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SOME PROPERTIES OF BOUNDED TRI-LINEAR MAPS

Abotaleb Sheikhali[1], Kazem Haghnejad Azar and Ali Ebadian

[1]**Independent researcher**
[2]**University of Mohaghegh Ardabili, Department of Mathematics and Applications, Ardabil, Iran**
[3]**University of Urmia, Department of Mathematics, Urmia, Iran**

Abstract. Let X, Y, Z and W be normed spaces and $f : X \times Y \times Z \longrightarrow W$ be a bounded tri-linear mapping. In this manuscript, we introduce the topological centers of bounded tri-linear mapping and we invistagate their properties. We study the relationships between weakly compactenss of bounded linear mappings and regularity of bounded tri-linear mappings. We extend some factorization property for bounded tri-linear mappings. We also establish the relations between regularity and factorization property of bounded tri-linear mappings.
Keywords: Arens product, Module action, Factors, Topological center and Tri-linear mappings

1. Introduction

Let X, Y, Z and W be normed spaces and $f : X \times Y \times Z \longrightarrow W$ be a bounded tri-linear mapping. One of the natural extensions of f can be derived by the following procedure:

1. $f^* : W^* \times X \times Y \longrightarrow Z^*$, given by $\langle f^*(w^*, x, y), z\rangle = \langle w^*, f(x, y, z)\rangle$, where $x \in X, y \in Y, z \in Z, w^* \in W^*$.

   The map f^* is a bounded tri-linear mapping and is called the adjoint of f.

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Corresponding Author: Abotaleb Sheikhali, Independent researcher | E-mail: Abotaleb.sheikhali.20@gmail.com
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2. $f^{**} = (f^*)^* : Z^{**} \times W^* \times X \longrightarrow Y^*$, given by $\langle f^{**}(z^{**}, w^*, x), y\rangle = \langle z^{**}, f^*(w^*, x, y)\rangle$, where $x \in X, y \in Y, z^{**} \in Z^{**}, w^* \in W^*$.
3. $f^{***} = (f^{**})^* : Y^{**} \times Z^{**} \times W^* \longrightarrow X^*$, given by $\langle f^{***}(y^{**}, z^{**}, w^*), x\rangle = \langle y^{**}, f^{**}(z^{**}, w^*, x)\rangle$, where $x \in X, y^{**} \in Y^{**}, z^{**} \in Z^{**}, w^* \in W^*$.
4. $f^{****} = (f^{***})^* : X^{**} \times Y^{**} \times Z^{**} \longrightarrow W^{**}$, given by $\langle f^{****}(x^{**}, y^{**}, z^{**}), w^*\rangle = \langle x^{**}, f^{***}(y^{**}, z^{**}, w^*)\rangle$, where $x^{**} \in X^{**}, y^{**} \in Y^{**}, z^{**} \in Z^{**}, w^* \in W^*$.

Now let $f^r : Z \times Y \times X \longrightarrow W$ be the flip of f defined by $f^r(z, y, x) = f(x, y, z)$, whenever $x \in X, y \in Y$ and $z \in Z$. Then f^r is a bounded tri-linear map and it may be extended as above to $f^{r****} : Z^{**} \times Y^{**} \times X^{**} \longrightarrow W^{**}$. When f^{****} and f^{r****r} are equal, then f is called regular. Regularity of f is equvalent to the following

$$w^* - \lim_{\alpha} w^* - \lim_{\beta} w^* - \lim_{\gamma} f(x_\alpha, y_\beta, z_\gamma) = w^* - \lim_{\gamma} w^* - \lim_{\beta} w^* - \lim_{\alpha} f(x_\alpha, y_\beta, z_\gamma),$$

where $\{x_\alpha\} \subset X, \{y_\beta\} \subset Y$ and $\{z_\gamma\} \subset Z$ and convergence to $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$ in the w^*-topologies, respectively. A bounded tri-linear mapping $f : X \times Y \times Z \longrightarrow W$ is regular whenever at least two of X, Y or Z are reflexive, see [19] and [20]. Also, we have naturally six different Aron-Berner extensions to the bidual spaces based on six different elements in S3 and compeletly regularity should be defined accordingly to the equalities of all these six Aron-Berner extensions, see [13].

Example 1.1. Let G be an infinite, compact Hausdorff group and let $1 < p < \infty$. By [9, pp 54], we know that $L^p(G) * L^1(G) \subset L^p(G)$, where

$$(k * g)(x) = \int_G k(y)g(y^{-1}x)dy, \qquad (x \in G, k \in L^p(G), g \in L^1(G)).$$

On the other hand, since the Banach space $L^p(G)$ is reflexive, the bounded tri-linear mapping

$$f : L^p(G) \times L^1(G) \times L^p(G) \longrightarrow L^p(G)$$

defined by $f(k, g, h) = (k * g) * h$, is regular for every $k, h \in L^p(G)$ and $g \in L^1(G)$, see [20].

A bounded bilinear(resp. tri-linear) mapping $m : X \times Y \longrightarrow Z$(resp. $f : X \times Y \times Z \longrightarrow W$) is said to be factor if is surjective, that is, $m(X \times Y) = Z$(resp. $f(X \times Y \times Z) = W$), see [5].

For a discussion of Arens regularity for Banach algebras and bounded bilinear maps, see [1], [2], [11], [12] and [18]. For example, every C^*-algebra is Arens regular, see [4]. Also $L^1(G)$ is Arens regular if and only if G is finite,[21].
The left topological center of m may be defined as follows:

$$Z_l(m) = \{x^{**} \in X^{**} : y^{**} \longrightarrow m^{***}(x^{**}, y^{**})\ is\ weak^* - to - weak^* - continuous\}.$$

Also the right topological center of turns out to be

$$Z_r(m) = \{y^{**} \in Y^{**} : x^{**} \longrightarrow m^{r***r}(x^{**}, y^{**})\ is\ weak^* - to - weak^* - continuous\}.$$

The subject of topological centers has been investigated in [6], [7] and [16]. In [14], Lau and Ulger gave several significant results related to the topological centers of certain dual algebras. In [11], authors extend some problems from Arens regularity and Banach algebras to module actions. They also extend the definitions of the left and right multiplier for module actions, see [10] and [12].

Let A be a Banach algebra, and let $\pi : A \times A \longrightarrow A$ denote the product of A, so that $\pi(a,b) = ab$ for every $a, b \in A$. The Banach algebra A is Arens regular whenever the map π is Arens regular. The first and second Arens products, denoted by $\Box$ and $\Diamond$ respectively, are definded by

$$a^{**}\Box b^{**} = \pi^{***}(a^{**}, b^{**}) \quad , \quad a^{**}\Diamond b^{**} = \pi^{r***r}(a^{**}, b^{**}) \quad , \quad (a^{**}, b^{**} \in A^{**}).$$

2. Module actions for bounded tri-linear maps

Let (π_1, X, π_2) be a Banach A-module and let $\pi_1 : A \times X \longrightarrow X$ and $\pi_2 : X \times A \longrightarrow X$ be the left and right module actions of A on X, respectively. If (π_1, X) (resp. (X, π_2)) is a left (resp. right) Banach A-module of A on X, then (X^*, π_1^*)(resp. (π_2^{r*r}, X^*)) is a right (resp. left) Banach A-module and $(\pi_2^{r*r}, X^*, \pi_1^*)$ is the dual Banach A-module of (π_1, X, π_2). We note also that $(\pi_1^{***}, X^{**}, \pi_2^{***})$ is a Banach $(A^{**}, \Box)$-module with module actions $\pi_1^{***} : A^{**} \times X^{**} \longrightarrow X^{**}$ and $\pi_2^{***} : X^{**} \times A^{**} \longrightarrow X^{**}$. Similary, $(\pi_1^{r***r}, X^{**}, \pi_2^{r***r})$ is a Banach $(A^{**}, \Diamond)$-module with module actions $\pi_1^{r***r} : A^{**} \times X^{**} \longrightarrow X^{**}$ and $\pi_2^{r***r} : X^{**} \times A^{**} \longrightarrow X^{**}$. If we continue dualizing we shall reach $(\pi_2^{***r*r}, X^{***}, \pi_1^{****})$ and $(\pi_2^{r*****r}, X^{***}, \pi_1^{r***r*})$ are the dual Banach $(A^{**}, \Box)$-module and Banach $(A^{**}, \Diamond)$-module of $(\pi_1^{***}, X^{**}, \pi_2^{***})$ and $(\pi_1^{r***r}, X^{**}, \pi_2^{r***r})$, respectively (see [15]). In [8], Eshaghi Gordji and Fillali show that if a Banach algebra A has a bounded left (or right) approximate identity, then the left (or right) module action of A on A^* is Arens regular if and only if A is reflexive.

