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EKELAND'S VARIATIONAL PRINCIPLE IN S^{JS}-METRIC SPACES

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Abstract. We prove Ekeland's variational principle in S^{JS} - metric spaces. A generalization of Caristi fixed point theorem on S^{JS} - metric spaces is obtained as a consequence.
Keywords:Ekeland's variational principle; S^{JS}- metric space; fixed point

1. Introduction

In his classic paper Ekeland [7] proved a theorem (Ekeland's variational principle) that asserts that there exists nearly optimal solutions to some optimization problems. Ekeland's variational principle can be applied when the lower level set of a minimization problems is not compact, so that the Bolzano–Weierstrass theorem cannot be used. Ekeland's principle relies on Cantor intersection theorem and axiom of choice. Ekeland's principle also leads to an elegant proof of the famous Caristi fixed point theorem [5]. For further generalizations and applications of Ekeland's variational principle we refere to [2, 8, 9, 11] and their references. Recently Beg et al. [1, 12, 13] introduced a very general notion of S^{JS} - metric spaces (see preliminaries) which does not satisfy the triangle inequality and symmetry, and obtained several interesting results with examples. In fact b - metric spaces [6], S_b- metric spaces [14], JS- metric spaces [10], and partial metric spaces [4] are special cases

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of S^{JS} - metric spaces. The aim of this paper is to prove a variant of Ekeland's variational principle in S^{JS} - metric spaces and then derive Caristi fixed point theorem as an application. The results above generalize/extend several results from the existing literature.

2. Preliminaries

In this section, we first give the notion of S^{JS}- metric space (X, J), due to [1], some notations and terminology and a lemma to use in next section.

Let X be a nonempty set and $J : X^3 \to [0, \infty]$ be a function. We define the set

$$S(J, X, x) = \{\{x_n\} \subset X : \lim_{n\to\infty} J(x, x, x_n) = 0\},$$

for all $x \in X$. If J satisfies

(*i*) $J(x, y, z) = 0$ implies $x = y = z$ for any $x, y, z \in X$;

(*ii*) there exists some $s > 0$ such that for any $(x, y, z) \in X^3$ and $\{z_n\} \in S(J, X, z)$, we have

$$J(x, y, z) \leq s \limsup_{n\to\infty}(J(x, x, z_n) + J(y, y, z_n)),$$

then the pair (X, J) is called an S^{JS}- metric space (with coefficient s). Several known examples of S^{JS}- metric spaces are given in [1] and [13], we give another examples of S^{JS}- metric spaces in the below.

Example 2.1. Let $X = \overline{\mathbb{R}}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = \exp(|x|) + \exp(|y|) + \exp(|z|) - 3$ for all $x, y, z \in X$, then clearly (J_1) is satisfied. For any $z \neq 0$, $S(J, X, z) = \emptyset$. For any $\{z_n\} \in S(J, X, 0)$, we see that

$$J(x, y, 0) \leq h \limsup_{n\to\infty}(J(x, x, z_n) + J(y, y, z_n)),$$

where $h \geqslant \frac{1}{2}$, for all $x, y \in X$. Then condition (J_2) is also satisfied. So J is an S^{JS}-metric. It is a non-symmetric S^{JS}-metric space.

Example 2.2. Let $X = \overline{\mathbb{R}}$ and $J : X^3 \to [0, \infty]$ be defined by $J(x, y, z) = |x-y|+|y|+2|z|$ for all $x, y, z \in X$, then clearly (J_1) is satisfied. For any $z \neq 0$, $S(J, X, z) = \emptyset$. If $z = 0$ then for any sequence $\{z_n\} \in S(J, X, 0)$, we get

$$J(x, y, 0) = |x - y| + |y| \leqslant |x| + 2|y| \leqslant 2(|x| + |y|) = 2 \limsup_{n\to\infty}(J(x, x, z_n) + J(y, y, z_n)),$$

for all $x, y \in X$. Therefore, the condition (J_2) is satisfied and J is an S^{JS}-metric on X. It is a non-symmetric S^{JS}-metric space.

