FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 5 (2021), 1129- 1142
https://doi.org/10.22190/FUMI210613082S
Original Scientific Paper

ON THE GEOMETRIC STRUCTURES OF GENERALIZED (k, μ)-SPACE FORMS

Jay Prakash Singh and Mohan Khatri

Department of Mathematics and Computer Science, Mizoram University, Aizawl-796004, India

Abstract. In this paper, the geometric structures of generalized (k, μ)-space forms and their quasi-umbilical hypersurface are analyzed. First ξ-Q and conformally flat generalized (k, μ)-space form are investigated and shown that a conformally flat generalized (k, μ)-space form is Sasakian. Next, we prove that a generalized (k, μ)-space form satisfying Ricci pseudosymmetry and Q-Ricci pseudosymmetry conditions is η-Einstein. We obtain the condition under which a quasi-umbilical hypersurface of a generalized (k, μ)-space form is a generalized quasi Einstein hypersurface. Also ξ-sectional curvature of a quasi-umbilical hypersurface of generalized (k, μ)-space form is obtained. Finally, the results obtained are verified by constructing an example of 3-dimensional generalized (k, μ)-space form.
Keywords:(k, μ)-space form, Q curvature, Hypersurface, Sasakian, η-Einstein.

1. Introduction

The curvature tensor R of the Riemannian manifold mostly determines the nature of the manifold and the sectional curvature of the manifold completely determines the curvature tensor R. A Riemannian manifold having a constant sectional curvature c is known as real space-form. The sectional curvature $K(X, \phi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a ϕ-sectional curvature. If the ϕ-sectional curvature of a Sasakian manifold is constant, then it is called Sasakian space form. Alegre et al. [2] introduced the notion of generalized Sasakian

Received June 13, 2021. accepted August 08, 2021.
Communicated by Uday Chand De
Corresponding Author: Jay Prakash Singh, Department of Mathematics and Computer Science, Mizoram University, Aizawl-796004, India | E-mail: jpsmaths@gmail.com
2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 53C35

space forms and gave many examples of it. Throughout the years, many geometers [3, 4, 13, 15, 16, 17] focused on generalized Sasakian space forms under different geometric conditions.

Blair et al. [5] introduced the notion of (k, μ)-contact metric manifolds. Following this, Koufogiorgos [23] introduced and studied (k, μ) space forms. The (k, μ) space forms are studied by [1, 14, 23, 30]. Carriazo et al. [8] introduced generalized (k, μ) space form which generalizes the notion of (k, μ) space forms. An almost contact metric manifold $(M^{2n+1}, \phi, \xi, g, \eta)$ is said to be a generalized (k, μ) space form if there exists differentiable functions $f_1, f_2, f_3, f_4, f_5, f_6$ on the manifold whose curvature tensor R is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \tag{1.1}$$

where $R_1, R_2, R_3, R_4, R_5, R_6$ are the following tensors:

$$\begin{aligned}
R_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y,\\
R_2(X,Y)Z &= g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z,\\
R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi,\\
R_4(X,Y)Z &= g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y,\\
R_5(X,Y)Z &= g(hY,Z)hX - g(hX,Z)hY + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX,\\
R_6(X,Y)Z &= \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi,
\end{aligned}$$

for any $X, Y, Z \in \chi(M)$. Here, h is a symmetric tensor given by $2h = \mathcal{L}_\xi \phi$, where $\mathcal{L}$ is Lie derivative. In particular, for $f_4 = f_5 = f_6 = 0$ it reduces to the generalized Sasakian space form [2]. It is obvious that (k, μ) space form is an example of generalized (k, μ) space form when

$$f_1 = \frac{c+3}{4}, f_2 = \frac{c-1}{4}, f_3 = \frac{c+3}{4} - k, f_4 = 1, f_5 = \frac{1}{2}, f_6 = 1 - \mu$$

are constants. In [8], the author studied generalized (k, μ) space forms in contact metric and Trans-Sasakian manifolds. Carriazo and Molina [9] studied D_α-homothetic deformations of generalized (k, μ)-space forms and found that deformed spaces are again generalized (k, μ)-space forms in dimension 3, but not in general. In recent years, many geometers studied generalized (k, μ)-space forms under several conditions [21, 28, 22, 20, 27, 29].

In [26], Mantica and Suh introduced and studied Q curvature tensor. In a $(2n+1)$-dimensional Riemannian manifold (M, g), the Q curvature tensor is given by

$$Q(X,Y)Z = R(X,Y)Z - \frac{v}{2n}\big[g(Y,Z)X - g(X,Z)Y\big], \tag{1.2}$$

for any $X, Y, Z \in \chi(M)$ and v is an arbitrary scalar function on M. If $v = \frac{r}{2n+1}$, then Q curvature tensor reduces to concircular curvature tensor [32]. In [13], De and Majhi studied Q curvature tensor in a generalized Sasakian space form.

One of the most important curvature tensors for analyzing the intrinsic properties of Riemannian manifold is the conformal curvature tensor introduced by Yano and Kon [33]. This curvature is invariant under conformal transformation. The conformal curvature C of type (1,3) on a $(2n+1)$-dimensional Riemannian manifold $(M, g), n > 1$, is defined by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}\big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)PX \\ -g(X,Z)PY\big] + \frac{r}{2n(2n-1)}\big[g(Y,Z)X - g(X,Z)Y\big], \end{aligned} \tag{1.3}$$

where R, S, P, r denote the Riemannian curvature tensor, the Ricci tensor, Ricci-operator and the scalar curvature of the manifold respectively. Kim [25] studied conformally flat generalized Sasakian space forms. De and Majhi [15] studied ϕ-conformal semisymmetric generalized Sasakian space forms.