We commence with the following definition for bounded tri-linear mapping.

Definition 2.1. Let X be a Banach space, A be a Banach algebra and $\Omega_1 : A \times A \times X \longrightarrow X$ be a bounded tri-linear map. Then the pair (Ω_1, X) is said to be a left Banach $A-$module when

$$\Omega_1(\pi(a,b), \pi(c,d), x) = \Omega_1(a, b, \Omega_1(c, d, x)),$$

for each $a, b, c, d \in A$ and $x \in X$. A right Banach $A-$module can be defined similarly. Let $\Omega_2 : X \times A \times A \longrightarrow X$ be a bounded tri-linear map. Then the pair (X, Ω_2) is said to be a right Banach $A-$module when

$$\Omega_2(x, \pi(a,b), \pi(c,d)) = \Omega_2(\Omega_2(x, a, b), c, d).$$

A triple (Ω_1, X, Ω_2) is said to be a Banach $A-$module when (Ω_1, X) and (X, Ω_2) are left and right Banach $A-$modules respectively, also

$$\Omega_2(\Omega_1(a, b, x), c, d) = \Omega_1(a, b, \Omega_2(x, c, d)).$$

Lemma 2.1. *If (Ω_1, X, Ω_2) is a Banach $A-$module, then $(\Omega_2^{r*r}, X^*, \Omega_1^*)$ is a Banach $A-$module.*

Proof. Since the pair (X, Ω_2) is a right Banach $A-$module, thus for every $a, b, c, d \in A$, $x \in X$ and $x^* \in X^*$ we have

$$\begin{aligned}
&\langle \Omega_2^{r*r}(\pi(a,b), \pi(c,d), x^*), x\rangle = \langle \Omega_2^{r*}(x^*, \pi(c,d), \pi(a,b)), x\rangle \\
&= \langle x^*, \Omega_2^{r}(\pi(c,d), \pi(a,b), x)\rangle = \langle x^*, \Omega_2(x, \pi(a,b), \pi(c,d))\rangle \\
&= \langle x^*, \Omega_2(\Omega_2(x,a,b), c, d)\rangle = \langle x^*, \Omega_2^{r}(d, c, \Omega_2(x,a,b))\rangle \\
&= \langle \Omega_2^{r*}(x^*, d, c), \Omega_2(x,a,b)\rangle = \langle \Omega_2^{r*r}(c, d, x^*), \Omega_2^{r}(b,a,x)\rangle \\
&= \langle \Omega_2^{r*}(\Omega_2^{r*r}(c,d,x^*), b, a), x\rangle = \langle \Omega_2^{r*r}(a, b, \Omega_2^{r*r}(c,d,x^*)), x\rangle.
\end{aligned}$$

Therefore $\Omega_2^{r*r}(\pi(a,b), \pi(c,d), x^*) = \Omega_2^{r*r}(a, b, \Omega_2^{r*r}(c,d,x^*))$, so (Ω_2^{r*r}, X) is a left Banach $A-$module. In the other hands, (Ω_1, X) is a left Banach $A-$module, thus we have

$$\begin{aligned}
&\langle \Omega_1^*(x^*, \pi(a,b), \pi(c,d)), x\rangle = \langle x^*, \Omega_1(\pi(a,b), \pi(c,d), x)\rangle \\
&= \langle x^*, \Omega_1(a, b, \Omega_1(c,d,x))\rangle = \langle \Omega_1^*(x^*, a, b), \Omega_1(c,d,x)\rangle \\
&= \langle \Omega_1^*(\Omega_1^*(x^*, a, b), c, d), x\rangle.
\end{aligned}$$

It follows that (X, Ω_1^*) is a right Banach $A-$module. Finally, we show that

$$\Omega_1^*(\Omega_2^{r*r}(a,b,x^*), c, d) = \Omega_2^{r*r}(a, b, \Omega_1^*(x^*, c, d)).$$

For every $x \in X$ we have

$$\begin{aligned}
&\langle \Omega_1^*(\Omega_2^{r*r}(a,b,x^*), c, d), x\rangle = \langle \Omega_2^{r*r}(a,b,x^*), \Omega_1(c,d,x)\rangle \\
&= \langle \Omega_2^{r*}(x^*, b, a), \Omega_1(c,d,x)\rangle = \langle x^*, \Omega_2^{r}(b, a, \Omega_1(c,d,x))\rangle \\
&= \langle x^*, \Omega_2(\Omega_1(c,d,x), a, b)\rangle = \langle x^*, \Omega_1(c, d, \Omega_2(x,a,b))\rangle \\
&= \langle \Omega_1^*(x^*, c, d), \Omega_2(x,a,b)\rangle = \langle \Omega_1^*(x^*, c, d), \Omega_2^{r}(b,a,x)\rangle \\
&= \langle \Omega_2^{r*}(\Omega_1^*(x^*, c, d), b, a), x\rangle = \langle \Omega_2^{r*r}(a, b, \Omega_1^*(x^*, c, d)), x\rangle.
\end{aligned}$$

Thus $(\Omega_2^{r*r}, X^*, \Omega_1^*)$ is a Banach $A-$module. □

Theorem 2.1. *Let (Ω_1, X, Ω_2) be a Banach $A-$module, then*

1. *The triple $(\Omega_1^{****}, X^{**}, \Omega_2^{****})$ is a Banach $(A^{**}, \Box, \Box)-$module.*
2. *The triple $(\Omega_1^{r****r}, X^{**}, \Omega_2^{r****r})$ is a Banach $(A^{**}, \Diamond, \Diamond)-$module.*

Proof. We prove only (1), the other part has the same argument. Let $\{a_\alpha\}, \{b_\beta\}, \{c_\gamma\}$ and $\{d_\theta\}$ are nets in A which converge to a^{**}, b^{**}, c^{**} and $d^{**} \in A^{**}$ in the

