi}\,\Gamma^2(\mu+1)} \int_0^{\frac{\pi}{2}} \frac{\cos^{2\mu-1}\phi}{\left[\left(\frac{1}{2}\right)^2 + r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}} d\phi - \frac{4}{\mu^2\,(1^2+r^2)^{2\mu}} \right]^{\frac{1}{2}}$$

$$\leq \kappa_{\tilde{O}} \left[\frac{2^{2\mu+3}}{\mu^2(\mu-\frac{1}{2}) \cdot B\left(\mu, \mu-\frac{1}{2}\right)} - \frac{4}{\mu^2\,(1^2+r^2)^{2\mu}} \right]^{\frac{1}{2}},$$

where,

$$\kappa_{\tilde{O}} = \left[G(1-G) + \frac{\pi^3}{32} \right]^{\frac{1}{2}} \tag{2.28}$$

and $B(x,y)$ is the Euler beta function given by (2.5)

The previous work investigating the generalised Mathieu series was extended to the alternating, odd, and even generalised Mathieu series.The odd $(\phi_{\mu}(r))$ and alternating $(\tilde{\phi}_{\mu}(r))$ generalised Mathieu series will be used to obtain other results conserning $L_{(4,1)}(\cdot)$ and $L_{(4,3)}(\cdot)$ as generators.

Let,

$$\Phi_{\mu}^{+}(r) := \frac{\phi_{\mu}(r) + \tilde{\phi}_{\mu}(r)}{2} = \sum_{n=0}^{\infty} \frac{2 \cdot (4n+1)}{\left((4n+1)^2 + r^2\right)^{1+\mu}} \tag{2.29}$$

$$= C_{\mu}(r) \cdot 2 \int_0^{\infty} \frac{xe^x}{e^{2x} - e^{-2x}} x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\, dx, \quad r, \mu > 0,$$

and,

$$\Phi_{\mu}^{-}(r) := \frac{\phi_{\mu}(r) - \tilde{\phi}_{\mu}(r)}{2} = \sum_{n=0}^{\infty} \frac{2 \cdot (4n+3)}{\left((4n+3)^2 + r^2\right)^{1+\mu}} \tag{2.30}$$

$$= C_\mu(r) \cdot 2 \int_0^\infty \frac{xe^{-x}}{e^{2x} - e^{-2x}} x^{\mu - \frac{1}{2}} J_{\mu - \frac{1}{2}}(rx)\, dx, \quad r, \mu > 0,$$

where,

$$C_\mu(r) = \frac{\sqrt{\pi}}{(2r)^{\mu - \frac{1}{2}} \Gamma(\mu + 1)} \cdot \tag{2.31}$$

Theorem 2.6. *(See [11] for the proof). For $\mu > 0$ and $r > 0$ the alternating odd generalized Mathieu series $\tilde{\phi}_\mu(r)$ satisfies the following relationship, namely,*

$$\left| \frac{\Phi_\mu^+(r)}{2C_\mu(r)} - \frac{(2r)^{\mu - \frac{1}{2}} \Gamma(\mu)}{\sqrt{\pi} (1^2 + r^2)^\mu} \cdot L_{(4,1)}(2) \right| \tag{2.32}$$

$$\leq \kappa_{\phi^+} \left[\frac{4\Gamma\left(2\mu - \frac{1}{2}\right)}{2^{2\mu - 1} \sqrt{\pi} \Gamma^2(\mu + 1)} \int_0^{\frac{\pi}{2}} \frac{\cos^{2\mu - 1} \phi}{\left[\left(\frac{1}{2}\right)^2 + r^2 \cos^2 \phi\right]^{2\mu - \frac{1}{2}}} d\phi - \frac{4}{\mu^2 (1^2 + r^2)^{2\mu}} \right]^{\frac{1}{2}}$$

$$\leq \kappa_{\phi^+} \left[\frac{2^{2\mu + 3}}{\mu^2 (\mu - \frac{1}{2}) \cdot B\left(\mu, \mu - \frac{1}{2}\right)} - \frac{4}{\mu^2 (1^2 + r^2)^{2\mu}} \right]^{\frac{1}{2}},$$

where,

$$\kappa_{\phi^+} = \left[L_{(4,1)}(2) \left(\frac{1}{2} - L_{(4,1)}(2)\right) + \frac{3}{2} L_{(4,1)}(3) \right]^{\frac{1}{2}} \tag{2.33}$$

and $B(x, y)$ is the Euler beta function given by (2.5).

Theorem 2.7. *(See [11] for the proof). For $\mu > 0$ and $r > 0$ the alternating odd generalized Mathieu series $\tilde{\phi}_\mu(r)$ satisfies the following relationship, namely,*

$$\left| \Phi_\mu^-(r) - \frac{(2r)^{\mu - \frac{1}{2}} \Gamma(\mu)}{\sqrt{\pi} (3^2 + r^2)^\mu} \cdot L_{(4,3)}(2) \right| \tag{2.34}$$

$$\leq \kappa_{\phi^-} \left[\frac{4\Gamma\left(2\mu - \frac{1}{2}\right)}{2^{2\mu - 1} \sqrt{\pi} \Gamma^2(\mu + 1)} \int_0^{\frac{\pi}{2}} \frac{\cos^{2\mu - 1} \phi}{\left[\left(\frac{3}{2}\right)^2 + r^2 \cos^2 \phi\right]^{2\mu - \frac{1}{2}}} d\phi - \frac{1}{3} \frac{(2r)^{\mu - \frac{1}{2}} \Gamma(\mu)}{\sqrt{\pi} (3^2 + r^2)^\mu} \right]^{\frac{1}{2}}$$

$$\leq \kappa_{\phi^-} \left[\frac{2^{2\mu + 3}}{\mu^2 (\mu - \frac{1}{2}) \cdot B\left(\mu, \mu - \frac{1}{2}\right)} - \frac{1}{3} \frac{(2r)^{\mu - \frac{1}{2}} \Gamma(\mu)}{\sqrt{\pi} (3^2 + r^2)^\mu} \right]^{\frac{1}{2}},$$

where,

$$\kappa_{\phi^-} = \left[\frac{1}{2} \left(L_{(4,3)}(2) + L_{(4,3)}(3) \right) - \frac{1}{3} \left(L_{(4,3)}(2) \right)^2 \right]^{\frac{1}{2}} \tag{2.35}$$

and $B(x, y)$ is the Euler beta function given by (2.5).

We may notice that the integral representation for the Dirichlet beta function family, may be represented by

$$2C_{\mu}(r)\int_0^{\infty} H(x)\cdot x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)\,dx, \qquad r,\mu>0 \tag{2.36}$$

where $C_{\mu}(r)$ is positive as defined in (1.7) and $H(x)$ is one of the following kernels (2.37)

$$H_{\phi}(x)=\frac{x}{e^{x}-e^{-x}},H_{\tilde{\phi}}(x)=\frac{x}{e^{x}+e^{-x}},H_{L_{(4,1)}}(x)=\frac{xe^{x}}{e^{2x}-e^{-2x}},H_{L_{(4,3)}}(x)=\frac{xe^{-x}}{e^{2x}-e^{-2x}}.$$

The subscripts relate to the generalized Mathieu beta family; odd, alternating, $L_{(4,1)}(x)$ and $L_{(4,3)}(x)$ Mathieu series integral representations, respectively.