Cartan [10] first initiated and completely classified complete simply connected locally symmetric spaces. A Riemannian manifold is said to be locally symmetric if the curvature tensor satisfies $\nabla R = 0$. The notion of local symmetry is weakened by many authors throughout the years. One such notion is pseudosymmetric spaces introduced by Deszcz [19]. It should be noted that pseudosymmetric spaces introduced by Deszcz is different from those introduced by Chaki [11]. In [31], authors obtained the necessary and sufficient condition for a Chaki pseudosymmetric manifold to be Deszcz pseudosymmetric. De and Samui [14] studied Ricci pseudosymmetric (k, μ)-contact space forms and show that it is an η-Einstein manifold.

The authors in [14], studied quasi-umbilical hypersurface on (k, μ)-space forms. A hypersurface $(\widetilde{M}^{2n+1}, \tilde{g})$ of a Riemannian manifold M^{2n+1} is called quasi-umbilical [12] if its second fundamental tensor has the form

$$H_\rho(X,Y) = \alpha g(X,Y) + \beta\omega(X)\omega(Y), \tag{1.4}$$

where ω is the 1-form, α, β are scalars and the vector field corresponding to the 1-form ω is a unit vector field. Here, the second fundamental tensor H_ρ is defined by $H_\rho(X,Y) = \widetilde{g}(A_\rho, Y)$, where A is (1,1) tensor and ρ is the unit normal vector field and X, Y are tangent vector fields.
A Riemannian manifold is called a generalized quasi-Einstein manifold [18] if its Ricci tensor S satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + c\lambda(X)\lambda(Y),$$

where a, b and c are non-zero scalars and η, λ are 1-forms. If $c = 0$, then the manifold reduces to a quasi-Einstein manifold.

The paper is organized as follows: After preliminaries, ξ-Q and conformally flat generalized (k,μ)-space forms are investigated in section 3. Next in section 4, it is shown that Q-Ricci pseudosymmetric and Ricci pseudosymmetric generalized (k,μ)-space forms are η-Einstein under certain conditions. Moreover, conformal Ricci pseudosymmetric generalized (k,μ)-space forms are studied. In section 5, quasi-umbilical hypersurface of generalized (k,μ)-space form are investigated and shown that it is a generalized quasi Einstein hypersurface. Also ξ-sectional curvature of a quasi-umbilical hypersurface of generalized (k,μ)-space form is obtained. Finally, the obtained results are verified by using an example of a 3-dimensional generalized (k,μ)-space form.

2. Preliminaries

In this section, we highlight some of the formulae and statements which will be used later in our studies.

A $(2n+1)$-dimensional smooth manifold M is said to be a contact metric manifold if there exists a global 1-form η, known as the contact form, such that $\eta\wedge(d\eta)^n \neq 0$ everywhere on M and there exists a unit vector field ξ, called the Reeb vector field, corresponding to 1-form η such that $d\eta(\xi,\cdot)=0$, a $(1,1)$ tensor field ϕ and Riemannian metric g such that

$$\phi^2X=-X+\eta(X)\xi,\quad \eta(X)=g(X,\xi),\quad d\eta(X,Y)=g(X,\phi Y), \tag{2.1}$$

for all $X,Y\in\chi(M)$, where $\chi(M)$ is the Lie-algebra of all vector fields on M. The metric g is called the associate metric and the structure (ϕ,ξ,η,g) is called contact metric structure. A Riemannian manifold M together with contact structure (ϕ,ξ,η,g) is called contact metric manifold. It follows from (2.1) that

$$\begin{aligned}\phi(\xi)=0,\quad \eta\cdot\phi=0,\quad g(X,\phi Y)=-g(\phi X,Y),\\ g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\end{aligned} \tag{2.2}$$

for any $X,Y\in\chi(M)$. Further we define two self-adjoint operators h and l by $h=\frac{1}{2}(\mathcal{L}_\xi\phi)$ and $l=R(\cdot,\xi)\xi$ respectively, where R is the Riemannian curvature of M. These operators satisfy

$$h\xi=l\xi=0,\quad h\phi+\phi h=0,\quad Tr.h=Tr.h\phi=0. \tag{2.3}$$

Here, "Tr." denotes trace. When unit vector ξ is Killing (i.e. $h=0$ or $Tr.l=2n$) then contact metric manifold is called K-contact. A contact structure is said to be normal if the almost complex structure J on $M\times\mathbb{R}$ defined by $J(X,f\frac{d}{dt})=(\phi X-f\xi,\eta(X)\frac{d}{dt})$, where t is the coordinate of $\mathbb{R}$ and f is a real function on $M\times\mathbb{R}$, is integrable. A normal contact metric manifold is called Sasakian. A Sasakian manifold is K-contact but the converse is true only in dimension 3. The (k,μ)-nullity distribution of a contact metric manifold $M(\phi,\xi,\eta,g)$ is a distribution