In an S^{JS}- metric space (X, J), a sequence $\{x_n\} \subset X$ is said to be convergent to an element $x \in X$ if $\{x_n\} \in S(J, X, x)$. A sequence $\{x_n\} \subset X$ is said to be Cauchy if $\lim_{n,m\to\infty} J(x_n, x_n, x_m) = 0$.

Space (X, J) is said to be complete if every Cauchy sequence in X is convergent. Open ball of center $x \in X$ and radius $r > 0$ in X is defined as follows:

$$B_J(x, r) = \{y \in X : J(x, x, y) < r\}.$$

A nonempty subset U of X, with the property that for any $x \in U$ there exists $r > 0$ such that $B_J(x, r) \subset U$ is called an open set. A subset B of X is called closed if B^c is open.

Lemma 2.1. *[1][Cantor's Intersection Theorem] Every complete S^{JS}- metric space has Cantor's intersection property.*

3. Ekeland's variational principle

Definition 3.1. In an S^{JS}-metric space (X, J), a mapping $\psi : X \to \overline{\mathbb{R}}$ is said to be lower semi-continuous at $t_0 \in X$ if for any $\epsilon > 0$ there exits some $\delta_\epsilon > 0$ such that $\psi(t_0) < \psi(t) + \epsilon$ for all $t \in B_J(t_0, \delta_\epsilon)$.

Definition 3.2. Let (X, J) be an S^{JS}-metric space and $\{A_n\}$ be a decreasing sequence of nonempty subsets of X. Then $\{A_n\}$ is said to have vanishing diameter property ($vd-$property) if for each $i \in \mathbb{N}$ there exists some fixed $a_i \in A_i$ such that $J(x, x, a_i) \leq J(a_i, a_i, a_i) + r_i$ for all $x \in A_i$, where $\{r_i\} \subset \mathbb{R}_+$ with $r_i \to 0$ as $i \to \infty$.

Definition 3.3. An S^{JS}-metric space (X, J) is said to have vanishing diameter property if for any decreasing sequence of nonempty subsets $\{A_n\}$ of X with $vd-$property we have $diam(A_n) \to 0$ as $n \to \infty$.

We now establish Ekeland's variational principle in an S^{JS}-metric space. Let us denote $d_J(x, y) = J(x, x, y)$ for all $x, y \in X$.

Theorem 3.1. *Let (X, J) be a complete S^{JS}-metric space with coefficient $s > 1$, such that d_J is continuous in both variables, $\sup\{J(x, x, x) : x \in X\} < \infty$ and X has vanishing diameter property. Now let, $f : X \to \overline{\mathbb{R}}$ be a lower semi-continuous, proper and lower bounded mapping. Then for every $x_0 \in X$ and $\epsilon > 0$ with*

$$f(x_0) \leq \inf_{x \in X} f(x) + \epsilon \tag{3.1}$$

there exists a sequence $\{x_n\} \subset X$ and $x_\epsilon \in X$ such that:

(i) $x_n \to x_\epsilon$ as $n \to \infty$,

(ii) For all $n \geq 1$,

$$J(x_\epsilon, x_\epsilon, x_n) - J(x_n, x_n, x_n) \leq \frac{\epsilon}{2^n}$$

(iii) For all $x \neq x_\epsilon$,

$$f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x, x, x_n) > f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n)$$

(iv)

$$\begin{aligned} f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n) &\leq f(x_0) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_n, x_n, x_n) \\ &\leq \inf_{x \in X} f(x) + \epsilon + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_n, x_n, x_n). \end{aligned}$$