Remark 2.4. It should be emphasized that the $H_{\cdot}(\cdot)$ in (2.36) represent the weights or kernel associated with the integral representation of the generalized Mathieu beta family and its companions. They satisfy the following conditions

$$\begin{aligned} &H_{L_{(4,3)}}(x)<H_{\tilde{\phi}}(x)<H_{L_{(4,1)}}(x)<H_{\phi}(x) \quad ,x>\frac{\ln(2)}{2} \\ &H_{\tilde{\phi}}(x)<H_{L_{(4,3)}}(x)<H_{L_{(4,1)}}(x)<H_{\phi}(x) \quad ,x<\frac{\ln(2)}{2}. \end{aligned} \tag{2.38}$$

The following theorem demonstrates the relationship for the generalized Mathieu series related to the Beta L-function family.This can be compared with the Zeta function family results were discussed in the previous subsection.

Theorem 2.8. *The following relationships hold,. Namely,*

$$\phi_{\mu}(r)=\left\{\begin{array}{c} 2\cdot\Phi_{\mu}^{+}(r)-\tilde{\phi}_{\mu}(r) \\ 2\cdot\Phi_{\mu}^{-}(r)+\tilde{\phi}_{\mu}(r) \end{array}\right\} \tag{2.39}$$

where $\phi_{\mu}(r)$ is defined in (2.9), $\tilde{\phi}_{\mu}(r)$ is given in (2.26) and, $\Phi_{\mu}^{+}(r)$ and $\Phi_{\mu}^{-}(r)$ are defined in (2.29) and (2.30) respectively.These entities represent the generalized Mathieu series propogated by series of reciprocal powers of odd numbers, alternating odd numbers,$L_{(4,1)}(\cdot)$ and $L_{(4,3)}(\cdot)$.

Proof. This is trivial since $\Phi_{\mu}^{+}(r) := \frac{\phi_{\mu}(r)+\tilde{\phi}_{\mu}(r)}{2}$ and $\Phi_{\mu}^{-}(r) := \frac{\phi_{\mu}(r)-\tilde{\phi}_{\mu}(r)}{2}$ are defined (2.29) and (2.30). □

Remark 2.5. If r is allowed to tend to zero for the first result at (2.39),namely, $\phi_{\mu}(r)=2\cdot\Phi_{\mu}^{+}(r)-\tilde{\phi}_{\mu}(r)$ then the relationship $\lambda(x)=2L_{(4,1)}(x)-\beta(x)$ where $x=2\mu+1$ results. Further, from the second result, $\phi_{\mu}(r)=2\cdot\Phi_{\mu}^{-}(r)+\tilde{\phi}_{\mu}(r)$ then the relationship $\lambda(x)=2L_{(4,3)}(x)+\beta(x)$ where $x=2\mu+1$ results.The similar process relating the Zeta function $\zeta(x)$ produced $\zeta(x)=2\lambda(x)-\eta(x)$,where $\lambda(x)$ is the odd zeta, $\eta(x)$ is the alternating zeta ,and $x=2\mu+1$ at (2.22).

3. Hardy upper bounds for Zeta and Beta families and double bounds for Eta and Dirichlet Beta

The current section develops a result by Hardy which enables a sharp upper bound of a well known function. The theorem is reproduced here since the result does not seem to have been utilised in the literature.The Zeta and Beta families which have been developed by the author, sharp lower and upper bounds as demonstrated to date. Further, upper bounds for some five of the Zeta and Beta families, bounds of Hardy -type have been developed within the remainder of this article.

3.1. An upper bound of Hardy for $\frac{1}{1-e^{-x}}$.

Some initial background is presented prior to the theorem of Hardy, below.

A theorem of Grüss [21] states that if $a > 0$, $F_1 \le f \le F_2$ and $G_1 \le g \le G_2$ where,

$$D(f,g) = \frac{1}{a}\int_0^a f(x)g(x)dx - \frac{1}{a^2}\int_0^a f(x)dx \cdot \int_0^a g(x)dx, \tag{3.1}$$

then,

$$|D(f,g)| \le \frac{1}{4}(F_2 - F_1)(G_2 - G_1) \tag{3.2}$$

for any bounded and integrable f, g.

Definition 3.1. f is a total-monotone in $(0, a)$ if

$$f \ge 0,\ f' \le 0,\ \ f'' \ge 0,\ \ f''' \ge 0.... \tag{3.3}$$

in $(0, a)$.

Definition 3.2. If f is a total-monotone in $(0, \infty)$ then a well known theorem of Bernstein

$$f(x) = \int_0^\infty e^{-xt} d\chi(t) \tag{3.4}$$

where $d\chi \ge 0$,for all positive x.

Theorem 3.1. *(Hardy [24]). If f and g are total-monoton in* $(0, \infty)$ *,then*

$$D(f,g) \le \frac{1}{12}(F_2 - F_1)(G_2 - G_1) \tag{3.5}$$

for every a. The constant $\frac{1}{12}$ is best possible.

Proof. If $F = \frac{1}{a}\int_0^a f(x)dx,\ G = \frac{1}{a}\int_0^a g(x)dx$ then (3.1) may be represented as that due to Sönin

$$D(f,g) = \frac{1}{a}\int_0^a (f(x)-F)\cdot(g(x)-G)dx \tag{3.6}$$

and so

$$D^2(f,g) \le \frac{1}{a}\int_0^a (f(x)-F)^2dx \cdot \frac{1}{a}\int_0^a (g(x)-G)^2dx = D(f,f)\cdot D(g,g) \tag{3.7}$$

and so it is enough to prove (3.5) when $f = g$ since $D(f,f) \ge 0$.

Hence from (3.4)

$$\frac{1}{a}\int_0^a f^{\,2}(x)dx = \frac{1}{a}\int_0^a \int_0^\infty e^{-xt}d\chi(t)\int_0^\infty e^{-xu}d\chi(u)dx \tag{3.8}$$

$$= \frac{1}{a}\int_0^\infty \int_0^\infty \frac{1-e^{-a(t+u)}}{t+u}d\chi(t)d\chi(u)$$

and further

$$\frac{1}{a}\int_0^a f\ (x)dx = \frac{1}{a}\int_0^\infty \frac{1-e^{-at}}{t}d\chi(t), \tag{3.9}$$

$$F_2 - F_1 = f(0) - f(a) = \int_0^\infty (1-e^{-at})d\chi(t)$$

Thus from (3.8) and (3.9) gives

$$D(f,f) = \int_0^\infty \int_0^\infty H(t,u)d\chi(t)d\chi(u),\ (F^2 - F^1) = \int_0^\infty \int_0^\infty K(t,u)d\chi(t)d\chi(u)$$

where

$$H(t,u) = \frac{1-e^{-a(t+u)}}{a(t+u)} - \frac{(1-e^{-at})(1-e^{-au})}{a^2tu},\ K(t,u) = (1-e^{-at})(1-e^{-au}),$$

and it is enough to prove that $H(t,u) \le \frac{1}{12}K(t,u)$,for all t,u.