$$\begin{aligned}N(k,\mu):p\to N_p(k,\mu)=\{Z\in\chi(M):R(X,Y)Z=k\{g(Y,Z)X\\ -g(X,Z)Y\}+\mu\{g(Y,Z)hX-g(X,Z)hY\}\},\end{aligned}$$

for any $X, Y, Z \in \chi(M)$ and real numbers k and μ. A contact metric manifold M with $\xi \in N(k, \mu)$ is called a (k, μ)-contact metric manifold.
In a generalized (k, μ)-space form (M^{2n+1}, g) the following relations hold [2]:

$$\begin{aligned} R(X,Y)\xi &= (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} \\ &+ (f_4 - f_6)\{\eta(Y)hX - \eta(X)hY\}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} PX &= (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi \\ &+ ((2n-1)f_4 - f_6)hX, \end{aligned} \tag{2.5}$$

$$r = 2n\{(2n+1)f_1 + 3f_2 - 2f_3\}, \tag{2.6}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y). \tag{2.7}$$

where, R, S, P, r are respectively the curvature tensor of type (1,3), the Ricci tensor, the Ricci operator i.e. $g(PX, Y) = S(X, Y)$, for any $X, Y \in \chi(M)$ and the scalar curvature of the manifold respectively.

3. Flatness of generalized (k, μ)-space form

De and Samui [14] studied conformally flat (k, μ) space form and De and Majhi [13] analyzed ξ-Q flatness of generalized Sasakian space form. Generalizing the results obtained, in this section we studied ξ-Q flat and conformally flat generalized (k, μ)-space form.

3.1. ξ-Q flat generalized (k, μ)-space form

Definition 3.1. A generalized (k, μ)-space form (M^{2n+1}, g), is said to be ξ-Q flat if $Q(X, Y)\xi = 0$, for any $X, Y \in \chi(M)$ on M.

We have, from (1.2)

$$Q(X,Y)\xi = R(X,Y)\xi - \frac{v}{2n}\big[\eta(Y)X - \eta(X)Y\big], \tag{3.1}$$

for any $X, Y \in \chi(M)$. Using (2.4) in (3.1) we get

$$\begin{aligned} Q(X,Y)\xi &= \big(f_1 - f_3 - \frac{v}{2n}\big)\big[\eta(Y)X - \eta(X)Y\big] \\ &+ (f_4 - f_6)[\eta(Y)hX - \eta(X)hY]. \end{aligned} \tag{3.2}$$

Suppose non-Sasakian generalized (k,μ)-space form is $\xi - Q$ flat. Then from (3.2) we get

$$(3.3)\quad \big(f_1 - f_3 - \frac{v}{2n}\big)\big[\eta(Y)X - \eta(X)Y\big] + (f_4 - f_6)[\eta(Y)hX - \eta(X)hY] = 0.$$

Taking $X = \phi X$ in (3.3), we obtain

$$(3.4)\quad \Big\{\big(f_1 - f_3 - \frac{v}{2n}\big)\phi X + (f_4 - f_6)h\phi X\Big\}\eta(Y) = 0.$$

Since $\eta(Y) \neq 0$ and taking inner product with U in (3.4) gives

$$(3.5)\quad \big(f_1 - f_3 - \frac{v}{2n}\big)g(\phi X, U) + (f_4 - f_6)g(\phi X, hU) = 0.$$

Since $g(\phi X, U) \neq 0$ and $g(\phi X, hU) \neq 0$, we see that $f_1 - f_3 = \frac{v}{2n}$ and $f_4 = f_6$. Conversely, taking $f_1 - f_3 = \frac{v}{2n}$ and $f_4 = f_6$, and putting these values in (3.2) gives $Q(X,Y)\xi = 0$ and hence M is $\xi - Q$ flat. Therefore, we can state the following:

Theorem 3.1. *A non-Sasakian generalized (k,μ)-space form (M^{2n+1}, g), is ξ-Q flat if and only if $f_1 - f_3 = \frac{v}{2n}$ and $f_4 = f_6$.*

In particular, if $v = \frac{r}{2n+1}$ then Q tensor reduces to concircular curvature tensor. Making use of (2.6) in the forgoing equation gives $v = \frac{2n\{(2n+1)f_1+3f_2-2f_3\}}{2n+1}$. In regard of Theorem 3.1, for ξ-concircularly flat we obtain $f_3 = \frac{3f_2}{1-2n}$ and hence we can state the following corollary:

Corollary 3.1. *A non-Sasakian generalized (k,μ)-space form (M^{2n+1}, g), is ξ-concircularly flat if and only if $f_3 = \frac{3f_2}{1-2n}$ and $f_4 = f_6$.*

We can easily see that Theorem 3.1 and Corollary 3.1 obtained by the geometers in [13], are particular cases of Theorem 3.1 and Corollary 3.1 respectively for $f_4 = f_5 = f_6 = 0$.

Substituting the values, $f_4 - f_6 = \mu$ and $f_1 - f_3 = k$ in Theorem 3.1, we obtained the following corollary:

Corollary 3.2. *A (k,μ)-space form (M^{2n+1}, g), is ξ-Q flat if and only if $k = \frac{v}{2n}$ and $\mu = 0$.*

3.2. Conformally flat generalized (k,μ)-space form

Definition 3.2. A generalized (k,μ)-space form $(M^{2n+1}, g), n > 1$, is said to be conformally flat if $C(X,Y)Z = 0$, for any $X, Y, Z \in \chi(M)$ on M.

