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FRACTIONAL TYPE HERMITE-HADAMARD INEQUALITIES FOR CONVEX AND AG(Log)-CONVEX FUNCTIONS *

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Abstract. In this paper, we give a new type of integral equality involving the left-sided and the right-sided Riemann-Liouville fractional integrals. Thereafter, some new fractional Hermite-Hadamard type inequalities are presented by using the above fractional integral equality involving the concepts of convex functions and s-convex functions and AG(log)-convex functions respectively.

Keywords: Fractional Hermite-Hadamard inequalities, Convex functions, s-convex functions, AG(Log)-convex functions.

1. Introduction and preliminaries

Concerning the problem of function approximation [1, 2, 3, 4, 5], integral inequality plays a very important role. Moreover, it is well-known that integral inequality and fixed-point theory arise in differential and integral equations [6], which provide a powerful tool for obtaining the estimation of solutions of such equations [7, 8]. There is a large number of interesting results which were inspired by the classical Hermite-Hadamard inequality. One can see the contributions [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and references therein.

Throughout this paper, let $I = [a, b] \subseteq R^+ \cup \{0\}$ be an interval and R^+ be a set of positive real numbers. First, we recall the concepts of convex and s-convex functions.

Definition 1.1. (see [20]) The function $f : I \subset R \to R$ is said to be convex, if for every $x, y \in I$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y).$$

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Definition 1.2. (see [21]) The function $f : I \to R$ is said to be s-convex, where $s \in (0, 1]$, if for every $x, y \in I$ and $t \in [0, 1]$, we have

$$f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y).$$

Next, we recall the concept of AG(Log)-convex functions.

Definition 1.3. (see [22]) A function $f : I \subseteq R^+ \to R^+$ is said to be arithmetic-geometric convex (or log-convex), if for every $x, y \in I$ and $\lambda \in [0, 1]$, we have

$$f((1 - \lambda)x + \lambda y) \leq f(x)^{1-\lambda} f(y)^{\lambda}.$$

In the sequel, we recall the concepts of the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in R^+$.

Definition 1.4. (see [23]) Let $f \in L[a, b]$ and $\Gamma(\cdot)$ be the Gamma function. The symbols ${}_{RL}J^{\alpha}_{a^+} f$ and ${}_{RL}J^{\alpha}_{b^-} f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in R^+$ and are defined by

$$({}_{RL}J^{\alpha}_{a^+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t)dt,\ (0 \leq a < x \leq b),$$

$$({}_{RL}J^{\alpha}_{b^-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t)dt,\ (0 \leq a \leq x < b),$$

respectively.

To end this section, we give a new type fractional integral equality which will be widely used in the sequel.

Lemma 1.1. *Let $f : [a, b] \to R$ be a differentiable function and $f' \in L[a, b]$. For any $0 < \alpha \leq 1$ and $0 < \lambda \leq 1$, the following equality for fractional integrals holds:*

$$\begin{aligned} & (1 + \lambda(1 - \alpha))f(b) + (1 - \lambda\alpha)f(a) + \lambda(2\alpha - 1)f((1 - \alpha)b + \alpha a) \\ & -\frac{\Gamma(\alpha + 1)}{(b - a)^{\alpha}}\left({}_{RL}J^{\alpha}_{a^+} f(b) + {}_{RL}J^{\alpha}_{b^-} f(a)\right) \\ = \ & (b - a)\left\{ \int_0^{1-\alpha} (t^{\alpha} - (1 - t)^{\alpha} + \alpha\lambda) f'(tb + (1 - t)a)dt \right. \\ & \left. + \int_{1-\alpha}^1 (t^{\alpha} - (1 - t)^{\alpha} + \lambda(1 - \alpha)) f'(tb + (1 - t)a)dt \right\}. \end{aligned}$$

Proof. It suffices to note that

$$I := \left\{ \int_0^{1-\alpha} (t^{\alpha} - (1 - t)^{\alpha} + \alpha\lambda) f'(tb + (1 - t)a)dt \right.$$

$$\begin{aligned}
&+\int_{1-\alpha}^{1}(t^{\alpha}-(1-t)^{\alpha}+\lambda(1-\alpha))f'(tb+(1-t)a)dt\Big\} \\
=\ &\int_{0}^{1-\alpha}\alpha\lambda f'(tb+(1-t)a)dt+\int_{1-\alpha}^{1}\lambda(1-\alpha)f'(tb+(1-t)a)dt \\
&+\int_{0}^{1}(t^{\alpha}-(1-t)^{\alpha})f'(tb+(1-t)a)dt \\
:=\ &I_1+I_2+I_3. \qquad (1.1)
\end{aligned}$$