It is sufficient to prove that

$$\frac{1-e^{-(x+y)}}{(x+y)(1-e^{-x})(1-e^{-y})} - \frac{1}{xy} \le \frac{1}{12}, \qquad x>0, y>0 \tag{3.10}$$

where $x = at$ and $y = au$.

Equation (3.10) may be expressed as

$$\frac{1}{1-e^{-x}} + \frac{1}{1-e^{-y}} - 1 \le \frac{1}{x} + \frac{1}{y} + \frac{x+y}{12}$$

or that

$$\varphi(x) + \varphi(y) \le 1 \tag{3.11}$$

where

$$\varphi(x) = \frac{1}{1 - e^{-x}} - \frac{1}{x} - \frac{x}{12} \tag{3.12}$$

and so $\varphi(x) \le \frac{1}{2}$. $\square$

Corollary 3.1. *The above Theorem of Hardy's effectively results in the following upper bound,*

$$\frac{1}{1-e^{-x}} \le \frac{1}{x} + \frac{1}{2} + \frac{x}{12} \ \ or \frac{1}{e^x - 1} \le e^{-x}(\frac{1}{x} + \frac{1}{2} + \frac{x}{12}), x > 0, \ \ and \tag{3.13}$$

$$\frac{x}{1-e^{-x}} \le 1 + \frac{x}{2} + \frac{x^2}{12} \ \ or \frac{x}{e^x - 1} \le e^{-x}(1 + \frac{x}{2} + \frac{x^2}{12}), x > 0.$$

3.2. An Identity and Bounds Involving the Eta and Related Functions

We note that there are functions that are closely related to the Zeta function ,$\zeta(x)$. Namely, the Dirichlet function eta, $\eta(x)$ and lamda, $\lambda(x)$ given by

$$\eta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{n^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t + 1} dt, x > 0, x \ne 1 \tag{3.14}$$

and

$$\lambda(x) = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t - e^{-t}} dt, x > 1, \tag{3.15}$$

These are related to $\zeta(x)$ by,

$$\eta(x) = (1 - 2^{1-x}) \cdot \zeta(x) \ \ \text{and} \ \ \lambda(x) = (1 - 2^{-x}) \cdot \zeta(x) \tag{3.16}$$

satisfying the well known identtity

$$\zeta(x) + \eta(x) = 2\lambda(x). \tag{3.17}$$

It is also obious from (3.16) that eliminating $\zeta(x)$ produces

$$\eta(x) = (1 - b(x))\lambda(x). \tag{3.18}$$

where

$$b(x) = \frac{1}{2^x - 1}. \tag{3.19}$$

The following lemma was developed in Cerone [9] to obtain sharp bounds for the eta function, $\eta(x)$ as given in Theorem 3.2.

Lemma 3.1. *The following identity for the eta function holds. Namely*

$$Q(x) := \frac{1}{\Gamma(x+1)} \int_0^\infty \frac{t^x}{(e^t+1)^2} dt = \eta(x+1) - \eta(x), \quad x > 0. \tag{3.20}$$

The following theorem presents sharp bounds for the secant slope $\eta(x)$ for a distance of one apart.

Theorem 3.2. *(Proven in [9]) For real numbers $x > 0$, we have*

$$\frac{c_\eta}{2^{x+1}} < \eta(x+1) - \eta(x) < \frac{d_\eta}{2^{x+1}} \tag{3.21}$$

with the best possible constants

$$c_\eta = 2\ln 2 - 1 = 0.3862943\ldots \quad \textit{and} \quad d_\eta = 1. \tag{3.22}$$

Given the sharp inequalities for $\eta(x+1) - \eta(x)$ in (3.21) – (3.22), then we may readily obtain sharp bounds for expressions involving the zeta function and the lambda function at a distance of one apart.

Corollary 3.2. *(Proven in [9]) For real numbers $x > 0$ we have*

$$\left(\ln 2 - \frac{1}{2}\right) b(x) < \zeta(x+1) - (1 - b(x))\zeta(x) < \frac{b(x)}{2}, \tag{3.23}$$

where $b(x)$ is as given by (3.19).

Proof. From Theorem 3.2 and (3.16) giving a relationship between $\eta(x)$ and $\zeta(x)$ we have

$$\eta(x+1) - \eta(x) = \left(1 - 2^{-x}\right)\zeta(x+1) - \left(1 - 2^{1-x}\right)\zeta(x)$$

and so from (3.21) and (3.22)

$$\frac{c_\eta}{2} \cdot b(x) < \zeta(x+1) - (1 - b(x))\zeta(x) < \frac{d_\eta}{2} \cdot b(x).$$

□

Remark 3.1. Cerone et al. [12] obtained the upper bound in (3.23) and a coarser lower bound of $\frac{b(x)}{8}$. Alzer [2] demonstrated that the constants $\ln 2 - \frac{1}{2}$ and $\frac{1}{2}$ in (3.23) are sharp.The Corollary 3.2 is developed via the eta function presented in Theorem 3.2, which is somewhat easier to procure.

Corollary 3.3. *(Proven in [9]) For real $x > 0$ we have*

$$\left(\ln 2 - \frac{1}{2}\right) b(x)\left(1 - 2^{-(x+1)}\right) < \lambda(x+1) - \left(\frac{1 - b(x)}{1 - b(x+1)}\right)\lambda(x) \tag{3.24}$$

$$< \frac{b(x)}{2} \cdot \left(1 - 2^{-(x+1)}\right),$$

where $b(x)$ *is from (3.19) and from (3.16) and (3.18) we have,*

$$\eta(x) = (1 - b(x))\lambda(x) \tag{3.25}$$

and so from (3.16) and (3.19)

$$\frac{2\ln 2 - 1}{2^{x+1}} < \eta(x+1) - \eta(x)$$

$$= (1 - b(x+1))\lambda(x+1) - (1 - b(x))\lambda(x) < \frac{1}{2^{x+1}}.$$

Division by $1 - b(x+1)$ *and some simplification readily produces (3.24).*

The advantage of having sharp inequalities such as (3.21), (3.23) and (3.24) involving function values at a distance of one apart is that if we place $x = 2n$, then since $\zeta(2n)$ is known explicitly, we may approximate $\zeta(2n+1)$ and provide explicit bounds. This is so for $\eta(\cdot)$ and $\lambda(\cdot)$ as well because of their relationship to $\zeta(\cdot)$ via (3.16) – (3.17).

In what follows, we investigate some numerical results associated with bounding the unknown $\zeta(2n+1)$ by expressions involving the explicitly known $\zeta(2n)$. The following corollaries hold.