w^*-topologies, respectively. Then by lemma 2.1 for every $x^* \in X^*$ we have

$$\begin{aligned}
&\langle \Omega_1^{****}(a^{**}, b^{**}, \Omega_1^{****}(c^{**}, d^{**}, x^{**})), x^* \rangle \\
&= \langle a^{**}, \Omega_1^{***}(b^{**}, \Omega_1^{****}(c^{**}, d^{**}, x^{**}), x^*) \rangle \\
&= \lim_\alpha \langle \Omega_1^{***}(b^{**}, \Omega_1^{****}(c^{**}, d^{**}, x^{**}), x^*), a_\alpha \rangle \\
&= \lim_\alpha \langle b^{**}, \Omega_1^{**}(\Omega_1^{****}(c^{**}, d^{**}, x^{**}), x^*, a_\alpha) \rangle \\
&= \lim_\alpha \lim_\beta \langle \Omega_1^{**}(\Omega_1^{****}(c^{**}, d^{**}, x^{**}), x^*, a_\alpha), b_\beta \rangle \\
&= \lim_\alpha \lim_\beta \langle \Omega_1^{****}(c^{**}, d^{**}, x^{**}), \Omega_1^{*}(x^*, a_\alpha, b_\beta) \rangle \\
&= \lim_\alpha \lim_\beta \langle c^{**}, \Omega_1^{***}(d^{**}, x^{**}, \Omega_1^{*}(x^*, a_\alpha, b_\beta)) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle \Omega_1^{***}(d^{**}, x^{**}, \Omega_1^{*}(x^*, a_\alpha, b_\beta)), c_\gamma \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle d^{**}, \Omega_1^{**}(x^{**}, \Omega_1^{*}(x^*, a_\alpha, b_\beta), c_\gamma) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \lim_\tau \langle \Omega_1^{**}(x^{**}, \Omega_1^{*}(x^*, a_\alpha, b_\beta), c_\gamma), d_\tau \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \lim_\tau \langle x^{**}, \Omega_1^{*}(\Omega_1^{*}(x^*, a_\alpha, b_\beta), c_\gamma, d_\tau) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \lim_\tau \langle x^{**}, \Omega_1^{*}(x^*, \pi(a_\alpha, b_\beta), \pi(c_\gamma, d_\tau)) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \lim_\tau \langle \Omega_1^{**}(x^{**}, x^*, \pi(a_\alpha, b_\beta)), \pi(c_\gamma, d_\tau) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \lim_\tau \langle \pi^*(\Omega_1^{**}(x^{**}, x^*, \pi(a_\alpha, b_\beta)), c_\gamma), d_\tau \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle d^{**}, \pi^*(\Omega_1^{**}(x^{**}, x^*, \pi(a_\alpha, b_\beta)), c_\gamma) \rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle \pi^{**}(d^{**}, \Omega_1^{**}(x^{**}, x^*, \pi(a_\alpha, b_\beta))), c_\gamma \rangle \\
&= \lim_\alpha \lim_\beta \langle c^{**}, \pi^{**}(d^{**}, \Omega_1^{**}(x^{**}, x^*, \pi(a_\alpha, b_\beta))) \rangle \\
&= \lim_\alpha \lim_\beta \langle \pi^{***}(c^{**}, d^{**}), \Omega_1^{**}(x^{**}, x^*, \pi(a_\alpha, b_\beta)) \rangle \\
&= \lim_\alpha \lim_\beta \langle \Omega_1^{***}(\pi^{***}(c^{**}, d^{**}), x^{**}, x^*), \pi(a_\alpha, b_\beta) \rangle \\
&= \lim_\alpha \lim_\beta \langle \pi^*(\Omega_1^{***}(\pi^{***}(c^{**}, d^{**}), x^{**}, x^*), a_\alpha), b_\beta \rangle \\
&= \lim_\alpha \langle b^{**}, \pi^*(\Omega_1^{***}(\pi^{***}(c^{**}, d^{**}), x^{**}, x^*), a_\alpha) \rangle \\
&= \lim_\alpha \langle \pi^{**}(b^{**}, \Omega_1^{***}(\pi^{***}(c^{**}, d^{**}), x^{**}, x^*)), a_\alpha \rangle \\
&= \langle a^{**}, \pi^{**}(b^{**}, \Omega_1^{***}(\pi^{***}(c^{**}, d^{**}), x^{**}, x^*)) \rangle \\
&= \langle \pi^{***}(a^{**}, b^{**}), \Omega_1^{***}(\pi^{***}(c^{**}, d^{**}), x^{**}, x^*) \rangle \\
&= \langle \Omega_1^{****}(\pi^{***}(a^{**}, b^{**}), \pi^{***}(c^{**}, d^{**}), x^{**}), x^* \rangle.
\end{aligned}$$

Thus $(\Omega_1^{****}, X^{**})$ is a left Banach $(A^{**}, \Box, \Box)-$module. Now we show that the pair $(X^{**}, \Omega_2^{****})$ is a right Banach $(A^{**}, \Box, \Box)-$module. Let $\{x_\eta\}$ be a net in X which converge to $x^{**} \in X^{**}$ in the w^*-topologies. The pair (X, Ω_2) is a right Banach

$A-$module, so we have

$$
\begin{aligned}
&\langle \Omega_2^{****}(\Omega_2^{****}(x^{**},a^{**},b^{**}),c^{**},d^{**}),x^*\rangle\\
&=\langle \Omega_2^{****}(x^{**},a^{**},b^{**}),\Omega_2^{***}(c^{**},d^{**},x^*)\rangle\\
&=\langle x^{**},\Omega_2^{***}(a^{**},b^{**},\Omega_2^{***}(c^{**},d^{**},x^*))\rangle\\
&=\lim_\eta \langle \Omega_2^{***}(a^{**},b^{**},\Omega_2^{***}(c^{**},d^{**},x^*)),x_\eta\rangle\\
&=\lim_\eta \langle a^{**},\Omega_2^{**}(b^{**},\Omega_2^{***}(c^{**},d^{**},x^*),x_\eta)\rangle\\
&=\lim_\eta\lim_\alpha \langle \Omega_2^{**}(b^{**},\Omega_2^{***}(c^{**},d^{**},x^*),x_\eta),a_\alpha\rangle\\
&=\lim_\eta\lim_\alpha \langle b^{**},\Omega_2^{*}(\Omega_2^{***}(c^{**},d^{**},x^*),x_\eta,a_\alpha)\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta \langle \Omega_2^{*}(\Omega_2^{***}(c^{**},d^{**},x^*),x_\eta,a_\alpha),b_\beta\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta \langle \Omega_2^{***}(c^{**},d^{**},x^*),\Omega_2(x_\eta,a_\alpha,b_\beta)\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta \langle c^{**},\Omega_2^{**}(d^{**},x^*,\Omega_2(x_\eta,a_\alpha,b_\beta))\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta\lim_\gamma \langle \Omega_2^{**}(d^{**},x^*,\Omega_2(x_\eta,a_\alpha,b_\beta)),c_\gamma\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta\lim_\gamma \langle d^{**},\Omega_2^{*}(x^*,\Omega_2(x_\eta,a_\alpha,b_\beta),c_\gamma)\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta\lim_\gamma\lim_\tau \langle \Omega_2^{*}(x^*,\Omega_2(x_\eta,a_\alpha,b_\beta),c_\gamma),d_\tau\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta\lim_\gamma\lim_\tau \langle x^*,\Omega_2(\Omega_2(x_\eta,a_\alpha,b_\beta),c_\gamma,d_\tau)\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta\lim_\gamma\lim_\tau \langle x^*,\Omega_2(x_\eta,\pi(a_\alpha,b_\beta),\pi(c_\gamma,d_\tau))\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta\lim_\gamma\lim_\tau \langle \Omega_2^{*}(x^*,x_\eta,\pi(a_\alpha,b_\beta)),\pi(c_\gamma,d_\tau)\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta\lim_\gamma\lim_\tau \langle \pi^*(\Omega_2^{*}(x^*,x_\eta,\pi(a_\alpha,b_\beta)),c_\gamma),d_\tau\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta\lim_\gamma \langle d^{**},\pi^*(\Omega_2^{*}(x^*,x_\eta,\pi(a_\alpha,b_\beta)),c_\gamma)\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta\lim_\gamma \langle \pi^{**}(d^{**},\Omega_2^{*}(x^*,x_\eta,\pi(a_\alpha,b_\beta)),c_\gamma\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta \langle c^{**},\pi^{**}(d^{**},\Omega_2^{*}(x^*,x_\eta,\pi(a_\alpha,b_\beta)))\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta \langle \pi^{***}(c^{**},d^{**}),\Omega_2^{*}(x^*,x_\eta,\pi(a_\alpha,b_\beta)))\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta \langle \Omega_2^{**}(\pi^{***}(c^{**},d^{**}),x^*,x_\eta),\pi(a_\alpha,b_\beta)\rangle\\
&=\lim_\eta\lim_\alpha\lim_\beta \langle \pi^*(\Omega_2^{**}(\pi^{***}(c^{**},d^{**}),x^*,x_\eta),a_\alpha),b_\beta\rangle\\
&=\lim_\eta\lim_\alpha \langle b^{**},\pi^*(\Omega_2^{**}(\pi^{***}(c^{**},d^{**}),x^*,x_\eta),a_\alpha)\rangle\\
&=\lim_\eta\lim_\alpha \langle \pi^{**}(b^{**},\Omega_2^{**}(\pi^{***}(c^{**},d^{**}),x^*,x_\eta)),a_\alpha\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_\eta \langle a^{**}, \pi^{**}(b^{**}, \Omega_2^{**}(\pi^{***}(c^{**}, d^{**}), x^*, x_\eta))\rangle \\
&= \lim_\eta \langle \pi^{***}(a^{**}, b^{**}), \Omega_2^{**}(\pi^{***}(c^{**}, d^{**}), x^*, x_\eta)\rangle \\
&= \lim_\eta \langle \Omega_2^{***}(\pi^{***}(a^{**}, b^{**}), \pi^{***}(c^{**}, d^{**}), x^*), x_\eta\rangle \\
&= \langle x^{**}, \Omega_2^{***}(\pi^{***}(a^{**}, b^{**}), \pi^{***}(c^{**}, d^{**}), x^*)\rangle \\
&= \langle \Omega_2^{****}(x^{**}, \pi^{***}(a^{**}, b^{**}), \pi^{***}(c^{**}, d^{**})), x^*\rangle.
\end{aligned}
$$