Integrating by parts

$$\begin{aligned}
I_1 &:= \int_{0}^{1-\alpha}\alpha\lambda f'(tb+(1-t)a)dt=\frac{\alpha\lambda}{b-a}\int_{0}^{1-\alpha}df(tb+(1-t)a) \\
&= \frac{\alpha\lambda}{b-a}[f((1-\alpha)b+\alpha a)-f(a)], \qquad (1.2)
\end{aligned}$$

and similarly we get

$$\begin{aligned}
I_2 &:= \int_{1-\alpha}^{1}\lambda(1-\alpha)f'(tb+(1-t)a)dt=\frac{\lambda(1-\alpha)}{b-a}\int_{1-\alpha}^{1}df(tb+(1-t)a) \\
&= \frac{\lambda(1-\alpha)}{b-a}[f(b)-f((1-\alpha)b+\alpha a)], \qquad (1.3)
\end{aligned}$$

$$\begin{aligned}
I_3 &:= \int_{0}^{1}(t^{\alpha}-(1-t)^{\alpha})f'(tb+(1-t)a)dt \\
&= \frac{1}{b-a}\int_{0}^{1}t^{\alpha}df(tb+(1-t)a)-\frac{1}{b-a}\int_{0}^{1}(1-t)^{\alpha}df(tb+(1-t)a) \\
&= \frac{f(b)}{b-a}-\frac{\alpha}{(b-a)^2}\int_{a}^{b}\left(\frac{x-a}{b-a}\right)^{\alpha-1}f(x)dx+ \\
&\quad \frac{f(a)}{b-a}-\frac{\alpha}{(b-a)^2}\int_{a}^{b}\left(\frac{b-u}{b-a}\right)^{\alpha-1}f(u)du \\
&= \frac{f(b)+f(a)}{b-a}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\,{}_{RL}J_{a^+}^{\alpha}f(b)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}\,{}_{RL}J_{b^-}^{\alpha}f(a). \qquad (1.4)
\end{aligned}$$

Submitting (1.2), (1.3) and (1.4) to (1.1), we have the result:

$$\begin{aligned}
I &:= \frac{(1+\lambda(1-\alpha))f(b)}{b-a}+\frac{(1-\lambda\alpha)f(a)}{b-a}+\frac{\lambda(2\alpha-1)f((1-\alpha)b+\alpha a)}{b-a} \\
&\quad -\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}({}_{RL}J_{a^+}^{\alpha}f(b)+{}_{RL}J_{b^-}^{\alpha}f(a)). \qquad (1.5)
\end{aligned}$$

Next, by multiplying both sides by $(b-a)$ for (1.5), we have the conclusion. □

2. Main results for convex functions

First, we give the inequalities for convex functions.

Theorem 2.1. *Let $f : [a, b] \to R$ be a differentiable.If $|f'| \in L[a, b]$ and $|f'|$ is a convex function. Then for any $0 < \alpha \leq 1, 0 < \lambda \leq 1$, the following inequalities for fractional integrals holds:*

Case 1: $\frac{1}{2} \leq \alpha \leq 1$.

$$\Big|(1+\lambda(1-\alpha))f(b) + (1-\lambda\alpha)f(a) + \lambda(2\alpha-1)f((1-\alpha)b+\alpha a)$$
$$-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}({}_{RL}J^\alpha_{a^+}f(b) + {}_{RL}J^\alpha_{b^-}f(a))\Big| \leq (b-a)\Big\{|f'(b)|\varphi(\alpha,\lambda) + |f'(a)|\phi(\alpha,\lambda)\Big\},$$

where

$$\varphi(\alpha,\lambda) = \frac{\alpha\lambda}{2}(1-\alpha)(3-2\alpha) + \frac{1}{\alpha+1} - \frac{1}{\alpha+2}(\frac{1}{2})^{\alpha+1},$$
$$\phi(\alpha,\lambda) = \frac{\alpha\lambda}{2}(1-\alpha)(2\alpha+1) + \frac{1}{\alpha+1}(1-(\frac{1}{2})^\alpha).$$

Case 2: $0 < \alpha < \frac{1}{2}$.