Proof. Consider the set

$$S_f(x_0) = \{x \in X : f(x) + d_J(x, x_0) \leq f(x_0) + d_J(x_0, x_0)\}.$$

Since $x_0 \in S_f(x_0)$ then $S_f(x_0)$ is nonempty. Let $\{z_n\} \subset S_f(x_0)$ be such that $\{z_n\}$ converges to some $z \in X$. Then $f(z_n) + d_J(z_n, x_0) \leq f(x_0) + d_J(x_0, x_0)$ for all $n \in \mathbb{N}$. Now f is lower semi-continuous at $z \in X$, so for any $\epsilon_1 > 0$, $f(z) < f(t) + \frac{\epsilon_1}{2}$ for all $t \in B_J(z, \delta_{\epsilon_1})$ for $\delta_{\epsilon_1} > 0$. Also $\{z_n\}$ converges to some z, so there exists $N_1 \geq 1$ such that $z_n \in B_J(z, \delta_{\epsilon_1})$ for all $n \geq N_1$. Therefore $f(z) < f(z_n) + \frac{\epsilon_1}{2}$ for all $n \geq N_1$. Now continuity of d_J implies that $d_J(z_n, x_0) \to d_J(z, x_0)$ as $n \to \infty$. Thus for all $n \geq N_2$

$$d_J(z, x_0) - \frac{\epsilon_1}{2} < d_J(z_n, x_0) < d_J(z, x_0) + \frac{\epsilon_1}{2}.$$

Therefore, for all $n \geq N = \max\{N_1, N_2\}$ we get,

$$\begin{aligned} f(z) + d_J(z, x_0) &< f(z_n) + d_J(z_n, x_0) + \epsilon_1 \forall n \geqslant N \\ &\leq f(x_0) + d_J(x_0, x_0) + \epsilon_1. \end{aligned} \tag{3.2}$$

Since $\epsilon_1 > 0$ is arbitrary, thus $f(z) + d_J(z, x_0) \leq f(x_0) + d_J(x_0, x_0)$. Therefore $z \in S_f(x_0)$. Hence $S_f(x_0)$ is closed. Also for any $y \in S_f(x_0)$ we get

$$\begin{aligned} d_J(y, x_0) - d_J(x_0, x_0) &\leq f(x_0) - f(y) \\ &\leq f(x_0) - \inf_{x \in X} f(x) \leq \epsilon. \end{aligned} \tag{3.3}$$

We choose $x_1 \in S_f(x_0)$ such that $f(x_1) + d_J(x_1, x_0) \leq \inf_{x \in S_f(x_0)}\{f(x) + d_J(x, x_0)\} + \frac{\epsilon}{2s}$ and let

$$S_f(x_1) = \{x \in X : f(x) + d_J(x, x_0) + \frac{1}{s} d_J(x, x_1) \leq f(x_1) + d_J(x_1, x_0) + \frac{1}{s} d_J(x_1, x_1)\}. \tag{3.4}$$

Thus $x_1 \in S_f(x_1)$ and in a similar way as above we can prove that $S_f(x_1)$ is also closed.

Inductively, we can suppose that $x_{n-1} \in S_f(x_{n-2})$ (for $n > 2$) was already chosen and we consider

$$S_f(x_{n-1}) \quad = \quad \{x \in S_f(x_{n-2}) : f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x, x_i) \leq f(x_{n-1}) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x_{n-1}, x_i)\}. \tag{3.5}$$

Let us choose $x_n \in S_f(x_{n-1})$ such that

$$f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x_n, x_i) \leq \inf_{x \in S_f(x_{n-1})} \{f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x, x_i)\} + \frac{\epsilon}{2^n s^n}$$

and we define the set

$$S_f(x_n) \quad = \quad \{x \in S_f(x_{n-1}) : f(x) + \sum_{i=0}^{n} \frac{1}{s^i} d_J(x, x_i) \leq f(x_n) + \sum_{i=0}^{n} \frac{1}{s^i} d_J(x_n, x_i)\}. \tag{3.6}$$

Clearly $x_n \in S_f(x_n)$ and $S_f(x_n)$ is also closed. Now for each $y \in S_f(x_n)$ we get

$$\begin{aligned}
\frac{1}{s^n} d_J(y, x_n) &\leq \{f(x_n) + \sum_{i=0}^{n} \frac{1}{s^i} d_J(x_n, x_i)\} - \{f(y) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(y, x_i)\} \\
&\leq \{f(x_n) + \sum_{i=0}^{n} \frac{1}{s^i} d_J(x_n, x_i)\} - \inf_{x \in S_f(x_{n-1})} \{f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} d_J(x, x_i)\} \\
&\leq \frac{1}{s^n} d_J(x_n, x_n) + \frac{\epsilon}{2^n s^n}. \qquad (3.7)
\end{aligned}$$