Suppose generalized (k,μ)-space form is conformally flat. Then from (1.3), we get

$$\begin{aligned} R(X,Y)Z - \frac{1}{2n-1}\big\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)PX - g(X,Z)PY\big\} \\ + \frac{r}{2n(2n-1)}\big\{g(Y,Z)X - g(X,Z)Y\big\} = 0. \end{aligned} \tag{3.6}$$

In consequence of taking $X = \xi$ in (3.6) and using (2.1), (2.4) and (2.5). Eq.(3.6) becomes

$$\begin{aligned} &(f_1 - f_3)\{g(Y,Z)\xi - \eta(Z)Y\} + (f_4 - f_6)\{g(hY,Z)\xi - \eta(Z)hY\} \\ &- \frac{1}{2n-1}\big\{S(Y,Z)\xi - 2n(f_1 - f_3)\eta(Z)Y + 2n(f_1 - f_3)g(Y,Z)\xi \\ &\quad - \eta(Z)PY\big\} + \frac{r}{2n(2n-1)}\big\{g(Y,Z)\xi - \eta(Z)Y\big\} = 0. \end{aligned} \tag{3.7}$$

Putting $Z = \phi Z$ in (3.7) and making use of (2.4), (2.5) and (2.6) results in the following

$$2(n+1)f_6 g(hY,\phi Z) = 0. \tag{3.8}$$

This shows that either $f_6 = 0$ or $\phi h = 0$. In the second case, from (2.1) we have $h = 0$. Therefore, we can state the following:

Theorem 3.2. *A generalized (k,μ)-space form $(M^{2n+1}, g), n > 1$, is conformally flat, then either $f_6 = 0$ or M is Sasakian.*

Corollary 3.3. *A (k,μ)-space form $(M^{2n+1}, g), n > 1$, is conformally flat, then $\mu = 1$ or M is Sasakian.*

4. Pseudosymmetric generalized (k,μ)-space form

In this section certain pseudo symmetry such as Ricci pseudo symmetry, Q-Ricci pseudo symmetry and conformal Ricci pseudo symmetry in the context of generalized (k,μ)-space form are studied. First, we review an important definition

Definition 4.1. [19, 31] A Riemannian manifold $(M,g), n \geq 1$, admitting a $(0,k)$-tensor field T is said to be T-pseudosymmetric if $R\cdot T$ and $D(g,T)$ are linearly dependent, i.e., $R \cdot T = L_T D(g,T)$ holds on the set $U_T = \{x \in M : D(g,T) \neq 0$ at $x\}$, where L_T is some function on U_T.

In particular, if $R\cdot R = L_R D(g,R)$ and $R\cdot S = L_S D(g,S)$ then the manifold is called pseudosymmetric and Ricci pseudosymmetric respectively. Moreover, if $L_R = 0$ (resp., $L_S = 0$) then pseudosymmetric (resp., Ricci pseudosymmetric) reduces to semisymmetric (resp., Ricci semisymmetric) introduced by Cartan in 1946.

4.1. Ricci pseudosymmetric generalized (k, μ)-space form

Definition 4.2. A generalized (k, μ)-space form (M^{2n+1}, g), is said to be Ricci pseudosymmetric if its Ricci curvature satisfies the following relation,

$$R \cdot S = f_{S_2} D(g, S),$$

holds on the set $U_{S_2} = \{x \in M : D(g, S) \neq 0 \text{ at } x\}$, where f_{S_2} is some function on U_{S_2}.

Suppose a generalized (k, μ)-space form (M^{2n+1}, g), is Ricci pseudosymmetric i.e.,

$$R \cdot S = f_{S_2} D(g, S),$$

which can be written as

$$\begin{aligned} S(R(X,Y)U,V) + S(U,R(X,Y)V) = -f_s\big[S(Y,V)g(X,U) \\ -S(X,V)g(Y,U) + S(U,Y)g(X,V) - S(U,X)g(Y,V)\big] \end{aligned} \tag{4.1}$$

Taking $X = U = \xi$ in (4.1) and using (2.4), (2.5) and (2.7), we get

$$\begin{aligned} \big(f_3 - f_1 + f_{S_2}\big)S(Y,V) + \big[2n(f_1 - f_3)(f_1 - f_3 - f_{S_2}) - (k-1)(f_4 \\ -f_6)((2n-1)f_4 - f_6)\big]g(Y,V) - (k-1)(f_4 - f_6)\big((2n-1)f_4 \\ -f_6\big)\eta(Y)\eta(V) + (f_4 - f_6)\big((1-2n)f_3 - 3f_2\big)g(hY,V) = 0. \end{aligned} \tag{4.2}$$

Considering $f_{S_2} \neq f_1 - f_3$ and further taking $(1-2n)f_3 - 3f_2 = 0$ in (4.2), the manifold is η-Einstein. Hence we can state the following:

Theorem 4.1. *A Ricci pseudosymmetric generalized (k, μ)-space form (M^{2n+1}, g), with $f_{S_2} \neq f_1 - f_3$, is η-Einstein manifold if $f_3 = \frac{3f_2}{1-2n}$.*