$$\Big|(1+\lambda(1-\alpha))f(b) + (1-\lambda\alpha)f(a) + \lambda(2\alpha-1)f((1-\alpha)b+\alpha a)$$
$$-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}({}_{RL}J^\alpha_{a^+}f(b) + {}_{RL}J^\alpha_{b^-}f(a))\Big| \leq (b-a)\Big\{|f'(b)|\varphi'(\alpha,\lambda) + |f'(a)|\phi'(\alpha,\lambda)\Big\},$$

where

$$\varphi'(\alpha,\lambda) = \frac{1}{\alpha+1}(\frac{1}{\alpha+2} - (\frac{1}{2})^\alpha) + \frac{\alpha^{\alpha+1}(3\alpha+2)}{(\alpha+1)(\alpha+2)} + \frac{\lambda\alpha}{2}(1-\alpha)(3-2\alpha),$$
$$\phi'(\alpha,\lambda) = \frac{1}{\alpha+1}(1-(\frac{1}{2})^\alpha) + \frac{1-(1-\alpha)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{\lambda\alpha}{2}(1+2\alpha)(1-\alpha).$$

Proof. Step 1. We check the results for Case 1.

Using Lemma 1.1, $|f'| \in L[a, b]$ and $|f'|$ is a convex function, we have

$$\begin{aligned} &\Big|(1+\lambda(1-\alpha))f(b) + (1-\lambda\alpha)f(a) + \lambda(2\alpha-1)f((1-\alpha)b+\alpha a) \\ &-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}({}_{RL}J^\alpha_{a^+}f(b) + {}_{RL}J^\alpha_{b^-}f(a))\Big| \\ \leq\ & (b-a)\Big\{\Big|\int_0^{1-\alpha}(t^\alpha - (1-t)^\alpha + \alpha\lambda)f'(tb+(1-t)a)dt\Big| \end{aligned}$$

$$
\begin{aligned}
&+\left|\int_{1-\alpha}^{\frac{1}{2}}(t^{\alpha}-(1-t)^{\alpha}+\lambda(1-\alpha))f'(tb+(1-t)a)dt\right| \\
&+\left|\int_{\frac{1}{2}}^{1}(t^{\alpha}-(1-t)^{\alpha}+\lambda(1-\alpha))f'(tb+(1-t)a)dt\right|\Bigg\} \\
\leq\ & (b-a)\Bigg\{\int_0^{1-\alpha}((1-t)^{\alpha}-t^{\alpha}+\alpha\lambda)\Big(t|f'(b)|+(1-t)|f'(a)|\Big)dt \\
&+\int_{1-\alpha}^{\frac{1}{2}}((1-t)^{\alpha}-t^{\alpha}+\lambda(1-\alpha))\Big(t|f'(b)|+(1-t)|f'(a)|\Big)dt \\
&+\int_{\frac{1}{2}}^{1}(t^{\alpha}-(1-t)^{\alpha}+\lambda(1-\alpha))\Big(t|f'(b)|+(1-t)|f'(a)|\Big)dt\Bigg\} \\
=\ & (b-a)\Big[K_1+K_2+K_3\Big]. \qquad (2.1)
\end{aligned}
$$

Integrating by parts

$$
\begin{aligned}
K_1 &:= \int_0^{1-\alpha}((1-t)^{\alpha}-t^{\alpha}+\alpha\lambda)\Big(t|f'(b)|+(1-t)|f'(a)|\Big)dt \\
&= |f'(b)|\Bigg\{\int_0^{1-\alpha}((1-t)^{\alpha}t-t^{\alpha+1}+\alpha\lambda t)dt\Bigg\} \\
&\quad +|f'(a)|\Bigg\{\int_0^{1-\alpha}((1-t)^{\alpha+1}-t^{\alpha}(1-t)+\alpha\lambda(1-t))dt\Bigg\} \\
&= |f'(b)|\Bigg\{-\frac{(1-\alpha)\alpha^{\alpha+1}}{\alpha+1}+\frac{1-\alpha^{\alpha+2}}{(\alpha+1)(\alpha+2)}-\frac{(1-\alpha)^{\alpha+2}}{\alpha+2}+\frac{\lambda\alpha(1-\alpha)^2}{2}\Bigg\} \\
&\quad +|f'(a)|\Bigg\{\frac{1-\alpha^{\alpha+2}}{\alpha+2}+\frac{\alpha(1-\alpha)^{\alpha+1}}{\alpha+1}-\frac{(1-\alpha)^{\alpha+2}}{(\alpha+1)(\alpha+2)}+\frac{\lambda\alpha(1-\alpha^2)}{2}\Bigg\}, \qquad (2.2)
\end{aligned}
$$