Therefore, for any $y \in S_f(x_n)$ we have

$$d_J(y, x_n) - d_J(x_n, x_n) \leq \frac{\epsilon}{2^n} \forall n \in \mathbb{N}.$$

Thus the decreasing sequence of nonempty closed subsets $\{S_f(x_n)\}_{n \geq 0}$ has $vd-$property. Since X has $vd-$ property therefore $diam(S_f(x_n)) \to 0$ as $n \to \infty$. Thus by Cantor's intersection theorem (See Lemma 2.1) we have $\cap_{n=0}^{\infty} S_f(x_n) = \{x_\epsilon\}$.

Now $d_J(x_\epsilon, x_n) \leq diam(S_f(x_n)) \to 0$ as $n \to \infty$ and we have $x_n \to x_\epsilon$ as $n \to \infty$. From (3.7) we see that

$$J(x_\epsilon, x_\epsilon, x_n) - J(x_n, x_n, x_n) \leq \frac{\epsilon}{2^n} \forall n \in \mathbb{N}.$$

Now

$$\begin{aligned}
f(x_1) + d_J(x_1, x_0) &\leq f(x_0) + d_J(x_0, x_0), \\
f(x_2) + d_J(x_2, x_0) + \frac{1}{s} d_J(x_2, x_1) &\leq f(x_1) + d_J(x_1, x_0) + \frac{1}{s} d_J(x_1, x_1)
\end{aligned}$$

$$\begin{aligned} &\leq f(x_0) + d_J(x_0, x_0) + \frac{1}{s} d_J(x_1, x_1) \\ &\quad \dots \\ f(x_m) + \sum_{i=0}^{m-1} \frac{1}{s^i} d_J(x_m, x_i) &\leq f(x_0) + \sum_{i=0}^{m-1} \frac{1}{s^i} d_J(x_i, x_i) \forall m > 1. \end{aligned} \tag{3.8}$$

Also $x_\epsilon \in S_f(x_q)$ for all $q \in \mathbb{N}$, therefore

$$\begin{aligned} f(x_\epsilon) + \sum_{i=0}^{q} \frac{1}{s^i} d_J(x_\epsilon, x_i) &\leq f(x_q) + \sum_{i=0}^{q} \frac{1}{s^i} d_J(x_q, x_i) \\ &\leq f(x_0) + \sum_{i=0}^{q} \frac{1}{s^i} d_J(x_i, x_i) \forall q \geq 1, \end{aligned} \tag{3.9}$$

which in turn implies that

$$\begin{aligned} f(x_\epsilon) + \sum_{i=0}^{\infty} \frac{1}{s^i} d_J(x_\epsilon, x_i) &\leq f(x_0) + \sum_{i=0}^{\infty} \frac{1}{s^i} d_J(x_i, x_i) \\ &\leq \inf_{x \in X} f(x) + \epsilon + \sum_{i=0}^{\infty} \frac{1}{s^i} d_J(x_i, x_i). \end{aligned} \tag{3.10}$$

Moreover for all $x \neq x_\epsilon$, we have $x \notin \cap_{n=0}^{\infty} S_f(x_n)$ and thus there exists $m \in \mathbb{N}$ such that $x \notin S_f(x_m)$. So $x \notin S_f(x_q)$ for all $q \geq m$. Therefore,

$$\begin{aligned} f(x) + \sum_{i=0}^{q} \frac{1}{s^i} d_J(x, x_i) &> f(x_q) + \sum_{i=0}^{q} \frac{1}{s^i} d_J(x_q, x_i) \\ &\geq f(x_\epsilon) + \sum_{i=0}^{q} \frac{1}{s^i} d_J(x_\epsilon, x_i) \forall q \geq m. \end{aligned} \tag{3.11}$$