If $f_{S_2} = 0$, then Ricci pseudosymmetric generalized (k, μ)-space form reduces to Ricci semisymmetric generalized (k, μ)-space form. In view of Theorem (4.1) we obtain the following:

Corollary 4.1. *A Ricci semisymmetric generalized (k, μ)-space form (M^{2n+1}, g), with $f_1 - f_3 \neq 0$ is η-Einstein manifold if $f_3 = \frac{3f_2}{1-2n}$.*

4.2. Q-Ricci pseudosymmetric generalized (k, μ)-space form

Definition 4.3. A generalized (k, μ)-space form (M^{2n+1}, g), is said to be Q-Ricci pseudosymmetric if

$$Q \cdot S = f_{S_3} D(g, S),$$

holds on the set $U_{S_3} = \{x \in M : D(g, S) \neq 0 \text{ at } x\}$, where f_{S_3} is any function on U_{S_3}.

Proceeding similarly as in Theorem 4.1, one can easily obtain the following relation:

Theorem 4.2. *A Q-Ricci pseudosymmetric generalized (k,μ)-space form (M^{2n+1},g), with $f_{S_3} \neq f_3 - f_1 - \frac{v}{2n}$ is η-Einstein manifold if $f_3 = \frac{3f_2}{1-2n}$.*

Taking $f_{S_3} = 0$ in Theorem 4.2, we easily obtain the following:

Corollary 4.2. *A Q-Ricci semisymmetric generalized (k,μ)-space form (M^{2n+1},g), with $f_3 - f_1 \neq \frac{v}{2n}$ is η-Einstein manifold if $f_3 = \frac{3f_2}{1-2n}$.*

4.3. Conformal Ricci pseudosymmetric generalized (k,μ)-space form

Definition 4.4. A generalized (k,μ)-space form $(M^{2n+1},g), n > 1$, is said to be conformal Ricci pseudosymmetric if

$$C \cdot S = f_{S_4} D(g,S),$$

holds on the set $U_{S_4} = \{x \in M : D(g,S) \neq 0 \text{ at } x\}$, where f_{S_4} is any function on U_{S_4}.

Suppose a generalized (k,μ)-space form is conformal Ricci pseudosymmetric. Then, we have

$$\begin{aligned} S(C(X,Y)U,V) + S(U,C(X,Y)V) = -f_{S_4}\big[S(Y,V)g(X,U) \\ -S(X,V)g(Y,U) + S(U,Y)g(X,V) - S(U,X)g(Y,V)\big]. \end{aligned} \tag{4.3}$$

Taking $X = U = \xi$ and $f_4 = f_6$ in (4.3) and making use of (1.3),(2.1) and (2.5), we obtain

$$\begin{aligned} S^2(Y,V) = \big(4nf_1 + 3f_2 - (2n+1)f_3 + 2n(2n-1)f_{S_4}\big)S(Y,V) \\ -(2n-1)f_{S_4}\eta(Y)\eta(V) - \big(2nf_1 + 3f_2 - f_3\big)g(Y,V). \end{aligned} \tag{4.4}$$

Thus, we can state the following:

Theorem 4.3. *If a generalized (k,μ)-space form $(M^{2n+1},g), n > 1$, is conformal Ricci pseudosymmetric with $f_4 = f_6$, then the relation(4.4) holds.*

5. Quasi-umbilical hypersurface of generalized (k,μ)-space form

Let us consider a quasi-umbilical hypersurface $\widetilde{M}$ of a generalized (k,μ)-space form. From Gauss [12], for any vector fields X,Y,Z,W tangent to the hypersurface we have

$$\begin{aligned} R(X,Y,Z,W) &= \widetilde{R}(X,Y,Z,W) - g(H(X,W),H(X,Z)) \\ &+ g(H(X,Z),H(Y,W)), \end{aligned} \tag{5.1}$$

where, $R(X,Y,Z,W) = g(R(X,Y)Z,W)$ and $\widetilde{R}(X,Y,Z,W) = g(\widetilde{R}(X,Y)Z,W)$. Here, H is the second fundamental tensor of $\widetilde{M}$ given by

$$H(X,Y) = \alpha g(X,Y)\rho + \beta\omega(X)\omega(Y)\rho, \tag{5.2}$$

where, ρ is the only unit normal vector field. Here, ω is the 1-form, the vector field corresponding to the 1-form ω is a unit vector field and α, β are scalars.
Using (5.2) in (5.1), we obtain the following result

$$\begin{aligned} & f_1\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] + f_2\big[g(X,\phi Z)g(\phi Y,W) \\ & -g(Y,\phi Z)g(\phi X,W) + 2g(X,\phi Y)g(\phi Z,W)\big] + f_3\big[\eta(X)\eta(Z)g(Y,W) \\ & -\eta(Y)\eta(Z)g(X,W) + g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)\big] \\ & +f_4\big[g(Y,Z)g(hX,W) - g(Y,Z)g(hY,W) + g(hY,Z)g(X,W) \\ & -g(hX,Z)g(Y,W)\big] + f_5\big[g(hY,Z)g(hX,W) - g(hX,Z)g(hY,W) \\ & +g(\phi hX,Z)g(\phi hY,W) - g(\phi hY,Z)g(\phi hX,W)\big] + f_6\big[\eta(X)\eta(Z)g(hY,W) \\ & -\eta(Y)\eta(Z)g(hX,W) + g(hX,Z)\eta(Y)\eta(W) - g(hY,Z)\eta(X)\eta(W)\big] \\ & = \widetilde{R}(X,Y,Z,W) - \alpha^2 g(X,W)g(Y,Z) - \alpha\beta g(X,W)\omega(Y)\omega(Z) \\ & -\alpha\beta g(Y,Z)\omega(X)\omega(W) + \alpha^2 g(Y,W)g(X,Z) + \alpha\beta g(Y,W)\omega(X)\omega(Z) \\ & +\alpha\beta g(X,Z)\omega(Y)\omega(W). \end{aligned} \tag{5.3}$$