Corollary 3.4. *The bound*

$$\left|\zeta(x+1) - (1 - b(x))\zeta(x) - \frac{\ln 2}{2} b(x)\right| \leq \frac{1 - \ln 2}{2} b(x), \quad x > 0 \tag{3.26}$$

holds, where $b(x)$ *is as given by (1.6).*

3.3. An Identity and Bounds Involving the Beta and Related Functions

We note that there are functions that are closely related to the Beta function ,$\beta(x)$. Namely, the Dirichlet functions $L_{(4,1)}(\cdot)$ and $L_{(4,3)}(\cdot)$ and, lamda, $\lambda(x)$ *are* given by

$$\lambda(x) = \sum_{n=1}^{\infty} \frac{1}{(2n+1)^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t - e^{-t}} dt, \quad x > 1, \tag{3.27}$$

$$\beta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{n^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t + e^{-t}} dt, \quad x > 0, x \neq 1 \tag{3.28}$$

and

$$L_{(4,1)}(x)=\sum_{n=1}^{\infty}\frac{1}{(4n+1)^x}=\frac{2}{\Gamma(x)}\int_0^{\infty}\frac{te^t}{e^{2t}-e^{-2t}}\cdot t^{x-1}dt,\quad x>0, \tag{3.29}$$

$$L_{(4,3)}(x)=\sum_{n=1}^{\infty}\frac{1}{(4n+3)^x}=\frac{2}{\Gamma(x)}\int_0^{\infty}\frac{te^{-t}}{e^{2t}-e^{-2t}}\cdot t^{x-1}dt,\quad x>0.$$

These are related to $\lambda(x)$ by,

$$\beta(x)=(1-2^{1-x})\cdot\lambda(x),\ L_{(4,1)}(x)=(1-2^{-x})\cdot\lambda(x)\text{ and }\ L_{(4,3)}(x)=2^{-x}\cdot\lambda(x) \tag{3.30}$$

satisfying the identities

$$\lambda(x)=2L_{(4,1)}(x)-\beta(x)\text{ and }\lambda(x)=2L_{(4,3)}(x)+\beta(x). \tag{3.31}$$

It is also obvious from (3.31) that eliminating $\lambda(x)$ produces

$$\beta(x)=(1-b(x))\cdot L_{(4,1)}(x),\text{ and }\beta(x)=\frac{2}{b(x)}\cdot L_{(4,3)}(x) \tag{3.32}$$

where,

$$b(x)=\frac{1}{2^x-1}. \tag{3.33}$$

The following lemma plays a significant role in obtaining bounds for the Dirichlet beta function, $\beta(x)$ as shown in the theorem below.

Lemma 3.2. *(See [8] for the proof) The following identity for the Dirichlet beta function holds. Namely,*

$$P(x):=\frac{2}{\Gamma(x+1)}\int_0^{\infty}\frac{e^{-t}}{(e^t+e^{-t})^2}\cdot t^x dt=\beta(x+1)-\beta(x). \tag{3.34}$$

The following theorem produces sharp bounds for the secant slope of $\beta(x)$.

Theorem 3.3. *(See [8] for the proof).For real numbers $x>0$, we have*

$$\frac{c_\beta}{3^{x+1}}<\beta(x+1)-\beta(x)<\frac{d_\beta}{3^{x+1}}, \tag{3.35}$$

with the best possible constants

$$c_\beta=3\left(\frac{\pi}{4}-\frac{1}{2}\right)=0.85619449\ldots\quad\textit{and}\quad d_\beta=2. \tag{3.36}$$

Given the sharp inequalities for $\beta(x+1)-\beta(x)$ in (3.35) – (3.36), then we may readily obtain sharp bounds for expressions involving the lambda function and the Dirichlet functions, $L_{(4,1)}(\cdot)$ and $L_{(4,3)}(\cdot)$ at a distance of one apart.

Corollary 3.5. *For real numbers $x > 0$ we have*

$$3\left(\frac{\pi}{2}-1\right)(\frac{2}{3})^{x+1}b\left(x\right) < \lambda\left(x+1\right)-\left(1-b\left(x\right)\right)\lambda\left(x\right) < (\frac{2}{3})^{x+1}b\left(x\right), \tag{3.37}$$

where $b(x)$ is as given by (3.33).

Proof. From Theorem 3.3 and (3.30) giving a relationship between $\beta(x)$ and $\lambda(x)$ we have

$$\beta\left(x+1\right)-\beta\left(x\right)=\left(1-2^{-x}\right)\lambda\left(x+1\right)-\left(1-2^{1-x}\right)\lambda\left(x\right)$$

and so from (3.35) and (3.36) produces

$$\frac{c_{\beta}}{3^{x+1}}\cdot\left(1+b\left(x\right)\right) < \lambda\left(x+1\right)-\left(1-b\left(x\right)\right)\lambda\left(x\right) < \frac{d_{\beta}}{3^{x+1}}\cdot\left(1+b\left(x\right)\right).$$

from which (3.37) is obtained after some simplification. $\square$

Corollary 3.6. *For real numbers $x > 0$ we have*

$$\frac{3}{4}\left(\frac{\pi}{2}-1\right)(\frac{2}{3})^{x+1}(1-2^{-(x+1)})b\left(x\right) < L_{(4,1)}\left(x+1\right)-\frac{1-b(x)}{1-b(x+1)}\cdot L_{(4,1)}(x) \tag{3.38}$$

$$< (\frac{2}{3})^{x+1}(1-2^{-(x+1)})b\left(x\right),$$

where $b(x)$ is as given by (3.33).

Proof. From Theorem 3.3 and (3.32) giving a relationship between $\beta(x)$ and $L_{(4,1)}(x)$ we have

$$\beta\left(x+1\right)-\beta\left(x\right)=\left(1-b(x+1)\right)\cdot L_{(4,1)}(x+1)-\frac{1-b(x)}{1-b(x+1)}\cdot L_{(4,3)}(x)$$

and so from (3.35) and (3.36) produces

$$\frac{c_{\beta}}{3^{x+1}(1-b(x+1))} < L_{(4,1)}(x+1)-\frac{1-b(x)}{1-b(x+1)}\cdot L_{(4,1)}(x) < \frac{d_{\beta}}{3^{x+1}(1-b(x+1))}$$

from which (3.38) is obtained after some simplification. $\square$

Corollary 3.7. *For real numbers $x > 0$ we have*

$$\frac{3}{4}\left(\frac{\pi}{2}-1\right)\cdot\frac{b\left(x\right)}{3^{x+1}} < L_{(4,3)}\left(x+1\right)-\frac{1-b(x)}{1-b(x+1)}\cdot L_{(4,3)}(x) < \frac{b\left(x\right)}{3^{x+1}}, \tag{3.39}$$

where $b(x)$ is as given by (3.33).