Hence we see that

$$f(x) + \sum_{i=0}^{\infty} \frac{1}{s^i} d_J(x, x_i) > f(x_\epsilon) + \sum_{i=0}^{\infty} \frac{1}{s^i} d_J(x_\epsilon, x_i).$$

□

Example 3.1. Let us consider $X = (-\infty, +\infty)$ and let $J : X^3 \to [0, \infty]$ be defined as $J(x, y, z) = |x - y|^2 + |y - z|^2$ for all $x, y, z \in X$. Then (X, J) is an S^{JS}-metric space for $s = 3$. Here $d_J(x, y) = |x - y|^2$, which is continuous in both the variables and $\sup\{J(x, x, x) : x \in X\} = 0$. Now we show that X has vanishing diameter property.

Let $\{E_n\}$ be a decreasing sequence of nonempty subsets of X such that it has $vd-$property. Then for any $i \in \mathbb{N}$ there exists some fixed $e_i \in E_i$ such that $J(x, x, e_i) = |x - e_i|^2 \leq J(e_i, e_i, e_i) + r_i = r_i$ for all $x \in E_i$, where $\{r_i\} \subset \mathbb{R}_+$ with $r_i \to 0$ as $i \to \infty$.

Let $x^{(i)}, y^{(i)}, z^{(i)} \in E_i$ be arbitrary. Then

$$\begin{aligned} J(x^{(i)}, y^{(i)}, z^{(i)}) &= |x^{(i)} - y^{(i)}|^2 + |y^{(i)} - z^{(i)}|^2 \\ &\leqslant 2[|x^{(i)} - e_i|^2 + |y^{(i)} - e_i|^2] + 2[|y^{(i)} - e_i|^2 + |z^{(i)} - e_i|^2] \\ &= 2[|x^{(i)} - e_i|^2 + 2|y^{(i)} - e_i|^2] + |z^{(i)} - e_i|^2] \\ &\leqslant 8r_i \to 0 \end{aligned}$$

as $i \to \infty$. This implies $diam(A_i) \leq 8r_i$. Since this is true for all $i \in \mathbb{N}$ we get $diam(A_i) \to 0$ as $r_i \to \infty$. Thus (X, J) has vanishing diameter property.

Let $f : X \to \overline{\mathbb{R}}$ be defined as $f(x) = e^{|x|} + x^2 + 4|x|$ for all $x \in X$. Then f is continuous and lower bounded. Let us take $\epsilon > 0$ as arbitrary and choose $x_0 \in X$ which satisfies $f(x_0) \leq \inf_{x \in X} f(x) + \epsilon$. Now let us consider $x_\epsilon = 0$, if $x_0 = 0$ then we have to choose $x_n = 0$ for all $n \geq 1$ and clearly Theorem 3.1 follows immediately. Now if $x_0 \neq 0$ then we choose $x_n = \sqrt{\frac{\epsilon}{Kr^n}}$, where $K \geq 1$ and $r > 2$ are chosen in such a way that

$$\epsilon \leq \min\{\frac{K(3r-1)}{3r}[f(x_0) - 1], K\}.$$

Then we have

(i) $x_n \to x_\epsilon$ as $n \to \infty$,

(ii) For all $n \geq 1$,

$$J(x_\epsilon, x_\epsilon, x_n) - J(x_n, x_n, x_n) = |x_\epsilon - x_n|^2 = \frac{\epsilon}{Kr^n} < \frac{\epsilon}{2^n}$$