Contracting over X and W in (5.3), we obtain

$$\begin{aligned} \widetilde{S}(Y,Z) &= \big(2nf_1 + 3f_2 - f_3 + 2n\alpha^2 + \alpha\beta\big)g(Y,Z) \\ & -\big(3f_2 + (2n+1)f_3\big)\eta(Y)\eta(Z) + \big((2n-1)f_4 - f_6\big)g(hY,Z) \\ & +\alpha\beta(2n-1)\omega(Y)\omega(Z). \end{aligned} \tag{5.4}$$

Hence, we can state the following:

Theorem 5.1. *A quasi-umbilical hypersurface of a generalized (k,μ)-space form is a generalized quasi Einstein hypersurface, provided $f_4 = \frac{f_6}{2n-1}$*

In particular, for a (k,μ)-space form, the above Theorem 5.1 reduces to the following:

Theorem 5.2. [14] *A quasi-umbilical hypersurface of a (k,μ)-contact space form is a generalized quasi-Einstein hypersurface, provided $\mu = 2 - 2n$.*

Corollary 5.1. *A quasi-umbilical hypersurface of a generalized Sasakian space form is a generalized quasi-Einstein hypersurface.*

For any vector fields X, Y, the tensor field $K(X,Y) = \widetilde{R}(X,Y,Y,X)$ is called the sectional curvature of $\widetilde{M}$ given by the sectional plane $\{X,Y\}$. The sectional curvature $K(X,\xi)$ of a sectional plane spanned by ξ and vector field X orthogonal to ξ is called the ξ-sectional curvature of $\widetilde{M}$.

Theorem 5.3. *A ξ-sectional curvature of a quasi-umbilical hypersurface of generalized (k,μ)-space form is given by*

$$K(X,\xi) = \big(f_1 - f_3 + \alpha^2\big)g(\phi X,\phi X) + (f_4 - f_6)g(hX,X) + \alpha\beta\big[(\omega(\xi))^2 + (\omega(X))^2\big] - 2\alpha\beta\eta(X)\omega(X)\omega(\xi).$$

Proof. Taking $W = X$ and $Z = Y$ in (5.3) results in following

$$\begin{aligned} & f_1\big[g(Y,Y)g(X,X) - g(X,Y)g(Y,X)\big] + f_2\big[g(X,\phi Y)g(\phi Y,X) \\ & -g(Y,\phi Y)g(\phi X,X) + 2g(X,\phi Y)g(\phi Y,X)\big] + f_3\big[\eta(X)\eta(Y)g(X,Y) \\ & -\eta(Y)\eta(Y)g(X,X) - g(X,Y)\eta(X)\eta(Y) - g(Y,Y)\eta(X)\eta(X)\big] \\ & +f_4\big[g(Y,Y)g(hX,X) - g(X,Y)g(hY,X) + g(hY,Y)g(X,X) \\ & -g(hX,Y)g(Y,X)\big] + f_5\big[g(hY,Y)g(hX,X) - g(hX,Y)g(hY,X) \\ & +g(\phi hX,Y)g(\phi hY,X) - g(\phi hY,Y)g(\phi hX,X)\big] + f_6\big[\eta(x)\eta(Y)g(hY,X) \\ & -\eta(Y)\eta(Y)g(hX,X) + g(hX,Y)\eta(Y)\eta(X) - g(hY,Y)\eta(X)\eta(X)\big] \\ & = K(X,Y) - \alpha^2 g(X,X)g(Y,Y) - \alpha\beta g(X,X)\omega(Y)\omega(Y) \\ & -\alpha\beta g(Y,Y)\omega(X)\omega(X) + \alpha^2 g(X,Y)g(X,Y) + \alpha\beta g(X,Y)\omega(X)\omega(Y) \\ & +\alpha\beta g(X,Y)\omega(Y)\omega(X). \end{aligned} \tag{5.5}$$

Putting $Y = \xi$ in (5.5) gives

$$K(X,\xi) = \big(f_1 - f_3 + \alpha^2\big)g(\phi X,\phi X) + (f_4 - f_6)g(hX,X) + \alpha\beta\big[(\omega(\xi))^2 + (\omega(X))^2\big] - 2\alpha\beta\eta(X)\omega(X)\omega(\xi).$$

This completes the proof. □

6. Examples of generalized (k,μ)-space forms

Now we will show the validity of obtained result by considering an example of a generalized (k,μ)-space form of dimension 3. Koufogiorgos and Tsichlias [24] constructed an example of generalized (k,μ)-space of dimension 3 which was later shown by Carriazo et al. [8] to be a contact metric generalized (k,μ)-space form $M^3(f_1,0,f_3,f_4,0,0)$ with non-constant f_1, f_3, f_4.