Proof. From Theorem 3.3 and (3.32) giving a relationship between $\beta(x)$ and $L_{(4,3)}(x)$ we have

$$\beta(x+1) - \beta(x) = \frac{2}{b(x)} \cdot L_{(4,3)}(x+1) - \frac{2}{b(x-1)} \cdot L_{(4,3)}(x)$$

and so from (3.35) and (3.36) produces

$$\frac{c_\beta}{3^{x+1}} \cdot \frac{b(x)}{2} < L_{(4,3)}(x+1) - \frac{b(x)}{b(x-1)} \cdot L_{(4,3)}(x) < \frac{d_\beta}{3^{x+1}} \cdot \frac{b(x)}{2}.$$

from which is obtained after some simplification. □

Corollary 3.8. *From (3.35)*

$$\left|\beta(x+1) - \beta(x) - \frac{d_\beta + c_\beta}{2 \cdot 3^{x+1}}\right| \leq \frac{d_\beta - c_\beta}{2 \cdot 3^{x+1}}, \quad x > 0 \tag{3.40}$$

the bound above holds where the lower and upper bound c_β and d_β respectively are given as in (3.36).

3.4. Application of Hardy upper Bounds for the Zeta and Beta Families

Consider the integral version of the Zeta and Beta families as

$$F(x) = \frac{1}{\Gamma(x)} \int_0^\infty W(t) \cdot t^{x-1} dt \tag{3.41}$$

where the weight $W(t)$ have subscripts to represent different Zeta and Beta families respectively, as shown below

$$W_\zeta(t) = \frac{1}{e^t - 1}, W_\eta(t) = \frac{1}{e^t + 1}, W_\lambda(t) = \frac{1}{e^t - e^{-t}}, W_E(t) = \frac{1}{e^{2t} - 1}, \tag{3.42}$$

$$W_\beta(t) = \frac{1}{e^t + e^{-t}}, W_{L_{(4,1)}}(t) = \frac{e^t}{e^t - e^{-t}}, W_{L_{(4,3)}}(t) = \frac{e^{-t}}{e^t - e^{-t}}.$$

However, the Hardy result cannot be used for the η and β subscripted cases.

Lemma 3.3. *The Hardy upper bounds for the appropriate bounds for (3.42) are given by*

$$W_\zeta(t) \leq B_\zeta(t) = e^{-t}\left(\frac{1}{t} + \frac{1}{2} + \frac{t}{12}\right) \tag{3.43}$$

and so, from (3.42) and (3.43) gives,

$$W_\lambda(t) \le \frac{e^t}{2} B_\zeta(2t) = B_\lambda(t) = \frac{e^{-t}}{2}(\frac{1}{t} + 1 + \frac{t}{3}), \tag{3.44}$$

$$W_E(t) \le \frac{e^t}{2} B_\zeta(2t) = B_E(t) = \frac{e^{-2t}}{2}(\frac{1}{t} + 1 + \frac{t}{3}),$$

$$W_{L_{(4,1)}}(t) \le \frac{e^{3t}}{4} B_\zeta(4t) = B_{L_{(4,1)}}(t) = \frac{e^{-t}}{4}(\frac{1}{t} + 2 + \frac{4}{3}t) \quad \textit{and,}$$

$$W_{L_{(4,3)}}(t) \le \frac{e^t}{4} B_\zeta(4t) = B_{L_{(4,3)}}(t) = \frac{e^{-3t}}{4}(\frac{1}{t} + 2 + \frac{4}{3}t).$$

Theorem 3.4. *The Hardy upper bounds derived from the above Lemma for the weights, provide the bounds for the respective Zeta and Beta families. Namely,*

$$\zeta(x) < \frac{1}{x} + \frac{1}{2} + \frac{x}{12}, \quad x > 1, \tag{3.45}$$

$$\lambda(x) < \frac{1}{2}(\frac{1}{x} + 1 + \frac{x}{3}), \quad x > 1,$$

$$E(x) < 2^{-x}(\frac{1}{x} + \frac{1}{2} + \frac{x}{12}), \quad x > 0,$$

$$L_{(4,1)}(x) < \frac{1}{4}(\frac{1}{x} + 2 + \frac{4}{3}x), \quad x > 0,$$

and

$$L_{(4,3)}(x) < \frac{3^{-x}}{4}(\frac{3}{x} + 2 + \frac{4}{9}x), \quad x > 0$$

where $E(x)$ is the even Zeta function given by $E(x) = \sum_{n=1}^{\infty} \frac{1}{(2n)^x} = 2^{-x}\zeta(x)$.

Proof. From (3.41) the zeta function, is defined as

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty W_\zeta(t) \cdot t^{x-1} dt$$

where (3.42) the weight function is $W_\zeta(t) = \frac{1}{e^t - 1}$ is bounded by $B_\zeta(t)$ (3.43) to give

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{1}{e^t - 1} \cdot t^{x-1} dt$$

$$< \frac{1}{\Gamma(x)} \int_0^\infty e^{-t}(\frac{1}{t} + \frac{1}{2} + \frac{t}{12}) \cdot t^{x-1} dt.$$

Now it is a well known fact that $\int_0^\infty e^{-pt} t^\alpha dt = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$ resulting in

$$\zeta(x) < \frac{1}{\Gamma(x)} \left\{ \frac{\Gamma(x-1)}{1^{x-1}} + \frac{\Gamma(x)}{2 \cdot 1^x} + \frac{\Gamma(x+1)}{12 \cdot 1^{x+1}} \right\}.$$

So upon using the result that $\Gamma(x+1) = x\Gamma(x)$ and simplifying, produces the first result in (3.45).

The rest of the results in (3.45) are obtained similarly. □

4. Application of Hardy-type Upper Bounds for Generalized Mathieu Series

The subsection upper bounds for Zeta and Beta families were obtained utilising Hardy [24] result as shown in subsection 3.1. In this Section it is intended to procure upper bounds for Generalzed Mathieu Series for the Zeta and Beta families except for alternating series relating to $H_A(x)$ and $H_{\tilde{\phi}}(x)$.

Let

$$G_\mu(r;H) = 2 \cdot C_\mu(r) \int_0^\infty H(x) \cdot x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\, dx, \qquad r,\mu > 0 \tag{4.1}$$

where, $C_\mu(r)$ is given in (1.7),and here,for convenience,

$$C_\mu(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\,\Gamma(\mu+1)}$$

and $H(\cdot)$ the generators of the generalized Mathieu Series,

$$H_M(x) = \frac{x}{e^x-1},\ H_A(x) = \frac{x}{e^x+1},\ H_O(x) = \frac{x}{e^x-e^{-x}},\ H_E(x) = \frac{x}{e^{2x}-1}. \tag{4.2}$$

$$H_\phi(x) = \frac{x}{e^x-e^{-x}}, H_{\tilde{\phi}}(x) = \frac{x}{e^x+e^{-x}},$$

$$H_{L_{(4,1)}}(x) = \frac{xe^x}{e^{2x}-e^{-2x}}, H_{L_{(4,3)}}(x) = \frac{xe^{-x}}{e^{2x}-e^{-2x}},$$

where, the first line relates to the generalized Mathieu series Zeta family and the second line represents the generalized Mathieu series Beta family. It should be noticed that $H_O(x)$ and $H_\phi(x)$ represent both the Zeta and Beta family.