and similarly we get

$$
\begin{aligned}
K_2 &:= \int_{1-\alpha}^{\frac{1}{2}}((1-t)^{\alpha}-t^{\alpha}+\lambda(1-\alpha))\Big(t|f'(b)|+(1-t)|f'(a)|\Big)dt \\
&= |f'(b)|\Bigg\{\int_{1-\alpha}^{\frac{1}{2}}((1-t)^{\alpha}t-t^{\alpha+1}+\lambda(1-\alpha)t)dt\Bigg\} \\
&\quad +|f'(a)|\Bigg\{\int_{1-\alpha}^{\frac{1}{2}}((1-t)^{\alpha+1}-t^{\alpha}(1-t)+\lambda(1-\alpha)(1-t))dt\Bigg\} \\
&= |f'(b)|\Bigg\{-\frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1}+\frac{\alpha^{\alpha+1}(2-\alpha^2)}{(\alpha+1)(\alpha+2)}+\frac{(1-\alpha)^{\alpha+2}}{\alpha+2} \\
&\quad +\frac{\lambda(1-\alpha)(1-4(1-\alpha)^2)}{8}\Bigg\}+|f'(a)|\Bigg\{-\frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1}
\end{aligned}
$$

$$+\frac{\alpha^2+\alpha+1}{(\alpha+1)(\alpha+2)}(1-\alpha)^{\alpha+1}+\frac{\alpha^{\alpha+2}}{\alpha+2}+\frac{\lambda(1-\alpha)(1-4\alpha^2)}{8}\Big\},$$

(2.3)

and

$$\begin{aligned}
K_3 &:= \int_{\frac{1}{2}}^{1}(t^\alpha-(1-t)^\alpha+\lambda(1-\alpha))\Big(t|f'(b)|+(1-t)|f'(a)|\Big)dt\\
&= |f'(b)|\Big\{\int_{\frac{1}{2}}^{1}(t^{\alpha+1}-(1-t)^\alpha t+\lambda(1-\alpha)t)dt\Big\}\\
&\quad +|f'(a)|\Big\{\int_{\frac{1}{2}}^{1}(t^\alpha(1-t)-(1-t)^{\alpha+1}+\lambda(1-\alpha)(1-t))dt\Big\}\\
&= |f'(b)|\Big\{\frac{1}{\alpha+2}+\frac{1}{(\alpha+1)(\alpha+2)}(\frac{1}{2})^{\alpha+1}+\frac{3\lambda(1-\alpha)}{8})\Big\}\\
&\quad +|f'(a)|\Big\{\frac{1}{(\alpha+1)(\alpha+2)}-\frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1}+\frac{\lambda(1-\alpha)}{8}\Big\}. \qquad (2.4)
\end{aligned}$$

Submitting (2.2), (2.3) and (2.4) to (2.1), we have the results for Case 1.

Step 2. We check the results for Case 2.

In fact, we have

$$\begin{aligned}
&\Big|(1+\lambda(1-\alpha))f(b)+(1-\lambda\alpha)f(a)+\lambda(2\alpha-1)f((1-\alpha)b+\alpha a)\\
&-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}({}_{RL}J^\alpha_{a^+}f(b)+{}_{RL}J^\alpha_{b^-}f(a))\Big|\\
\le\ & (b-a)\Big[\int_0^{\frac{1}{2}}((1-t)^\alpha-t^\alpha+\alpha\lambda)\Big|f'(tb+(1-t)a)\Big|dt\\
&+\int_{\frac{1}{2}}^{1-\alpha}(t^\alpha-(1-t)^\alpha+\lambda(1-\alpha))\Big|f'(tb+(1-t)a)\Big|dt\\
&+\int_{1-\alpha}^{1}(t^\alpha-(1-t)^\alpha+\lambda(1-\alpha))\Big|f'(tb+(1-t)a)\Big|dt\Big]\\
\le\ & (b-a)\Big[\int_0^{\frac{1}{2}}((1-t)^\alpha-t^\alpha+\alpha\lambda)\Big(t|f'(b)|+(1-t)|f'(a)|\Big)dt\\
&+\int_{\frac{1}{2}}^{1-\alpha}(t^\alpha-(1-t)^\alpha+\lambda\alpha)\Big(t|f'(b)|+(1-t)|f'(a)|\Big)dt\\
&+\int_{1-\alpha}^{1}(t^\alpha-(1-t)^\alpha+\lambda(1-\alpha))\Big(t|f'(b)|+(1-t)|f'(a)|\Big)dt\Big]\\
=\ & (b-a)\Big[K_4+K_5+K_6\Big]. \qquad (2.5)
\end{aligned}$$