Finally, we show that

$$
\Omega_2^{****}(\Omega_1^{****}(a^{**}, b^{**}, x^{**}), c^{**}, d^{**}) = \Omega_1^{****}(a^{**}, b^{**}, \Omega_2^{****}(x^{**}, c^{**}, d^{**})).
$$

Next we have

$$
\begin{aligned}
&\langle \Omega_2^{****}(\Omega_1^{****}(a^{**}, b^{**}, x^{**}), c^{**}, d^{**}), x^*\rangle \\
&= \langle \Omega_1^{****}(a^{**}, b^{**}, x^{**}), \Omega_2^{***}(c^{**}, d^{**}, x^*)\rangle \\
&= \langle a^{**}, \Omega_1^{***}(b^{**}, x^{**}, \Omega_2^{***}(c^{**}, d^{**}, x^*))\rangle \\
&= \lim_\alpha \langle \Omega_1^{***}(b^{**}, x^{**}, \Omega_2^{***}(c^{**}, d^{**}, x^*)), a_\alpha\rangle \\
&= \lim_\alpha \langle b^{**}, \Omega_1^{**}(x^{**}, \Omega_2^{***}(c^{**}, d^{**}, x^*), a_\alpha)\rangle \\
&= \lim_\alpha \lim_\beta \langle \Omega_1^{**}(x^{**}, \Omega_2^{***}(c^{**}, d^{**}, x^*), a_\alpha), b_\beta\rangle \\
&= \lim_\alpha \lim_\beta \langle x^{**}, \Omega_1^{*}(\Omega_2^{***}(c^{**}, d^{**}, x^*), a_\alpha, b_\beta)\rangle \\
&= \lim_\alpha \lim_\beta \lim_\eta \langle \Omega_1^{*}(\Omega_2^{***}(c^{**}, d^{**}, x^*), a_\alpha, b_\beta), x_\eta\rangle \\
&= \lim_\alpha \lim_\beta \lim_\eta \langle \Omega_2^{***}(c^{**}, d^{**}, x^*), \Omega_1(a_\alpha, b_\beta, x_\eta)\rangle \\
&= \lim_\alpha \lim_\beta \lim_\eta \langle c^{**}, \Omega_2^{**}(d^{**}, x^*, \Omega_1(a_\alpha, b_\beta, x_\eta))\rangle \\
&= \lim_\alpha \lim_\beta \lim_\eta \lim_\gamma \langle \Omega_2^{**}(d^{**}, x^*, \Omega_1(a_\alpha, b_\beta, x_\eta)), c_\gamma\rangle \\
&= \lim_\alpha \lim_\beta \lim_\eta \lim_\gamma \langle d^{**}, \Omega_2^{*}(x^*, \Omega_1(a_\alpha, b_\beta, x_\eta), c_\gamma)\rangle \\
&= \lim_\alpha \lim_\beta \lim_\eta \lim_\gamma \lim_\tau \langle \Omega_2^{*}(x^*, \Omega_1(a_\alpha, b_\beta, x_\eta), c_\gamma), d_\tau\rangle \\
&= \lim_\alpha \lim_\beta \lim_\eta \lim_\gamma \lim_\tau \langle x^*, \Omega_2(\Omega_1(a_\alpha, b_\beta, x_\eta), c_\gamma, d_\tau)\rangle \\
&= \lim_\alpha \lim_\beta \lim_\eta \lim_\gamma \lim_\tau \langle x^*, \Omega_1(a_\alpha, b_\beta, \Omega_2(x_\eta, c_\gamma, d_\tau))\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\alpha}\lim_{\beta}\lim_{\eta}\lim_{\gamma}\lim_{\tau}\langle \Omega_1^*(x^*, a_\alpha, b_\beta), \Omega_2(x_\eta, c_\gamma, d_\tau)\rangle \\
&= \lim_{\alpha}\lim_{\beta}\lim_{\eta}\lim_{\gamma}\langle d^{**}, \Omega_2^*(\Omega_1^*(x^*, a_\alpha, b_\beta), x_\eta, c_\gamma)\rangle \\
&= \lim_{\alpha}\lim_{\beta}\lim_{\eta}\lim_{\gamma}\langle \Omega_2^{**}(d^{**}, \Omega_1^*(x^*, a_\alpha, b_\beta), x_\eta), c_\gamma\rangle \\
&= \lim_{\alpha}\lim_{\beta}\lim_{\eta}\langle c^{**}, \Omega_2^{**}(d^{**}, \Omega_1^*(x^*, a_\alpha, b_\beta), x_\eta)\rangle \\
&= \lim_{\alpha}\lim_{\beta}\lim_{\eta}\langle \Omega_2^{***}(c^{**}, d^{**}, \Omega_1^*(x^*, a_\alpha, b_\beta)), x_\eta\rangle \\
&= \lim_{\alpha}\lim_{\beta}\langle x^{**}, \Omega_2^{***}(c^{**}, d^{**}, \Omega_1^*(x^*, a_\alpha, b_\beta))\rangle \\
&= \lim_{\alpha}\lim_{\beta}\langle \Omega_2^{****}(x^{**}, c^{**}, d^{**}), \Omega_1^*(x^*, a_\alpha, b_\beta)\rangle \\
&= \lim_{\alpha}\lim_{\beta}\langle \Omega_1^{**}(\Omega_2^{****}(x^{**}, c^{**}, d^{**}), x^*, a_\alpha), b_\beta\rangle \\
&= \lim_{\alpha}\langle b^{**}, \Omega_1^{**}(\Omega_2^{****}(x^{**}, c^{**}, d^{**}), x^*, a_\alpha)\rangle \\
&= \lim_{\alpha}\langle \Omega_1^{***}(b^{**}, \Omega_2^{****}(x^{**}, c^{**}, d^{**}), x^*), a_\alpha\rangle \\
&= \langle a^{**}, \Omega_1^{***}(b^{**}, \Omega_2^{****}(x^{**}, c^{**}, d^{**}), x^*)\rangle \\
&= \langle \Omega_1^{****}(a^{**}, b^{**}, \Omega_2^{****}(x^{**}, c^{**}, d^{**})), x^*\rangle.
\end{aligned}
$$

as claimed. □

Example 2.1. Let A be a Banach algebra, for any $a, b \in A$ the symbol $[a, b] = ab - ba$ stands for multiplicative commutator of a and b. Let $M_{n\times n}(C)$ be the Banach algebra of all $n \times n$ matrices. We define

$$A = \{\begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{C})|\ u, v \in \mathbb{C}\}, \quad X = \{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in M_{2\times 2}(\mathbb{C})|\ x, y, z \in \mathbb{C}\}.$$

Now let $\Omega_1 : A \times A \times X \longrightarrow X$ to be the bounded tri-linear map given by

$$\Omega_1(a, b, x) = -[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, abx] \quad , \quad (a, b \in A, \quad x \in X).$$

For every $a = \begin{pmatrix} u_1 & v_1 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} u_2 & v_2 \\ 0 & 0 \end{pmatrix}, c = \begin{pmatrix} u_3 & v_3 \\ 0 & 0 \end{pmatrix}, d = \begin{pmatrix} u_4 & v_4 \\ 0 & 0 \end{pmatrix} \in A$ and $x =$