Lemma 4.1. *The folowing integral is evaluated as*

$$B_\nu(\beta,H) := \int_0^\infty e^{-\alpha x}\left[A+Bx+Cx^2\right] x^\nu J_\nu(\beta x)\, dx \tag{4.3}$$

$$= \frac{(2\beta)^\nu}{\sqrt{\pi}} \frac{\Gamma(\nu+\frac{1}{2})}{\left[\alpha^2+\beta^2\right]^{\nu+\frac{1}{2}}} \left\{ \begin{array}{c} A + \left[B(2\alpha)-2C\right]\frac{(\nu+\frac{1}{2})}{[\alpha^2+\beta^2]^1} \\ +C(2\nu)^2\frac{(\nu+\frac{3}{2})(\nu+\frac{1}{2})}{[\alpha^2+\beta^2]^2} \end{array} \right\}.$$

Proof. Gradshtein and Ryzhik [23] on page 712 quotes the reult

$$\int_0^\infty e^{-\alpha x} x^\nu J_\nu(\beta x)\, dx = \frac{(2\beta)^\nu \Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\left[\alpha^2+\beta^2\right]^{\nu+\frac{1}{2}}}, \mathrm{Re}(\nu) > -\frac{1}{2}, \mathrm{Re}(\alpha) > \left|\mathrm{Im}(\beta)\right|,$$

which accounts for the coefficient of A . Differentiating with respect to α gives the coefficient of B, namely

$$\int_0^\infty e^{-\alpha x} x^{^{\nu+1}} J_\nu\left(\beta x\right) dx = \frac{(2\beta)^\nu \Gamma(\nu+\frac{3}{2})}{\sqrt{\pi}\left[\alpha^2+\beta^2\right]^{\nu+\frac{3}{2}}}, \mathrm{Re}(\nu)>-1, \mathrm{Re}(\alpha)>|\mathrm{Im}(\beta)|$$

and, the coefficient of C is procured by differentiating again with respective to α to give,

$$\int_0^\infty e^{-\alpha x} x^{^{\nu+2}} J_\nu\left(\beta x\right) dx = \frac{(2\beta)^\nu}{\sqrt{\pi}} \left\{ \frac{(2\alpha)^2\Gamma(\nu+\frac{5}{2})}{\left[\alpha^2+\beta^2\right]^{\nu+\frac{5}{2}}} - \frac{\Gamma(\nu+\frac{3}{2})}{\left[\alpha^2+\beta^2\right]^{\nu+\frac{3}{2}}} \right\}.$$

Rearanging the three coefficients of the powers, produces (4.3). $\square$

Lemma 4.2. *The upper bound for the equation (4.1) is given by*
(4.4)

$$G_\mu(r;H) \le \frac{1}{[\alpha^2+r^2]^\mu} \left\{ \frac{A}{\mu} + [B(2\alpha)-2C]\frac{1}{[\alpha^2+r^2]^1} + C(2\alpha)^2 \frac{(\mu+1)}{[\alpha^2+r^2]^2} \right\}$$
$$:= B_\mu(r,H),$$

the Hardy theorem leading to the result (3.13).

Proof. If in (4.3) let $\nu=\mu-\frac{1}{2}$, $\beta=r$,then after simple manipulation produces the upper bound depending on (4.2) without $H_A(x)$ and $H_{\tilde{\phi}}(x)$, the drivers of alternating Mathieu series . $\square$

Theorem 4.1. *The Generalized Mathieu positive series upper bounds obtained via intergral equivalence for the Zeta and Beta families*

$$G_\mu(r;H) = 2\cdot C_\mu(r)\int_0^\infty H(x)\cdot x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}\left(rx\right) dx,$$

where

$$S_\mu(r) = G_\mu(r;H_M),\ \ H_M(x) = \frac{x}{e^x-1}$$

and

$$B_M(x) = e^{-x}(1+\frac{x}{2}+\frac{x^2}{12}).$$

(i) So with $\alpha=1,\ A=1,\ B=\frac{1}{2}$ *and* $C=\frac{1}{12}$ *gives the upper bound*

$$S_\mu(r) \le \frac{1}{[1^2+r^2]^\mu}\left\{ \frac{1}{\mu} + \frac{5}{6}\frac{1}{[1^2+r^2]^1} + \frac{1}{3}\frac{(\mu+1)}{[1^2+r^2]^2} \right\} := B_\mu(r,H_M).$$

(ii) For

$$\phi_\mu(r) = G_\mu(r; H_\phi),\ \ H_\phi(x) = \frac{x}{e^x - e^{-x}} = \frac{e^x}{2}\ H_M(2x)$$

and

$$B_\phi(x) = \frac{e^{-x}}{2}(1 + x + \frac{x^2}{3})$$

with $\alpha = 1,\ A = \frac{1}{2},\ B = \frac{1}{2}$ *and* $C = \frac{1}{6}$ *gives the upper bound*

$$\phi_\mu(r) \le \frac{1}{[1^2 + r^2]^\mu}\left\{\frac{1}{\mu} + \frac{4}{3}\frac{1}{[1^2 + r^2]^1} + \frac{4}{3}\frac{(\mu+1)}{[1^2 + r^2]^2}\right\} := B_\mu(r, H_\phi).$$

(iii) For

$$\psi_\mu(r) = G_\mu(r; H_E),\ \ H_E(x) = \frac{x}{e^{2x} - 1} = \frac{1}{2}\ H_M(2x)\ \ \textit{and}$$

$$B_E(x) = \frac{e^{-2x}}{2}(1 + x + \frac{x^2}{3})$$

with $\alpha = 2, A = \frac{1}{2}, B = \frac{1}{2}$ *and* $C = \frac{1}{6}$ *gives the upper bound*

$$\psi_\mu(r) \le \frac{1}{[2^2 + r^2]^\mu}\left\{\frac{1}{\mu} + \frac{10}{3}\frac{1}{[2^2 + r^2]^1} + \frac{16}{3}\frac{(\mu+1)}{[2^2 + r^2]^2}\right\} := B_\mu(r, H_E).$$

(iv) For

$$\Phi^+_\mu(r) = G_\mu(r; H_{L_{(4,1)}}),\ \ H_{L_{(4,1)}}(x) = \frac{xe^x}{e^{2x} - e^{-2x}} = \frac{e^{3x}}{4}\ H_M(4x)$$

and,

$$B_{L_{(4,1)}}(x) = \frac{e^{-x}}{4}(1 + 2x + \frac{4}{3}\ x^2)$$

with $\alpha = 1,\ A = \frac{1}{4},\ B = \frac{1}{2}$ *and* $C = \frac{1}{3}$ *gives the upper bound*

$$\Phi^+_\mu(r) \le \frac{1}{[1^2 + r^2]^\mu}\left\{\frac{1}{2\mu} + \frac{2}{3}\frac{1}{[1^2 + r^2]^1} + \frac{8}{3}\frac{(\mu+1)}{[1^2 + r^2]^2}\right\} := B_\mu(r, H_{L_{(4,1)}}).$$