Integrating by parts

$$\begin{aligned}
K_4 &:= \int_0^{\frac{1}{2}} \Big((1-t)^\alpha - t^\alpha + \alpha\lambda\Big)\Big(t|f'(b)| + (1-t)|f'(a)|\Big)dt \\
&= |f'(b)| \int_0^{\frac{1}{2}} \Big((1-t)^\alpha t - t^{\alpha+1} + \alpha\lambda t\Big)dt \\
&\quad + |f'(a)| \int_0^{\frac{1}{2}} \Big((1-t)^{\alpha+1} - t^\alpha(1-t) + \alpha\lambda(1-t)\Big)dt \\
&= |f'(b)|\Big\{ -\frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1} + \frac{1}{(\alpha+1)(\alpha+2)} + \frac{\alpha\lambda}{8}\Big\} \\
&\quad + |f'(a)|\Big\{\frac{1}{\alpha+2} - \frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1} + \frac{3\alpha\lambda}{8}\Big\}, \qquad (2.6)
\end{aligned}$$

and similarly we get

$$\begin{aligned}
K_5 &:= \int_{\frac{1}{2}}^{1-\alpha} (t^\alpha - (1-t)^\alpha + \alpha\lambda)\Big(t|f'(b)| + (1-t)|f'(a)|\Big)dt \\
&= |f'(b)| \int_{\frac{1}{2}}^{1-\alpha} (t^{\alpha+1} - (1-t)^\alpha t + \alpha\lambda t)dt \\
&\quad + |f'(a)| \int_{\frac{1}{2}}^{1-\alpha} (t^\alpha(1-t) - (1-t)^{\alpha+1} + \alpha\lambda(1-t))dt \\
&= |f'(b)|\Big\{ -\frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1} + \frac{\alpha^{\alpha+1}(2-\alpha^2)}{(\alpha+1)(\alpha+2)} + \frac{(1-\alpha)^{\alpha+2}}{\alpha+2} \\
&\quad + \frac{\alpha\lambda}{2}((1-\alpha)^2 - \frac{1}{4})\Big\} + |f'(a)|\Big\{ -\frac{1}{\alpha+1}(\frac{1}{2})^{\alpha+1} + \frac{\alpha(1-\alpha)^{\alpha+1}}{\alpha+1} \\
&\quad + \frac{\alpha^{\alpha+2}}{\alpha+2} + \frac{1}{(\alpha+1)(\alpha+2)} + \frac{\lambda\alpha}{2}(\frac{1}{4} - \alpha^2)\Big\} \qquad (2.7)
\end{aligned}$$

and

$$\begin{aligned}
K_6 &:= \int_{1-\alpha}^{1} (t^\alpha - (1-t)^\alpha + \lambda(1-\alpha))\Big(t|f'(b)| + (1-t)|f'(a)|\Big)dt \\
&= |f'(b)| \int_{1-\alpha}^{1} (t^{\alpha+1} - (1-t)^\alpha t + \lambda(1-\alpha)t)dt \\
&\quad + |f'(a)| \int_{1-\alpha}^{1} (t^\alpha(1-t) - (1-t)^{\alpha+1} + \lambda(1-\alpha)(1-t))dt \\
&= |f'(b)|\Big\{\frac{1-(1-\alpha)^{\alpha+2}}{\alpha+2} + \frac{\alpha^{\alpha+1}(\alpha^2+2\alpha-2)}{(\alpha+1)(\alpha+2)} + \frac{\lambda(1-\alpha)}{2}(1-(1-\alpha)^2)\Big\} \\
&\quad + |f'(a)|\Big\{\frac{1-(1+\alpha)\alpha^{\alpha+2}}{(\alpha+1)(\alpha+2)} - \frac{(1-\alpha)^{\alpha+1}}{(\alpha+1)(\alpha+2)}(\alpha^2+\alpha+1) + \alpha^2\frac{\lambda(1-\alpha)}{2}\Big\}. \qquad (2.8)
\end{aligned}$$

Submitting (2.6), (2.7) and (2.8) to (2.5), we derive the results for Case 2. □

Secondly, we give the inequalities for s-convex functions.