(iii) For all $x \neq x_\epsilon$,

$$\begin{aligned} & f(x) + \textstyle\sum_{n=0}^{\infty} \frac{1}{s^n} J(x, x, x_n) \\ = \;& e^{|x|} + x^2 + 4|x| + \textstyle\sum_{n=0}^{\infty} \frac{1}{3^n} |x - \sqrt{\frac{\epsilon}{Kr^n}}|^2 \\ = \;& e^{|x|} + x^2 + 4|x| + \tfrac{3}{2}x^2 - 2\sqrt{\tfrac{\epsilon}{K}} \frac{3r^{\frac{1}{2}}}{3r^{\frac{1}{2}} - 1} x + \tfrac{\epsilon}{K} \frac{3r}{3r-1} \\ \geqslant \;& e^{|x|} + x^2 + 4|x| + \tfrac{3}{2}x^2 - 2\frac{3r^{\frac{1}{2}}}{3r^{\frac{1}{2}} - 1} x + \tfrac{\epsilon}{K} \frac{3r}{3r-1} \\ > \;& 1 + \tfrac{\epsilon}{K} \frac{3r}{3r-1} = f(x_\epsilon) + \textstyle\sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n). \end{aligned}$$

(iv)

$$\begin{aligned} f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n) &= 1 + \frac{\epsilon}{K} \frac{3r}{3r-1} \leqslant f(x_0) \\ &= f(x_0) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_n, x_n, x_n) \\ &\leq \inf_{x \in X} f(x) + \epsilon + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_n, x_n, x_n). \end{aligned}$$

Next we have the following consequence of Ekeland's variational principle in S^{JS}-metric spaces.

Corollary 3.1. *Let (X, J) be a complete S^{JS}-metric space with coefficient $s > 1$, such that d_J is continuous in both variables,* $\sup\{J(x,x,x) : x \in X\} < \infty$ *and X has vanishing diameter property. Now let, $f : X \to \overline{\mathbb{R}}$ be a lower semi-continuous, proper and lower bounded mapping. Then for every $\epsilon > 0$ there exists a sequence $\{x_n\} \subset X$ and $x_\epsilon \in X$ such that:*

(i) $x_n \to x_\epsilon$ *as* $n \to \infty$,

(ii) $f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x, x, x_n) \geq f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n)$ *for every* $x \in X$,

(iii) $f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n) \leq \inf_{x \in X} f(x) + \epsilon + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_n, x_n, x_n)$.

As an application of Theorem 3.1 we now prove Caristi's fixed point theorem in the context of S^{JS}-metric spaces.

Theorem 3.2. *Let (X, J) be a complete S^{JS}-metric space with coefficient $s > 1$, such that d_J is continuous in both variables,* $\sup\{J(x,x,x) : x \in X\} < \infty$ *and X has vanishing diameter property. Let $T : X \to X$ be an operator for which there exists a lower semi-continuous mapping, proper and lower bounded mapping $f : X \to \overline{\mathbb{R}}$ such that*

$$J(u, u, v) + sJ(u, u, Tu) \geq J(Tu, Tu, v) \tag{3.12}$$

and

$$\frac{s^2}{s-1} J(u, u, Tu) \leq f(u) - f(Tu) \forall u, v \in X. \tag{3.13}$$

Then T has at least one fixed point in X.

Proof. Let us assume that for all $x \in X$, $Tx \neq x$. Using Corollary 3.1 for f, we obtain that for each $\epsilon > 0$ there exists a sequence $\{x_n\} \subset X$ such that $x_n \to x_\epsilon$ as $n \to \infty$ and

$$f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x, x, x_n) > f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} J(x_\epsilon, x_\epsilon, x_n) \forall x \neq x_\epsilon.$$

If in the above inequality, we put $x = T(x_\epsilon)$ then, since $T(x_\epsilon) \neq x_\epsilon$, we get that

$$\begin{aligned} f(x_\epsilon) - f(Tx_\epsilon) &< \sum_{n=0}^{\infty} \frac{1}{s^n} [d_J(Tx_\epsilon, x_n) - d_J(x_\epsilon, x_n)] \\ &< \sum_{n=0}^{\infty} \frac{1}{s^n} s d_J(x_\epsilon, Tx_\epsilon)(5.15) \\ &= s \sum_{n=0}^{\infty} \frac{1}{s^n} d_J(x_\epsilon, Tx_\epsilon) \\ &= \frac{s^2}{s-1} d_J(x_\epsilon, Tx_\epsilon). \end{aligned} \tag{3.14}$$