(v) For

$$\Phi^-_\mu(r) = G_\mu(r; H_{L_{(4,3)}}),\quad H_{L_{(4,3)}}(x) = \frac{xe^{-x}}{e^{2x} - e^{-2x}} = \frac{e^x}{4}\ H_M(4x)$$

$$\textit{and,}\ \ B_{L_{(4,3)}}(x) = \frac{e^{-3x}}{4}(1 + 2x + \frac{4}{3}\ x^2)$$

with $\alpha = 3, A = \frac{1}{4}, B = \frac{1}{2}$ *and* $C = \frac{1}{3}$ *gives the upper bound*

$$\Phi^-_\mu(r) \le \frac{1}{[3^2 + r^2]^\mu}\left\{\frac{1}{2\mu} + \frac{2}{3}\frac{1}{[3^2 + r^2]^1} + \frac{8}{3}\frac{(\mu+1)}{[3^2 + r^2]^2}\right\} := B_\mu(r, H_{L_{(4,3)}}).$$

Remark 4.1. The upper bounds of the five generalized Mathieu series, which are generated by the Hardy bounds, have equivalent behaviour.For example from (i),

$$\lim_{r\to 0^+} B_\mu(r, H_M) = B_\mu(0, H_M) = \frac{1}{\mu} + \frac{5}{6} + \frac{(\mu+1)}{3}\ ,\ \mu > 0$$

and

$$\lim_{r\to\infty} B_\mu(r, H_M) = 0.$$

All the upper bounds of the above, tend to zero as $r \to \infty$ however they differ for $0 < r < \infty$.

As mentioned, the main goal is to develop bounds of generalized Mathieu series involving generators of Zeta and Beta families with an emphasis of the Hardy result, in Section 3. It is worthwhile to compare bounds for the Mathieu series, $S(r)$ (1.1) and the integral version (1.4) with :Alzer *et al.*(1.2) [1], Čebyšev functional method (1.10) and, the Hardy approach for the upper bound as in Theorem 15 part (i) with $\mu = 1$, namely

$$(4.5)\qquad \begin{cases} \frac{1}{r^2+2\zeta(3)} < S_A(r) < \frac{1}{r^2+\frac{1}{6}}, \qquad x \neq 0 \\ \left| S_{\check{C}}(r) - \frac{\pi^2}{12\left(r^2+\frac{1}{4}\right)} \right| \le 2\sqrt{2}\cdot\kappa \left\{ \frac{2}{1+(4r)^2} - \frac{1}{\left[1+(2r)^2\right]^2} \right\}^{\frac{1}{2}} \\ S_H(r) \le \frac{1}{[1^2+r^2]} \left\{ 1 + \frac{5}{6}\cdot\frac{1}{[1^2+r^2]} + \frac{2}{3}\cdot\frac{1}{[1^2+r^2]^2} \right\} \end{cases}$$

where the subscripts of $S(r)$ represent the inequations above and,

$$\kappa = \left[\pi^2\left(1 - \frac{\pi^2}{72}\right) - 7\zeta(3)\right]^{\frac{1}{2}} = 0.3198468959....$$

Lemma 4.3. *Let the first and third upper bounds of (4.5) are derived by $D_{AH}(\cdot)$ then,*

$$(4.6)\qquad D_{AH}(r) = S_A(r) - S_H(r) = \frac{r^2+21}{36(r^2+1)^3(r^2+\frac{1}{6})} > 0, \quad r \ge 0$$

so that $S_H(r)$ is a superior (tighter) upper bound than the upper bound of $S_A(r)$. Further, $S_H(0) = 2.5$, $S_A(0) = 6$ and, both upproach zero as $r \to \infty$.

Proof. Let $\sigma = r^2 + 1$then $S_A(\sigma) = \frac{1}{\sigma - \frac{5}{6}}$ and $S_H(\sigma) = \frac{1}{\sigma} + \frac{5}{6}\cdot\frac{1}{\sigma^2} + \frac{2}{3}\cdot\frac{1}{\sigma^3}$.

Now, $\frac{1}{\sigma-\frac{5}{6}} - \frac{1}{\sigma} = \frac{5}{6\sigma}\cdot\frac{1}{\sigma-\frac{5}{6}}$ and so

$$D_{AH}(\sigma) = S_A(\sigma) - S_H(\sigma)$$

$$= \frac{5}{6\sigma}\cdot\frac{1}{\sigma-\frac{5}{6}} - \frac{5}{6}\cdot\frac{1}{\sigma^2} - \frac{2}{3}\cdot\frac{1}{\sigma^3}$$

$$= \frac{5}{6\sigma}.[\frac{1}{\sigma - \frac{5}{6}} - \frac{1}{\sigma}] - \frac{2}{3}.\frac{1}{\sigma^3} = \left(\frac{5}{6\sigma}\right)^2.\frac{1}{\sigma - \frac{5}{6}} - \frac{2}{3}.\frac{1}{\sigma^3}$$

$$= \frac{1}{\sigma^2}\left\{\frac{5^2}{6^2(\sigma - \frac{5}{6})} - \frac{2}{3\sigma}\right\} = \frac{\sigma + 20}{36\sigma^3(\sigma - \frac{5}{6})}.$$

Hence, return to r gives $D_{AH}(r) > 0$ as at (4.6). $\square$

Remark 4.2. The Mathieu series upper bound from the second in (4.5) is given by

$$S_{\check{C}}(r) \leq \frac{\pi^2}{12\left(r^2 + \frac{1}{4}\right)} + 2\sqrt{2}\cdot\kappa\left\{\frac{2}{1 + (4r)^2} - \frac{1}{\left[1 + (2r)^2\right]^2}\right\}^{\frac{1}{2}} \tag{4.7}$$

where $S_{\check{C}}(r)$ represents the Čebyšev functional method. Consider now, from the expression

$$D_{\check{C}H}(r) = S_{\check{C}}(r) - S_H(r).$$

Now, from (4.5),

$$S_{\check{C}}(0) = \frac{\pi^2}{3} + 2\sqrt{2}\left[\pi^2\left(1 - \frac{\pi^2}{72}\right) - 7\zeta(3)\right]^{\frac{1}{2}} = 4.1945317698984431729$$

and $S_H(0) = 2.5$. Further, the functions cross only at $r^* = 0.62102384511259743548$

$$S_{\check{C}}(r^*) = S_H(r^*) = 1.4062510257077055102.$$

Hence,

$$S_H(r) < S_{\check{C}}(r), \quad 0 \leq r \leq r^*, \; S_{\check{C}}(r) < S_H(r), \qquad r > r^*$$

so that $S_H(r)$ is sharper than $S_{\check{C}}(r)$ in $0 \leq r < r^*$ and, for $r > r^*$, $S_{\check{C}}(r)$ is slightly lower than $S_H(r)$ as both tend to *zero* as $r \to \infty$.

The work has been undertaken using Maple-2021 with 20 digit calculation.