Theorem 2.2. *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$. If $|f'| \in L[a,b]$ and $|f'|$ is an s-convex function, then for any $0 < \alpha \leq 1$ and $0 < \lambda \leq 1$ the following inequality for fractional integrals holds:*

$$\Big|(1+\lambda(1-\alpha))f(b) + (1-\lambda\alpha)f(a) + \lambda(2\alpha-1)f((1-\alpha)b+\alpha a) - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}({}_{RL}J^\alpha_{a^+} f(b) + {}_{RL}J^\alpha_{b^-} f(a))\Big| \leq (b-a)\Big\{\widetilde{\varphi}(\alpha,\lambda) + \widetilde{\phi}(\alpha,\lambda)\Big\},$$

where

$$\widetilde{\varphi}(\alpha,\lambda) = \frac{\alpha\lambda}{b-a}\Big|f((1-\alpha)b+\alpha a) - f(a)\Big| + \frac{\lambda(1-\alpha)}{b-a}\Big|f(b) - f((1-\alpha)b+\alpha a)\Big|,$$

$$\widetilde{\phi}(\alpha,\lambda) = \left(\frac{1-(\frac{1}{2})^{\alpha p}}{\alpha p+1}\right)^{\frac{1}{p}}\left\{\left[\frac{|f'(b)|^q}{8} + \frac{3|f'(a)|^q}{8}\right]^{1/q} + \left[\frac{3|f'(b)|^q}{8} + \frac{|f'(a)|^q}{8}\right]^{1/q}\right\},$$

and $1/p + 1/q = 1$.

Proof. Using Lemma 1.1 via $f' \in L[a,b]$ and $|f'|$ is a convex function, we have

$$\begin{aligned} &\Big|(1+\lambda(1-\alpha))f(b) + (1-\lambda\alpha)f(a) + \lambda(2\alpha-1)f((1-\alpha)b+\alpha a) \\ &-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}({}_{RL}J^\alpha_{a^+} f(b) + {}_{RL}J^\alpha_{b^-} f(a))\Big| \\ \leq\ & (b-a)\Big\{\lambda\alpha\Big|\int_0^{1-\alpha} f'(tb+(1-t)a)dt\Big| + \lambda(1-\alpha)\Big|\int_{1-\alpha}^1 f'(tb+(1-t)a)dt\Big| \\ &+\Big|\int_0^1 (t^\alpha-(1-t)^\alpha)f'(tb+(1-t)a)dt\Big|\Big\} \\ &\leq (b-a)[S_1+S_2+S_3]. \end{aligned} \tag{2.9}$$

Integrating by parts

$$\begin{aligned} S_1 &:= \lambda\alpha\Big|\int_0^{1-\alpha} f'(tb+(1-t)a)dt\Big| = \frac{\lambda\alpha}{b-a}\Big|\int_a^{(1-\alpha)b+\alpha a} f'(x)dx\Big| \\ &= \frac{\alpha\lambda}{b-a}\Big|f((1-\alpha)b+\alpha a) - f(a)\Big|, \end{aligned} \tag{2.10}$$

and similarly we get

$$\begin{aligned} S_2 &:= \lambda(1-\alpha)\Big|\int_{1-\alpha}^1 f'(tb+(1-t)a)dt\Big| = \frac{\lambda(1-\alpha)}{b-a}\Big|\int_{(1-\alpha)b+\alpha a}^b f'(x)dx\Big| \\ &= \frac{\lambda(1-\alpha)}{b-a}\Big|f(b) - f((1-\alpha)b+\alpha a)\Big|, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
S_3 &:= \left|\int_0^1 (t^\alpha-(1-t)^\alpha)f'(tb+(1-t)a)dt\right| \\
&\le \left|\int_0^{\frac12} (t^\alpha-(1-t)^\alpha)f'(tb+(1-t)a)dt\right| \\
&\quad+\left|\int_{\frac12}^1 (t^\alpha-(1-t)^\alpha)f'(tb+(1-t)a)dt\right| \\
&\le \left(\int_0^{\frac12}((1-t)^\alpha-t^\alpha)^p dt\right)^{\frac1p}\left(\int_0^{\frac12}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac1q} \\
&\quad+\left(\int_{\frac12}^1 (t^\alpha-(1-t)^\alpha)^p dt\right)^{\frac1p}\left(\int_{\frac12}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac1q} \\
&\le \left(\int_0^{\frac12}((1-t)^{\alpha p}-t^{\alpha p})dt\right)^{\frac1p}\left(\int_0^{\frac12}[t|f'(b)|^q+(1-t)|f'(a)|^q]dt\right)^{\frac1q} \\
&\quad+\left(\int_{\frac12}^1 (t^{\alpha p}-(1-t)^{\alpha p})dt\right)^{\frac1p}\left(\int_{\frac12}^1[t|f'(b)|^q+(1-t)|f'(a)|^q]dt\right)^{\frac1q} \\
&\le \left(\frac{1-(\frac12)^{\alpha p}}{\alpha p+1}\right)^{\frac1p}\left\{\left[\frac{\left|f'(b)\right|^q}{8}+\frac{3\left|f'(a)\right|^q}{8}\right]^{1/q}+\left[\frac{3\left|f'(b)\right|^q}{8}+\frac{\left|f'(a)\right|^q}{8}\right]^{1/q}\right\}. \qquad (2.12)
\end{aligned}$$