$\begin{pmatrix} x_1 & y_1 \\ 0 & z_1 \end{pmatrix} \in X$, we have

$$\Omega_1(\pi(a,b),\pi(c,d),x) = \Omega_1(\begin{pmatrix} u_1u_2 & u_1v_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_3u_4 & u_3v_4 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_1 & y_1 \\ 0 & z_1 \end{pmatrix})$$
$$= -[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_1u_2u_3u_4x_1 & u_1u_2u_3u_4y_1 + u_1u_2u_3v_4z_1 \\ 0 & 0 \end{pmatrix}]$$
$$= \begin{pmatrix} 0 & u_1u_2u_3u_4x_1 \\ 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & u_1u_2u_3u_4x_1 \\ 0 & 0 \end{pmatrix}$$
$$= -[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & u_1u_2u_3u_4x_1 \\ 0 & 0 \end{pmatrix}] = \Omega_1(\begin{pmatrix} u_1 & v_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_2 & v_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & u_3u_4x_1 \\ 0 & 0 \end{pmatrix})$$
$$= \Omega_1(\begin{pmatrix} u_1 & v_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_2 & v_2 \\ 0 & 0 \end{pmatrix}, (-\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & u_3u_4x_1 \\ 0 & 0 \end{pmatrix}))$$
$$= \Omega_1(\begin{pmatrix} u_1 & v_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_2 & v_2 \\ 0 & 0 \end{pmatrix}, -[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_3u_4x_1 & u_3u_4y_1 + u_3v_4z_1 \\ 0 & 0 \end{pmatrix}])$$
$$= \Omega_1(\begin{pmatrix} u_1 & v_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_2 & v_2 \\ 0 & 0 \end{pmatrix}, \Omega_1(\begin{pmatrix} u_3 & v_3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_4 & v_4 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_1 & y_1 \\ 0 & z_1 \end{pmatrix}))$$
$$= \Omega_1(a,b,\Omega_1(c,d,x)),$$

Therefore (Ω_1, X) is a left Banach $A-$module.

Theorem 2.2. *Let $a,b,c,d \in A$, $x^* \in X^*$, $x^{**} \in X^{**}$ and $b^{**}, c^{**} \in A^{**}$. Then*

1. *If (Ω_1, X) is a left Banach $A-$module, then*
$$\Omega_1^{***}(b^{**}, \Omega_1^{****}(c,d,x^{**}), x^*) = \pi^{**}(b^{**}, \Omega_1^{***}(\pi^{***}(c,d), x^{**}, x^*)),$$
2. *If (X, Ω_2) is a right Banach $A-$module, then*
$$\Omega_2^{r***r}(x^*, \Omega_2^{r****r}(x^{**}, a, b), c^{**}) = \pi^{r**}(c^{**}, \Omega_2^{r***r}(x^*, x^{**}, \pi^{***}(a,b)).$$

Proof. (1) Since the pair (Ω_1, X) is a left Banach $A-$module, thus for every $x \in X$ we have

$$\langle \Omega_1^*(x^*, \pi(a,b), \pi(c,d)), x\rangle = \langle x^*, \Omega_1(\pi(a,b), \pi(c,d), x)\rangle$$
$$= \langle x^*, \Omega_1(a,b,\Omega_1(c,d,x))\rangle = \langle \Omega_1^*(x^*,a,b), \Omega_1(c,d,x)\rangle$$
$$= \langle \Omega_1^*(\Omega_1^*(x^*,a,b),c,d), x\rangle.$$

Hence $\Omega_1^*(x^*, \pi(a,b), \pi(c,d)) = \Omega_1^*(\Omega_1^*(x^*,a,b),c,d)$, which implies that

$$\langle \pi^*(\Omega_1^{***}(\pi^{***}(c,d), x^{**}, x^*), a), b\rangle = \langle \Omega_1^{***}(\pi^{***}(c,d), x^{**}, x^*), \pi(a,b)\rangle$$
$$= \langle \pi^{***}(c,d), \Omega_1^{**}(x^{**}, x^*, \pi(a,b))\rangle = \langle c, \pi^{**}(d, \Omega_1^{**}(x^{**}, x^*, \pi(a,b)))\rangle$$
$$= \langle d, \pi^*(\Omega_1^{**}(x^{**}, x^*, \pi(a,b)), c)\rangle = \langle \Omega_1^{**}(x^{**}, x^*, \pi(a,b)), \pi(c,d)\rangle$$
$$= \langle x^{**}, \Omega_1^*(x^*, \pi(a,b), \pi(c,d))\rangle = \langle x^{**}, \Omega_1^*(\Omega_1^*(x^*,a,b),c,d)\rangle$$
$$= \langle \Omega_1^{**}(x^{**}, \Omega_1^*(x^*,a,b),c), d\rangle = \langle \Omega_1^{***}(d, x^{**}, \Omega_1^*(x^*,a,b)), c\rangle$$
$$= \langle \Omega_1^{***}(c,d,x^{**}), \Omega_1^*(x^*,a,b)\rangle = \langle \Omega_1^{**}(\Omega_1^{***}(c,d,x^{**}), x^*, a), b\rangle.$$

Thus $\pi^*(\Omega_1^{***}(\pi^{***}(c,d),x^{**},x^*),a) = \Omega_1^{**}(\Omega_1^{***}(c,d,x^{**}),x^*,a)$. Finally, we have

$$\begin{aligned}\langle \Omega_1^{***}(b^{**},\Omega_1^{****}(c,d,x^{**}),x^*),a\rangle &= \langle b^{**},\Omega_1^{**}(\Omega_1^{****}(c,d,x^{**}),x^*,a\rangle \\ &= \langle b^{**},\pi^*(\Omega_1^{***}(\pi^{***}(c,d),x^{**},x^*),a)\rangle \\ &= \langle \pi^{**}(b^{**},\Omega_1^{***}(\pi^{***}(c,d),x^{**},x^*)),a\rangle.\end{aligned}$$

A similar argument applies for (2). □

3. Topological centers of bounded tri-linear maps

In this section, we shall investigate the topological centers of bounded tri-linear maps. The main definition of this section is as follows.

Definition 3.1. Let $f : X \times Y \times Z \longrightarrow W$ be a bounded tri-linear map. We define the topological centers of f by

$Z_l^1(f) = \{x^{**} \in X^{**} |\ y^{**} \longrightarrow f^{****}(x^{**},y^{**},z^{**})$ *is weak*$^* -$ *to* $-$ *weak*$^* -$ *continuous*$\}$,

$Z_l^2(f) = \{x^{**} \in X^{**} |\ z^{**} \longrightarrow f^{****}(x^{**},y^{**},z^{**})$ *is weak*$^* -$ *to* $-$ *weak*$^* -$ *continuous*$\}$,

$Z_r^1(f) = \{z^{**} \in Z^{**} |\ x^{**} \longrightarrow f^{r*****r}(x^{**},y^{**},z^{**})$ *is weak*$^* -$ *to* $-$ *weak*$^* -$ *continuous*$\}$,

$Z_r^2(f) = \{z^{**} \in Z^{**} |\ y^{**} \longrightarrow f^{r*****r}(x^{**},y^{**},z^{**})$ *is weak*$^* -$ *to* $-$ *weak*$^* -$ *continuous*$\}$,

$Z_c^1(f) = \{y^{**} \in Y^{**} |\ x^{**} \longrightarrow f^{r*****r}(x^{**},y^{**},z^{**})$ *is weak*$^* -$ *to* $-$ *weak*$^* -$ *continuous*$\}$.

$Z_c^2(f) = \{y^{**} \in Y^{**} |\ z^{**} \longrightarrow f^{****}(x^{**},y^{**},z^{**})$ *is weak*$^* -$ *to* $-$ *weak*$^* -$ *continuous*$\}$.