Also from (3.13) we get $\frac{s^2}{s-1} d_J(x_\epsilon, Tx_\epsilon) \leq f(x_\epsilon) - f(Tx_\epsilon)$, a contradiction. Therefore there exists at least one $x^* \in X$ such that $Tx^* = x^*$. □

Definition 3.4. [14] Let X be a nonempty set and $s \geq 1$ be a given number. Also let a function $S_b : X^3 \to [0, \infty)$ satisfy the following conditions, for each $x, y, z, w \in X$:

(i) $S_b(x, y, z) = 0$ if and only if $x = y = z$;

(ii) $S_b(x, y, z) \leq s[S_b(x, x, w) + S_b(y, y, w) + S_b(z, z, w)]$.

The pair (X, S_b) is called an S_b-metric space.

Souayah and Mlaiki [14, Theorem 2.4] follows from our Theorem 3.1 as an immediate corollary.

Corollary 3.2. *Let (X, S_b) be a complete S_b-metric space with coefficient $s > 1$, such that the S_b-metric is continuous and $f : X \to \overline{\mathbb{R}}$ is a lower semi-continuous, proper and lower bounded mapping. Then for every $x_0 \in X$ and $\epsilon > 0$ with*

$$f(x_0) \leq \inf_{x \in X} f(x) + \epsilon, \tag{3.15}$$

there exists a sequence $\{x_n\} \subset X$ and $x_\epsilon \in X$ such that:

(i) $x_n \to x_\epsilon$ as $n \to \infty$,

(ii) $S_b(x_\epsilon, x_\epsilon, x_n) \leq \frac{\epsilon}{2^n}$ for all $n \geq 1$,

(iii) $f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x, x, x_n) > f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_\epsilon, x_\epsilon, x_n)$ for every $x \neq x_\epsilon$,

(iv) $f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_\epsilon, x_\epsilon, x_n) \leq f(x_0) \leq \inf_{x \in X} f(x) + \epsilon$.

Proof. Let $\{A_n\}$ be a decreasing sequence of nonempty subsets of X such that it has $vd-$property. Then for each $i \in \mathbb{N}$ there exists some fixed $a_i \in A_i$ such that $S_b(x, x, a_i) \leq S_b(a_i, a_i, a_i) + r_i = r_i$ for all $x \in A_i$, where $\{r_i\} \subset \mathbb{R}_+$ with $r_i \to 0$ as $i \to \infty$.

Let $x^{(i)}, y^{(i)}, z^{(i)} \in A_i$ be arbitrary. Then

$$\begin{aligned} S_b(x^{(i)}, y^{(i)}, z^{(i)}) &\leq s[S_b(x^{(i)}, x^{(i)}, a_i) + S_b(y^{(i)}, y^{(i)}, a_i) + S_b(z^{(i)}, z^{(i)}, a_i)] \\ &\leq 3sr_i. \end{aligned} \tag{3.16}$$

It implies $diam(A_i) \leq 3sr_i$. Since this is true for all $i \in \mathbb{N}$ we get $diam(A_i) \to 0$ as $r_i \to \infty$. Thus (X, S_b) has vanishing diameter property. Therefore all the conditions of Theorem 3.1 are satisfied and the result follows immediately. □

Corollary 3.3. *Let (X, S_b) be a complete S_b-metric space with coefficient $s > 1$, such that the S_b-metric is continuous and let $T : X \to X$ be an operator for which* *there exists a lower semi-continuous, proper and lower bounded mapping* $f : X \to \overline{\mathbb{R}}$*, such that:*

$$S_b(u,u,v) + sS_b(u,u,Tu) \geq S_b(Tu,Tu,v) \tag{3.17}$$

and

$$\frac{s^2}{s-1} S_b(u,u,Tu) \leq f(u) - f(Tu) \forall u, v \in X. \tag{3.18}$$

Then T *has at least one fixed point in* X.

Proof. Using Theorem 3.2 and Corollary 3.2 we get the required proof. □

Remark 3.1. [3, Theorem 2.2] is a particular case of our Theorem 3.1.

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