Example 6.1: Let M^3 be the manifold $M = \{(x_1,x_2,x_3) \in \mathbb{R}^3 | x_3 \neq 0\}$ where (x_1,x_2,x_3) are standard coordinates on $\mathbb{R}^3$. Consider the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = -2x_2x_3\frac{\partial}{\partial x_1} + \frac{2x_1}{x_3^2}\frac{\partial}{\partial x_2} - \frac{1}{x_3^2}\frac{\partial}{\partial x_3}, \quad e_3 = \frac{1}{x_3}\frac{\partial}{\partial x_2},$$

are linearly independent at each point of M and are related by

$$[e_1,e_2] = \frac{2}{x_3^2}e_3, \quad [e_2,e_3] = 2e_1 + \frac{1}{x_3^3}e_3, \quad [e_3,e_1] = 0.$$

Let g be the Riemannian metric defined by $g(e_i, e_j) = \delta_{ij}, i, j = 1, 2, 3$ and η be the 1-form defined by $\eta(X) = g(X, e_1)$ for any X on M. Also, let ϕ be the $(1, 1)$-tensor field defined by $\phi e_1 = 0, \quad \phi e_2 = e_3 \quad \phi e_3 = -e_2$. Therefore, (ϕ, e_1, η, g) defines a contact metric structure on M. Put $\lambda = \frac{1}{x_3^2}, k = 1 - \frac{1}{x_3^4}$ and $\mu = 2(1 - \frac{1}{x_3^2})$, then symmetric tensor h satisfies $he_1 = 0, he_2 = \lambda e_2, he_3 = -\lambda e_3$. The non-vanishing components of the Riemannian curvature are as follows:

$$\begin{aligned} R(e_1, e_2)e_1 &= -(k + \lambda\mu)e_2, \quad R(e_1, e_2)e_2 = (k + \lambda\mu)e_1, \\ R(e_1, e_3)e_1 &= (-k + \lambda\mu)e_3, \quad R(e - 1, e_3)e_3 = (k - \lambda\mu)e_1, \\ R(e_2, e_3)e_2 &= (k + \mu - 2\lambda^3)e_3, \quad R(e_2, e_3)e_3 = -(k + \mu - 2\lambda^3)e_2. \end{aligned}$$

Therefore, M is a generalized (k, μ)-space with k, μ not constant. As a contact metric generalized (k, μ)-space is a generalized (k, μ)-space form with $k = f_1 - f_3$ and $\mu = f_4 - f_6$ (Theorem 4.1, [8]), the manifold under consideration is a generalized (k, μ)-space form $M^3(f_1, 0, f_3, f_4, 0, 0)$ where

$$\begin{aligned} f_1 &= -3 + \frac{2}{x_3^2} + \frac{1}{x_3^4} + \frac{2}{x_3^6}, \\ f_3 &= -4 + \frac{2}{x_3^2} + \frac{2}{x_3^4} + \frac{2}{x_3^6} \\ f_4 &= 2(1 - \frac{1}{x_3^2}). \end{aligned}$$

Next we obtain the non-vanishing components of Q-curvature tensor for arbitrary function v as follows:

$$\begin{aligned} Q(e_1, e_2)e_1 = -(k + \lambda\mu - \frac{v}{2})e_2, &\quad Q(e_1, e_2)e_2 = (k + \lambda\mu - \frac{v}{2})e_1, \\ Q(e_1, e_3)e_1 = (-k + \lambda\mu + \frac{v}{2})e_3, &\quad Q(e_1, e_3)e_3 = (k - \lambda\mu - \frac{v}{2})e_1, \\ Q(e_2, e_3)e_2 = (k + \mu - 2\lambda^3 + \frac{v}{2})e_3, &\quad Q(e_2, e_3)e_3 = -(k + \mu - 2\lambda^3 + \frac{v}{2})e_2. \end{aligned}$$

From the above equations we see that $Q(X, Y)e_1 = 0$ for all X, Y on M if and only if $v = 2(1 - \frac{1}{x_3^4})$ and $x_3^2 = 1$. Hence, Theorem 3.1 is verified.

Example 6.2: In [2], it was shown that the warped product $\mathbb{R} \times_f \mathbb{C}^m$ with

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

is a generalized Sasakian space form. Since every generalized Sasakian space form is a particular case of generalized (k, μ)-space form, $\mathbb{R} \times_f \mathbb{C}^m$ with f_1, f_2, f_3 define as above and $f_4 = f_5 = f_6 = 0$ is a generalized (k, μ)-space form.

Acknowledgment

The authors are grateful to the referees for their valuable suggestions towards the improvement of the paper. The second author is thankful to the Department of Science and Technology, New Delhi, India for financial support in the form of Inspire Fellowship (DST/INSPIRE Fellowship/2018/IF180830).