Lemma 3.1. *For a bounded tri-linear map $f : X \times Y \times Z \longrightarrow W$, we have*

1. *The map f^{****} is the extension of f such that $x^{**} \longrightarrow f^{****}(x^{**},y^{**},z^{**})$ is weak$^*-$weak* continuous for each $y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$.*
2. *The map f^{****} is the extension of f such that $y^{**} \longrightarrow f^{****}(x,y^{**},z^{**})$ is weak$^*-$weak* continuous for each $x \in X$ and $z^{**} \in Z^{**}$.*
3. *The map f^{****} is the extension of f such that $z^{**} \longrightarrow f^{****}(x,y,z^{**})$ is weak$^*-$weak* continuous for each $x \in X$ and $y \in Y$.*
4. *The map $f^{r*****r}$ is the extension of f such that $z^{**} \longrightarrow f^{r*****r}(x^{**},y^{**},z^{**})$ is weak$^*-$weak* continuous for each $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$.*
5. *The map $f^{r*****r}$ is the extension of f such that $x^{**} \longrightarrow f^{r*****r}(x^{**},y,z)$ is weak$^*-$weak* continuous for each $y \in Y$ and $z \in Z$.*

6. *The map* $f^{r*****r}$ *is the extension of* f *such that* $y^{**} \longrightarrow f^{r*****r}(x^{**}, y^{**}, z)$ *is* weak*−weak* *continuous for each* $x^{**} \in X^{**}$ *and* $z \in Z$.

Proof. See [19] and [20]. □

As immediate consequences, we give the next Theorem.

Theorem 3.1. *If* $f : X \times Y \times Z \longrightarrow W$ *is a bounded tri-linear map, then* $X \subseteq Z_l^1(f)$ *and* $Z \subseteq Z_r^2(f)$.

The mapping f^{****} is the extension of f such that $x^{**} \longrightarrow f^{****}(x^{**}, y^{**}, z^{**})$ from X^{**} into W^{**} is weak$^*-$ to $-$ weak* continuous for every $y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$, hence for first right topological center of f we have

$Z_r^1(f) \supseteq \{z^{**} \in Z^{**} | f^{r*****r}(x^{**}, y^{**}, z^{**}) = f^{****}(x^{**}, y^{**}, z^{**}),\ for\ every\ x^{**} \in X^{**}, y^{**} \in Y^{**}\}$.

The mapping $f^{r*****r}$ is the extension of f such that $z^{**} \longrightarrow f^{r*****r}(x^{**}, y^{**}, z^{**})$ from Z^{**} into W^{**} is weak$^*-$ to $-$ weak* continuous for every $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$, hence for second left topological center of f we have

$Z_l^2(f) \supseteq \{x^{**} \in X^{**} | f^{r*****r}(x^{**}, y^{**}, z^{**}) = f^{****}(x^{**}, y^{**}, z^{**}),\ for\ every\ y^{**} \in Y^{**}, z^{**} \in Z^{**}\}$.

Example 3.1. Let G be a finite locally compact Hausdorff group. Then

$$f : L^1(G) \times L^1(G) \times L^1(G) \longrightarrow L^1(G)$$

defined by $f(k, g, h) = k * g * h$, is regular for every k, g and $h \in L^1(G)$. So $L^1(G) \subseteq Z_r^1(f)$.

Theorem 3.2. *Let* A *be a Banach algebra. Then*

1. *If* (Ω_1, X) *is a left Banach* $A-$*module and* $\Omega_1^{***}, \pi^{***}(A, A)$ *are factors, then* $Z_l^1(\Omega_1) \subseteq Z_l(\pi)$.

2. *If* (X, Ω_2) *is a right Banach* $A-$*module and* $\Omega_2^{r***r}, \pi^{***}(A, A)$ *are factors, then* $Z_r^2(\Omega_2) \subseteq Z_r(\pi)$.

Proof. We prove only (1), the other one has the same argument. Let $a^{**} \in Z_l^1(\Omega_1)$, we show that $a^{**} \in Z_l(\pi)$. Let $\{b_\alpha^{**}\}$ be a net in A^{**} which converges to $b^{**} \in A^{**}$ in the w^*-topologies. We must show that $\pi^{***}(a^{**}, b_\alpha^{**})$ converges to $\pi^{***}(a^{**}, b^{**})$ in the w^*-topologies. Let $a^* \in A^*$, since Ω_1^{***} factors, so there exists $x^* \in X^*, x^{**} \in X^{**}$ and $c^{**} \in A^{**}$ such that $a^* = \Omega_1^{***}(c^{**}, x^{**}, x^*)$. In the other hands $\pi^{***}(A, A)$ factors, thus there exists $c, d \in A$ such that $\pi^{***}(c, d) = c^{**}$. Because $a^{**} \in Z_l^1(\Omega_1)$ thus $\Omega_1^{****}(a^{**}, b_\alpha^{**}, x^{**})$ converges to $\Omega_1^{****}(a^{**}, b^{**}, x^{**})$ in the w^*-topologies.

In partiqular $\Omega_1^{****}(a^{**}, b_\alpha^{**}, \Omega_1^{****}(c, d, x^{**}))$ converges to $\Omega_1^{****}(a^{**}, b^{**}, \Omega_1^{****}(c, d, x^{**}))$ in the w^*-topologies. Now by Theorem 2.2, we have

$$\begin{aligned}
\lim_\alpha \langle \pi^{***}(a^{**}, b_\alpha^{**}), a^*\rangle &= \lim_\alpha \langle \pi^{***}(a^{**}, b_\alpha^{**}), \Omega_1^{***}(c^{**}, x^{**}, x^*)\rangle \\
&= \lim_\alpha \langle \pi^{***}(a^{**}, b_\alpha^{**}), \Omega_1^{***}(\pi^{***}(c, d), x^{**}, x^*)\rangle \\
&= \lim_\alpha \langle a^{**}, \pi^{**}(b_\alpha^{**}, \Omega_1^{***}(\pi^{***}(c, d), x^{**}, x^*))\rangle \\
&= \lim_\alpha \langle a^{**}, \Omega_1^{***}(b_\alpha^{**}, \Omega_1^{****}(c, d, x^{**}), x^*)\rangle \\
&= \lim_\alpha \langle \Omega_1^{****}(a^{**}, b_\alpha^{**}, \Omega_1^{****}(c, d, x^{**}), x^*\rangle \\
&= \langle \Omega_1^{****}(a^{**}, b^{**}, \Omega_1^{****}(c, d, x^{**}), x^*\rangle \\
&= \langle a^{**}, \Omega_1^{***}(b^{**}, \Omega_1^{****}(c, d, x^{**}), x^*)\rangle \\
&= \langle a^{**}, \pi^{**}(b^{**}, \Omega_1^{***}(\pi^{***}(c, d), x^{**}, x^*))\rangle \\
&= \langle a^{**}, \pi^{**}(b^{**}, \Omega_1^{***}(c^{**}, x^{**}, x^*))\rangle \\
&= \langle a^{**}, \pi^{**}(b^{**}, a^*)\rangle \\
&= \langle \pi^{***}(a^{**}, b^{**}), a^*\rangle.
\end{aligned}$$

Therefore $\pi^{***}(a^{**}, b_\alpha^{**})$ converges to $\pi^{***}(a^{**}, b^{**})$ in the w^*-topologies, as required. $\square$

Theorem 3.3. *Let A be a Banach algebra and $\Omega : A \times A \times A \longrightarrow A$ be a bounded tri-linear mapping. Then for every $a \in A, a^* \in A^*$ and $a^{**} \in A^{**}$,*

1. *If A has a bounded right approximate identity and bounded linear map $T : A^* \longrightarrow A^*$ given by $T(a^*) = \pi^{**}(a^{**}, a^*)$ is weakly compactenss, then Ω is regular.*
2. *If A has a bounded left approximate identity and bounded linear map $T : A \longrightarrow A^*$ given by $T(a) = \pi^{r*r*}(a^{**}, a)$ is weakly compactenss, then Ω is regular.*