Milovanović and Pogany [32] , obtain a Mathieu series upper bound over different zone

$$S_{MP}(r) \leq \begin{cases} \frac{1}{r^2 + \frac{1}{4}}, & 0 \leq r \leq \frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{4r^2+1} - 1}, & r > \frac{\sqrt{3}}{2} \end{cases}$$

and compare with $S_A(r) < \frac{1}{r^2 + \frac{1}{6}}$ .Using a different approach they obtain a sharper bound over $0 < r \leq r_1 = \sqrt{\frac{1}{6}(5 + 2\sqrt{3})} \approx$ 1.18772 and for $r > r_1$,a better result with $\frac{1}{r^2 + \frac{1}{6}}$.

The current result relies on the Mathieu series, whereas the generalized Mathieu series involving generators of Zeta and Beta families are extended to Čebyšev functional methods to abtain lower and upper bounds, and with Hardy- type upper bounds.

5. Some further results using Hardy-type upper bounds

In the literature, there are many applications for which the Hardy theorem leads to the result (3.13) to produce a best possible upper bound. Before undertaking some examples it is worthwhile to outline some deffinitions.

The Hurwitz zeta function $\zeta(s,a)$ is an analytic function of s everywhere in the complex s-plane (except for a simple pole at $s=1$ with residue 1) and is defined by the series (see [34] , p 607)

$$\zeta(s,a)=\sum_{n=1}^{\infty}\frac{1}{(n+a)^s},...R(s)>1,a\neq 0,-1,-2....$$

The Riemann zeta function is a special case of the Hurwitz zeta function with $a=1$ and has its only singularity a simple pole, with residue 1, at the point $s=1$.

$\zeta(s,a)$ is an analytic function of s in the half-plane $R(s)=\sigma>1$, and have the integral representation

$$\Gamma(s)\,\zeta(s,a)=\int_0^{\infty}\frac{x^{s-1}e^{-(a-1)x}}{e^x-1}dx,...R(s)>1,R(a)>0. \tag{5.1}$$

The digamma(or Psi) function (see [34] , p 139-140, by 5.7.6 and 5.9.16) are given by

$$\psi(z)+\gamma=\left\{\begin{array}{l}\sum_{n=0}^{\infty}\frac{z}{n(n+z)},...z\neq 0,-1,-2....\\ \int_0^{\infty}\frac{e^{-t}-e^{-zt}}{1-e^{-t}}dt,...,R(z)>0\end{array}\right\} \tag{5.2}$$

where γ is the *Euler constant*

$$\gamma=-\psi(1)=\int_0^{\infty}\left(\frac{1}{e^t-1}-\frac{1}{te^t}\right)dt \tag{5.3}$$

The function $\psi^{(n)}(z),n=1,2,...$are the *polygamma functions* and given by

$$\psi^{(n)}(z)=(-1)^{n+1}\int_0^{\infty}\frac{t^ne^{-zt}}{1-e^{-t}}dt,n=1,2,3...R(z)>0 \tag{5.4}$$

Theorem 5.1. *The Hardy -type upper bound are obtained utilizing the above functions are given below:*

(i) The Hurwitz zeta function upper bounds are given from (5.1)

$$\zeta(s,a)\leq\frac{1}{sa^{s-1}}\left\{1+\frac{s}{2a}+\frac{s(s-1)}{12a^2}\right\}.,.R(s)>1. \tag{5.5}$$

(ii)The digamma(or Psi) function from (5.2) the upper bounds are given as

$$\psi(z)+\gamma\leq\frac{1}{12z^2}(z-1)(z+7).,.R(z)>0, \tag{5.6}$$

and the Hardy bound of the Euler constant by

$$\gamma \leq \frac{7}{12}. \tag{5.7}$$

(iii)The polygamma function integral from (5.4) the upper bounds are given by

$$\left|\psi^{(n)}(z)\right| \leq \frac{\Gamma(n)}{12z^{n+2}}\left[\left(z+\frac{n}{4}\right)^2+\frac{(3n+4)\,n}{48}\right].,.R(z)>0. \tag{5.8}$$

Proof. It is well known facts that

$$\int_0^\infty e^{-pt}t^\alpha dt = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}} \text{ and } \Gamma(x+1) = x\Gamma(x). \tag{5.9}$$

Further, the Hardy theorem which leads to the result (3.13) to provide the upper bounds for the proofs.

(i) From (5.1)

$$\begin{aligned}
\Gamma(s)\,\zeta(s,a) &= \int_0^\infty \frac{x^{s-1}e^{-ax}}{1-e^{-x}}dx \\
&\leq \int_0^\infty e^{-ax}x^{s-1}\left\{\frac{1}{x}+\frac{1}{2}+\frac{x}{12}\right\}dx \\
&= \int_0^\infty e^{-ax}x^{s-1}\left\{x^{s-2}+\frac{x^{s-1}}{2}+\frac{x^s}{12}\right\}dx \\
&= \frac{\Gamma(s-1)}{a^{s-1}}+\frac{1}{2}\cdot\frac{\Gamma(s)}{a^s}+\frac{1}{12}\cdot\frac{\Gamma(s+1)}{a^{s+1}},
\end{aligned}$$

where the Hardy upper bound is used and the last two lines are due from (5.9) to give the result,(5.5).

(ii) From the integrals (5.2) and (5.3)

$$\begin{aligned}
\psi(z)+\gamma &= \int_0^\infty \frac{e^{-t}-e^{-zt}}{1-e^{-t}}dt..,..R(z)>0 \\
&\leq \int_0^\infty (e^{-t}-e^{-zt})\left\{\frac{1}{t}+\frac{1}{2}+\frac{t}{12}\right\}dt \\
&= \frac{1}{2}(1-\frac{1}{z})+\frac{1}{12}(1-\frac{1}{z^2}) = \frac{1}{12}(1-\frac{1}{z})(7+\frac{1}{z}),
\end{aligned}$$

where the Hardy upper bound is used and (5.6).

Note that

$$\gamma = \int_0^\infty (\frac{1}{e^t-1}-\frac{e^{-t}}{t})dt \leq \int_0^\infty e^{-t}(\frac{1}{2}+\frac{t}{12})dt = \frac{1}{2}\left(\frac{\Gamma(1)}{1^1}+\frac{\Gamma(2)}{6.1^2}\right)$$

which is as given in (5.7).

(iii) From (5.4)

$$\left|\psi^{(n)}(z)\right| \le \int_0^\infty e^{-zt}t^n\left(\frac{1}{t}+\frac{1}{2}+\frac{t}{12}\right)dt,\ n=1,2,3...R(z)>0$$

$$=\int_0^\infty e^{-zt}\left(t^{n-1}+\frac{t^n}{2}+\frac{t^{n+1}}{12}\right)dt$$

$$=\frac{\Gamma(n)}{z^n}+\frac{\Gamma(n+1)}{2\cdot z^{n+1}}+\frac{\Gamma(n+2)}{12\cdot z^{n+2}}$$

$$=\frac{\Gamma(n)}{z^n}\left(1+\frac{n}{2\cdot z^1}+\frac{(n+1)n}{12\cdot x^2}\right)$$

$$=\frac{\Gamma(n)}{12\cdot z^{n+2}}\left[12z^2+6nz+(n+1)n\right].$$

The polygamma function integral upper bound is given by (5.8). □

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