REFERENCES

1. A. Akbar and A. Sarkar: *Some curvature properties of (k, μ)-contact space forms.* Malaya J. Mat. **3**(1) (2015), 45–50.
2. P. Alegre, D. E. Blair and A. Carriazo: *Generalized Sasakian space forms.* Israel J. Math. **141** (2004), 157–183.
3. P. Alegre and A. Carriazo: *Submanifolds of generalized Sasakian space forms.* Taiwanese J. Math. **13** (2009), 923–911.
4. P. Alegre and A. Carriazo: *Structures on generalized Sasakian space forms.* Diff. Geom. Appl. **26** (2008), 656–666.
5. D. E. Blair, T. Koufogiorgos and B. J. Papantoniou: *Contact Metric Manifolds satisfying a nullity condition.* Israel J. Math. **19**(1-3) (1995), 189–214.
6. D. E. Blair: *Contact manifolds in Riemannian geometry.* Lecture note in Mathematics, 509. Springer-Verlag, Berlin, New-York, 1976.
7. D. E. Blair: *Riemannian geometry of contact and sympletic manifolds.* Birkhauser Boston, 2002.
8. A. Carriazo, V. M. Molina and M. M. Tripathi: *Generalized (k, μ) space forms.* Mediterr. J. Math. **10** (2013), 175–196.
9. A. Carriazo and V. M. Molina: *Generalized (k, μ)-space forms and D_α-homothetic deformations.* Balkan J. Geo. Appl. **16**(1) (2011), 37–47.
10. E. Cartan: *Sur une classe remarquable d'espaces de Riemann.* Bull. Soc. Math. France **54** (1926), 214–264.
11. M. C. Chaki: *On pseudosymmetric manifolds.* An. Stiint. Univ. AL.I. Cuza din Iasi Sect. I-a Math. N. S. **33**(1) (1987), 53–58.
12. B. Y. Chen: *Geometry of submanifolds.* Marcel Dekker, Inc. New York, 1973.
13. U. C. De and P. Majhi: *On the Q curvature tensor of a generalized sasakian-space-form.* Krag. J. Math. **13**(3) (2019), 333–319.
14. U. C. De and S. Samui: *The structure of some classes of (k, μ)-contact space forms.* Diff. Geom.-Dynamical System, **18** (2016), 1–13.
15. U. C. De and P. Majhi: *ϕ-semisymmetric generalized Sasakian space-form.* Arab J. Math. Sci. **21** (2015), 170–178.
16. U. C. De and A. Sarkar: *Some results on generalized Sasakian space forms.* Thai J. Math. **8** (2010), 1–10.
17. U. C. De and A. Yildiz: *Certain curvature conditions on generalized Sasakian space forms.* Quaest. Math. **38**(1) (2015), 195–504.

18. U. C. De and G. C. Ghosh: *On generalized quasi Einstein manifolds.* Kyungpook Math. J. **44** (2001), 607–615.

19. R. Deszcz: *On pseudosymmetric spaces.* Bull. Belg. Math. Soc., Ser. A, **44** (1992), 1–34.

20. M. Faghfouri and N. Ghaffarzadeh: *On doubly warped product submanifolds of generalized* (k, μ)*-space forms.* Afrika Mat. **26**(7-8) (2015), 1443–1455.

21. S. K. Hui, S. Uddin and P. Mandal: *Submanifolds of generalized* (k, μ)*-space-forms.* Per. Math. Hung. **77** (2018), 329–339.

22. D. L. Kumar and H. G. Nagaraja: *Second Order Parallel Tensor and Ricci solitons on Generalized* (k, μ)*-Space forms.* Math. Adv. Pure Appl. Sci. **2**(1) (2019), 1–7.

23. T. Koufogiorgos: *Contact Riemannian manifolds with constant* ϕ*-sectional curvature.* Tokyo J. Math. **20** (1997), 13–22.

24. T. Koufogiorgos and C. Tsichlias: *On the existence of a new class of contact metric manifolds.* Canad. Math. Bull. **20**(1) (2000), 400–447.

25. U. K. Kim: *Conformally flat generalized Sasakian space-forms and locally symmetric generalized Sasakian space forms.* Note Mat. **25** (2006), 55–67.

26. C. A. Mantica and Y. J. Suh: *Pseudo Q-symmetric Riemannian manifolds.* Int. J. Geom. Methods Mod. Phys. **10**(5) (2013), 1–25.

27. B. Shammukha and V. Venkatesha: *Projective curvature tensor on generalized* (k, μ)*-space forms.* Italian J. Pure Appl. Math. **12** (2019), 810–850.

28. G. S. Shivaprasanna: *Some results on generalized* (k, μ) *space-forms.* Int. J. sci. Engg. Appl. Sci. **2**(7) (2016), 184–191.

29. G. S. Shivaprasanna, Y. B. Maralabhavi and G. Somashekhara: *On Semi-Symmetric Metric Connection in a Generalized* (k, μ) *Space Forms.* Int. J. Math. Trends Tech. **9**(3) (2014), 173–188.

30. S. Shashidhar and H. G. Nagaraja: η*-Einstein* (k, μ)*-space forms.* J. Info. Math. Sci. **7**(2) (2015), 109–120.

31. A. A. Shaikh, R. Deszcz, M. Hotlos and J. Jelwicki: *On pseudosymmetric manifolds.* Publicationes Mathematicae, **86**(3-4) (2015), 433–456.

32. K. Yano: *Concircular geometry I. Concircular transformation.* Proc. Imp. Acad. Tokyo, **16** (1910), 195–200.

33. K. Yano and M. Kon: *Structures on manifolds.* Series in Pure Math. Vol 3, World Scientific Publ. Co., Singapore, 1984.