Submitting (2.10), (2.11) and (2.12) to (2.9), we have the desired result. □

Theorem 2.3. *Let $f:[a,b]\to R$ be differentiable on (a,b) with $a\ge 0$, $f'\in L[a,b]$. If $|f'|$ is an s-convex function, then for any $0<\alpha\le 1$ we have*

$$\begin{aligned}
&\Bigg|(1+\lambda(1-\alpha))f(b)+(1-\lambda\alpha)f(a)+\lambda(2\alpha-1)f((1-\alpha)b+\alpha a) \\
&-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}({}_{RL}J_{a^+}^\alpha f(b)+{}_{RL}J_{b^-}^\alpha f(a))\Bigg| \\
\le\ & (b-a)\Bigg\{\frac{|f'(b)|}{s+1}(\lambda(1-\alpha)^{s+1}(2\alpha-1)+\lambda(1-\alpha))+\frac{|f'(a)|}{s+1}(\lambda\alpha^{s+1}-\lambda\alpha) \\
&\Big(|f'(b)|+|f'(a)|\Big)\left(\frac{1}{2^{s+1}(s+1)}-\frac{1}{2^{\alpha+1}(\alpha+1)}-\frac{1}{2^{\alpha+s}(\alpha+s+1)}+\frac{1}{\alpha+s+1}\right)\Bigg\}.
\end{aligned}$$

Proof. Using Lemma 1.1 via $|f'|$ is an s-convex function, we have

$$\begin{aligned}
&\Bigg|(1+\lambda(1-\alpha))f(b)+(1-\lambda\alpha)f(a)+\lambda(2\alpha-1)f((1-\alpha)b+\alpha a) \\
&-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}({}_{RL}J_{a^+}^\alpha f(b)+{}_{RL}J_{b^-}^\alpha f(a))\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq (b-a)\Big\{\lambda\alpha\Big|\int_0^{1-\alpha} f'(tb+(1-t)a)dt\Big| + \lambda(1-\alpha)\Big|\int_{1-\alpha}^1 f'(tb+(1-t)a)dt\Big| \\
&\quad + \Big|\int_0^1 (t^\alpha-(1-t)^\alpha) f'(tb+(1-t)a)dt\Big|\Big\} \\
&\leq (b-a)\Big\{\lambda\alpha\int_0^{1-\alpha} |f'(tb+(1-t)a)|dt + \lambda(1-\alpha)\int_{1-\alpha}^1 |f'(tb+(1-t)a)|dt \\
&\quad + \int_0^1 |(t^\alpha-(1-t)^\alpha) f'(tb+(1-t)a)|dt\Big\} \\
&= (b-a)\Big\{H_1+H_2+H_3\Big\}. \qquad (2.13)
\end{aligned}
$$

Integrating by parts

$$
\begin{aligned}
H_1 &:= \lambda\alpha\int_0^{1-\alpha} |f'(tb+(1-t)a)|dt \leq \lambda\alpha\int_0^{1-\alpha} (t^s|f'(b)|+(1-t)^s|f'(a)|)dt \\
&\leq \lambda\alpha\Big\{\frac{(1-\alpha)^{s+1}}{s+1}|f'(b)| - \frac{1-\alpha^{s+1}}{s+1}|f'(a)|\Big\}, \qquad (2.14)
\end{aligned}
$$