Proof. We only prove (1). Let T be weakly compact, then $T^{**}(A^{***}) \subseteq A^*$. On the other hand, a direct verification reveals that $T^{**}(A^{***}) = \pi^{*****}(A^{**}, A^{***})$. Thus $\pi^{*****}(A^*, A^{***}) \subseteq A^*$. Now let $a^{**}, b^{**} \in A^{**}$, $a^{***} \in A^{***}$ and let $\{a_\alpha\}$, $\{a_\beta^*\}$ be nets in A and A^* which convergence to a^{**}, a^{***} in the w^*-topologies, respectively. Then we have

$$\begin{aligned}
\langle \pi^{*r****r}(a^{***}, a^{**}), b^{**}\rangle &= \langle \pi^{*r****}(a^{**}, a^{***}), b^{**}\rangle = \langle a^{**}, \pi^{*r***}(a^{***}, b^{**})\rangle \\
&= \lim_\alpha \langle \pi^{*r**}(a^{***}, b^{**}), a_\alpha\rangle = \lim_\alpha \langle a^{***}, \pi^{*r*}(b^{**}, a_\alpha)\rangle \\
&= \lim_\alpha \lim_\beta \langle \pi^{*r*}(b^{**}, a_\alpha), a_\beta^*\rangle = \lim_\alpha \lim_\beta \langle b^{**}, \pi^{*r}(a_\alpha, a_\beta^*)\rangle \\
&= \lim_\alpha \lim_\beta \langle b^{**}, \pi^*(a_\beta^*, a_\alpha)\rangle = \lim_\alpha \lim_\beta \langle \pi^{**}(b^{**}, a_\beta^*), a_\alpha\rangle \\
&= \lim_\alpha \lim_\beta \langle \pi^{***}(a_\alpha, b^{**}), a_\beta^*\rangle = \lim_\alpha \langle a^{***}, \pi^{***}(a_\alpha, b^{**})\rangle \\
&= \lim_\alpha \langle \pi^{****}(a^{***}, a_\alpha), b^{**}\rangle = \lim_\alpha \langle \pi^{*****}(b^{**}, a^{***}), a_\alpha\rangle \\
&= \langle a^{**}, \pi^{*****}(b^{**}, a^{***})\rangle = \langle \pi^{****}(a^{***}, a^{**}), b^{**}\rangle.
\end{aligned}$$

Therefore π^* is Arens regular. It follows that A is reflexive, see [8, Theorem 2.1]. Thus Ω is regular. □

4. Factors of bounded tri-linear mapping

We commence with the following definition.

Definition 4.1. Let X, Y, Z, S_1, S_2 and S_3 be normed spaces, $f : X \times Y \times Z \longrightarrow W$ and $g : S_1 \times S_2 \times S_3 \longrightarrow W$ be bounded tri-linear mappings. Then we say that f factors through g by bounded linear mappings $h_1 : X \longrightarrow S_1$, $h_2 : Y \longrightarrow S_2$ and $h_3 : Z \longrightarrow S_3$, if $f(x, y, z) = g(h_1(x), h_2(y), h_3(z))$.

The following theorem gives some necessary and sufficient conditions under which for factorization of the first and second extension of a bouneded tri-linear mappings.

Theorem 4.1. *Let $f : X \times Y \times Z \longrightarrow W$ and $g : S_1 \times S_2 \times S_3 \longrightarrow W$ be bounded tri-linear mapping. Then*

1. *The map f factors through g if and only if f^{****} factors through g^{****},*
2. *The map f factors through g if and only if f^{r****r} factors through g^{r****r}.*

Proof. (1) Let f factor through g by bounded linear mappings $h_1 : X \longrightarrow S_1, h_2 : Y \longrightarrow S_2$ and $h_3 : Z \longrightarrow S_3$, then $f(x, y, z) = g(h_1(x), h_2(y), h_3(z))$ for every $x \in X, y \in Y$ and $z \in Z$. Let $\{x_\alpha\}, \{y_\beta\}$ and $\{z_\gamma\}$ be nets in X, Y and Z which converge to $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$ in the w^*-topologies, respectively. Then for every $w^* \in W^*$ we have

$$\begin{aligned}
\langle f^{****}(x^{**}, y^{**}, z^{**}), w^*\rangle &= \lim_\alpha \lim_\beta \lim_\gamma \langle w^*, f(x_\alpha, y_\beta, z_\gamma)\rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle w^*, g(h_1(x_\alpha), h_2(y_\beta), h_3(z_\gamma))\rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle g^*(w^*, h_1(x_\alpha), h_2(y_\beta)), h_3(z_\gamma)\rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle h_3^*(g^*(w^*, h_1(x_\alpha), h_2(y_\beta))), z_\gamma\rangle \\
&= \lim_\alpha \lim_\beta \langle z^{**}, h_3^*(g^*(w^*, h_1(x_\alpha), h_2(y_\beta)))\rangle \\
&= \lim_\alpha \lim_\beta \langle h_3^{**}(z^{**}), g^*(w^*, h_1(x_\alpha), h_2(y_\beta))\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \lim_\alpha \lim_\beta \langle g^{**}(h_3^{**}(z^{**}), w^*, h_1(x_\alpha)), h_2(y_\beta)\rangle \\
&= \lim_\alpha \lim_\beta \langle h_2^*(g^{**}(h_3^{**}(z^{**}), w^*, h_1(x_\alpha))), y_\beta\rangle \\
&= \lim_\alpha \langle y^{**}, h_2^*(g^{**}(h_3^{**}(z^{**}), w^*, h_1(x_\alpha)))\rangle \\
&= \lim_\alpha \langle h_2^{**}(y^{**}), g^{**}(h_3^{**}(z^{**}), w^*, h_1(x_\alpha))\rangle \\
&= \lim_\alpha \langle g^{***}(h_2^{**}(y^{**}), h_3^{**}(z^{**}), w^*), h_1(x_\alpha)\rangle \\
&= \lim_\alpha \langle h_1^*(g^{***}(h_2^{**}(y^{**}), h_3^{**}(z^{**}), w^*)), x_\alpha\rangle \\
&= \langle x^{**}, h_1^*(g^{***}(h_2^{**}(y^{**}), h_3^{**}(z^{**}), w^*))\rangle \\
&= \langle h_1^{**}(x^{**}), g^{***}(h_2^{**}(y^{**}), h_3^{**}(z^{**}), w^*)\rangle \\
&= \langle g^{****}(h_1^{**}(x^{**}), h_2^{**}(y^{**}), h_3^{**}(z^{**})), w^*\rangle.
\end{aligned}
$$

Therefore f^{****} factors through g^{****}.

Conversely, suppose that f^{****} factors through g^{****}, thus

$$f^{****}(x^{**}, y^{**}, z^{**}) = g^{****}(h_1^{**}(x^{**}), h_2^{**}(y^{**}), h_3^{**}(z^{**})),$$

in particular, for $x \in X, y \in Y$ and $z \in Z$ we have

$$f^{****}(x, y, z) = g^{****}(h_1^{**}(x), h_2^{**}(y), h_3^{**}(z)).$$

Then for every $w^* \in W^*$ we have

$$
\begin{aligned}
&\langle w^*, f(x, y, z)\rangle = \langle f^*(w^*, x, y), z\rangle \\
&= \langle f^{**}(z, w^*, x), y\rangle = \langle f^{***}(y, z, w^*), x\rangle \\
&= \langle f^{****}(x, y, z), w^*\rangle = \langle g^{****}(h_1^{**}(x), h_2^{**}(y), h_3^{**}(z)), w^*\rangle \\
&= \langle h_1^{**}(x), g^{***}(h_2^{**}(y), h_3^{**}(z), w^*)\rangle = \langle x, h_1^*(g^{***}(h_2^{**}(y), h_3^{**}(z), w^*))\rangle \\
&= \langle g^{***}(h_2^{**}(y), h_3^{**}(z), w^*), h_1(x)\rangle = \langle h_2^{**}(y), g^{**}(h_3^{**}(z), w^*, h_1(x))\rangle \\
&= \langle y, h_2^*(g^{**}(h_3^{**}(z), w^*, h_1(x)))\rangle = \langle g^{**}(h_3^{**}(z), w^*, h_1(x)), h_2(y)\rangle \\
&= \langle h_3^{**}(z), g^*(w^*, h_1(x), h_2(y))\rangle = \langle z, h_3^*(g^*(w^*, h_1(x), h_2(y)))\rangle \\
&= \langle g^*(w^*, h_1(x), h_2(y)), h_3(z)\rangle = \langle w^*, g(h_1(x), h_2(y)), h_3(z))\rangle.
\end{aligned}
$$

It follows that f factors through g and proof follows.

(2) The proof is similar to (1). □

Corollary 4.1. *Let $f : X \times Y \times Z \longrightarrow W$ and $g : S_1 \times S_2 \times S_3 \longrightarrow W$ be bounded tri-linear map and let f factors through g. If g is regular then f is also regular.*

Proof. Let g be regular then $g^{****} = g^{r*****r}$. Since the f factors through g then for every $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$ we have

$$
\begin{aligned}
f^{****}(x^{**}, y^{**}, z^{**}) &= g^{****}(h_1^{**}(x^{**}), h_2^{**}(y^{**}), h_3^{**}(z^{**})) \\
&= g^{r*****r}(h_1^{**}(x^{**}), h_2^{**}(y^{**}), h_3^{**}(z^{**})) \\
&= f^{r*****r}(x^{**}, y^{**}, z^{**}).
\end{aligned}
$$

Therefore $f^{****} = f^{r*****r}$, as claimed. □

5. Approximate identity and Factorization properties

Let X be a Banach space, A and B be Banach algebras with bounded left approximate identitis $\{e_\alpha\}$ and $\{e_\beta\}$, respactively. Then a bounded tri-linear mapping $K_1 : A \times B \times X \longrightarrow X$ is said to be left approximately unital if

$$w^* - \lim_\beta w^* - \lim_\alpha K_1(e_\alpha, e_\beta, x) = x,$$

and K_1 is said left unital if there exists $e_1 \in A$ and $e_2 \in B$ such that $K_1(e_1, e_2, x) = x$, for every $x \in X$. Similarly, bounded tri-linear mapping $K_2 : X \times B \times A \longrightarrow X$ is said to be right approximately unital if

$$w^* - \lim_\beta w^* - \lim_\alpha K_1(x, e_\beta, e_\alpha) = x,$$

and K_2 is also said to be right unital if $K_2(x, e_2, e_1) = x$.

Lemma 5.1. *Let X be a Banach space, A and B be Banach algebras. Then bounded tri-linear mapping*

1. *$K_1 : A \times B \times X \longrightarrow X$ is left approximately unital if and only if $K_1^{r*****r} : A^{**} \times B^{**} \times X^{**} \longrightarrow X^{**}$ is left unital.*
2. *$K_2 : X \times B \times A \longrightarrow X$ is right approximately unital if and only if $K_2^{****} : X^{**} \times B^{**} \times A^{**} \longrightarrow X^{**}$ is right unital.*

Proof. We prove only (1), the other part has the same argument. Let K_1 be a left approximately unital. Thus there exists bounded left approximate identitys $\{e_\alpha\} \subseteq A$ and $\{e_\beta\} \subseteq B$ such that

$$w^* - \lim_\beta w^* - \lim_\alpha K_1(e_\alpha, e_\beta, x) = x,$$

for every $x \in X$. Let $\{e_\alpha\}$ and $\{e_\beta\}$ converge to $e_1^{**} \in A^{**}$ and $e_2^{**} \in B^{**}$ in the w^*-topologies, respectively. On the other hand, for every $x^{**} \in X^{**}$, let $\{x_\gamma\} \subseteq X$ converge to x^{**} in the w^*-topologies, then we have

$$\begin{aligned}
&\langle K_1^{r*****r}(e_1^{**}, e_2^{**}, x^{**}), x^*\rangle = \langle K_1^{r****}(x^{**}, e_2^{**}, e_1^{**}), x^*\rangle \\
&= \langle x^{**}, K_1^{r***}(e_2^{**}, e_1^{**}, x^*)\rangle = \lim_\gamma \langle K_1^{r***}(e_2^{**}, e_1^{**}, x^*), x_\gamma\rangle \\
&= \lim_\gamma \langle e_2^{**}, K_1^{r**}(e_1^{**}, x^*, x_\gamma)\rangle = \lim_\gamma \lim_\beta \langle K_1^{r**}(e_1^{**}, x^*, x_\gamma), e_\beta\rangle \\
&= \lim_\gamma \lim_\beta \langle e_1^{**}, K_1^{r*}(x^*, x_\gamma, e_\beta)\rangle = \lim_\gamma \lim_\beta \lim_\alpha \langle K_1^{r*}(x^*, x_\gamma, e_\beta), e_\alpha\rangle \\
&= \lim_\gamma \lim_\beta \lim_\alpha \langle x^*, K_1^{r}(x_\gamma, e_\beta, e_\alpha)\rangle = \lim_\gamma \lim_\beta \lim_\alpha \langle x^*, K_1(e_\alpha, e_\beta, x_\gamma)\rangle \\
&= \lim_\gamma \langle x^*, x_\gamma\rangle = \langle x^{**}, x^*\rangle.
\end{aligned}$$

Therefore $K_1^{r*****r}(e_1^{**}, e_2^{**}, x^{**}) = x^{**}$. It follows that $K_1^{r*****r}$ is left unital.

Conversely, suppose that $K_1^{r*****r}$ is left unital. So there exists $e_1^{**} \in A^{**}$ and $e_2^{**} \in b^{**}$ such that $K_1^{r*****r}(e_1^{**}, e_2^{**}, x^{**}) = x^{**}$ for every $x^{**} \in X^{**}$. Now let $\{e_\alpha\}, \{e_\beta\}$ and $\{x_\gamma\}$ be nets in A, B and X converging to e_1^{**}, e_2^{**} and x^{**} in the w^*-topologies, respectively. Thus

$$\begin{aligned} w^* - \lim_\gamma w^* - \lim_\beta w^* - \lim_\alpha K_1(e_\alpha, e_\beta, x_\gamma) &= K_1^{r*****r}(e_1^{**}, e_2^{**}, x^{**}) \\ &= x^{**} = w^* - \lim_\gamma x_\gamma. \end{aligned}$$

Therefore K_1 is left approximately unital and proof follows. □

Remark 5.1. It should be remarked that in contrast to the situation occurring for $K_1^{r*****r}$ and K_2^{****} in the above lemma, K_1^{****} and $K_2^{r*****r}$ are not necessarily left and right unital respectively, in general.

Theorem 5.1. *Suppose X, S are Banach spaces and A, B are Banach algebras.*

1. *Let $K_1 : A \times B \times X \longrightarrow X$ be left approximately unital and factors through $g_r : A \times B \times S \longrightarrow X$ from rigth by $h : X \longrightarrow S$. If h is weakly compactenss, then X is reflexive.*
2. *Let $K_2 : X \times B \times A \longrightarrow X$ be right approximately unital and factors through $g_l : S \times B \times A \longrightarrow X$ from left by $h : X \longrightarrow S$. If h is weakly compactenss, then X is reflexive.*

Proof. We only give the proof for (1). Since K_1 is left approximately unital, there exists $e_1^{**} \in A^{**}$ and $e_2^{**} \in B^{**}$ such that

$$K_1^{r*****r}(e_1^{**}, e_2^{**}, x^{**}) = x^{**},$$

for every $x^{**} \in X^{**}$. On the other hand, the bounded tri-linear mapping K_1 factors through g_r from right, so by Theorem 4.1, $K_1^{r*****r}$ factors through $g_r^{r*****r}$ from right. Thus

$$K_1^{r*****r}(e_1^{**}, e_2^{**}, x^{**}) = g_r^{r*****r}(e_1^{**}, e_2^{**}, h_3^{**}(x^{**})).$$

Then for every $x^{***} \in X^{***}$ we have

$$\begin{aligned} \langle x^{***}, x^{**} \rangle &= \langle x^{***}, K_1^{r*****r}(e_1^{**}, e_2^{**}, x^{**}) \rangle \\ &= \langle x^{***}, g_r^{r*****r}(e_1^{**}, e_2^{**}, h^{**}(x^{**})) \rangle \\ &= \langle g_r^{r*****r*}(x^{***}, e_1^{**}, e_2^{**}), h^{**}(x^{**}) \rangle \\ &= \langle h^{***}(g_r^{r*****r*}(x^{***}, e_1^{**}, e_2^{**})), x^{**} \rangle. \end{aligned}$$

Therefore $x^{***} = h^{***}(g_r^{r*****r*}(x^{***}, e_1^{**}, e_2^{**}))$. The weak compactness of h implies that $h^{***}(S^{***}) \subseteq X^*$. In particular $h^{***}(g_r^{r*****r*}(x^{***}, e_1^{**}, e_2^{**})) \subseteq X^*$, that is, X^* is reflexive. So X is reflexive. □

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