and similarly we get

$$
\begin{aligned}
H_2 &:= \lambda(1-\alpha)\int_{1-\alpha}^1 |f'(tb+(1-t)a)|dt \\
&\leq \lambda(1-\alpha)\int_{1-\alpha}^1 (t^s|f'(b)|+(1-t)^s|f'(a)|)dt \\
&\leq \lambda(1-\alpha)\Big\{\frac{1-(1-\alpha)^{s+1}}{s+1}|f'(b)| + \frac{\alpha^{s+1}}{s+1}|f'(a)|\Big\}, \qquad (2.15)
\end{aligned}
$$

and

$$
\begin{aligned}
H_3 &:= \int_0^1 |(t^\alpha-(1-t)^\alpha) f'(tb+(1-t)a)|dt \\
&\leq \int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha)(t^s|f'(b)|+(1-t)^s|f'(a)|)dt \\
&\quad + \int_{\frac{1}{2}}^1 (t^\alpha-(1-t)^\alpha)(t^s|f'(b)|+(1-t)^s|f'(a)|)dt \\
&\leq \int_0^{\frac{1}{2}} (t^s - t^{\alpha+s})|f'(b)|dt + \int_0^{\frac{1}{2}} ((1-t)^{\alpha+s} - t^\alpha)|f'(a)|dt \\
&\quad + \int_{\frac{1}{2}}^1 (t^{\alpha+s} - (1-t)^\alpha)|f'(b)|dt + \int_{\frac{1}{2}}^1 ((1-t)^s - (1-t)^{\alpha+s})|f'(a)|dt \\
&\leq \Big(|f'(b)|+|f'(a)|\Big)\Big(\frac{1}{2^{s+1}(s+1)}
\end{aligned}
$$

$$(2.16) \qquad -\frac{1}{2^{\alpha+1}(\alpha+1)} - \frac{1}{2^{\alpha+s}(\alpha+s+1)} + \frac{1}{\alpha+s+1}\Bigg).$$

Submitting (2.14), (2.15) and (2.16) to (2.13), we have the result. This completes the proof. □

3. Main results for AG(Log)-convex functions

Lemma 3.1. *Let $A > 0$, $B > 0$, $\alpha \in (0,1)$. The following equality holds:*

$$\int_0^{1-\alpha} B^t A^{1-t} dt = \frac{A}{\ln B - \ln A}\left[\left(\frac{B}{A}\right)^{1-\alpha} - 1\right].$$

Proof.

$$\begin{aligned}\int_0^{1-\alpha} B^t A^{1-t} dt &= A\int_0^{1-\alpha} B^t A^{-t} dt = A\int_0^{1-\alpha}\left(\frac{B}{A}\right)^t dt \\ &= \frac{A}{\ln(B)-\ln(A)}\int_0^{1-\alpha} d\left(\frac{B}{A}\right)^t = \frac{A}{\ln(B)-\ln(A)}\left[\left(\frac{B}{A}\right)^{1-\alpha} - 1\right].\end{aligned}$$

□

Now, we give the inequalities for AG(log)-convex functions.

Theorem 3.1. *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$ and $|f'|$ are AG(log)-convex functions, $|f'(a)| > 0$, $|f'(b)| > 0$, $|f'(a)| \neq |f'(b)|$, then for any $0 < \alpha \leq 1$ and $0 < \lambda \leq 1$, $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, the following inequality for fractional integrals holds:*

$$\begin{aligned}&\Big|(1+\lambda(1-\alpha))f(b) + (1-\lambda\alpha)f(a) + \lambda(2\alpha-1)f((1-\alpha)b+\alpha a) \\ &-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left({}_{RL}J^\alpha_{a^+}f(b) + {}_{RL}J^\alpha_{b^-}f(a)\right)\Big| \leq (b-a)\Big\{Q_1' + Q_2' + Q_3'\Big\},\end{aligned}$$

where

$$\begin{aligned}Q_1' &= \frac{\lambda\alpha|f'(a)|}{\ln|f'(b)| - \ln|f'(a)|}\left\{\left(\frac{|f'(b)|}{|f'(a)|}\right)^{1-\alpha} - 1\right\}, \\ Q_2' &= \frac{\lambda(1-\alpha)|f'(a)|}{\ln|f'(b)| - \ln|f'(a)|}\left(\frac{|f'(b)|}{|f'(a)|}\right)^{1-\alpha}\left\{\left(\frac{|f'(b)|}{|f'(a)|}\right)^{\alpha} - 1\right\}. \\ Q_3' &= \left(\frac{1-(\frac{1}{2})^{\alpha p}}{\alpha p+1}\right)^{\frac{1}{p}}\left\{1 + \left(\frac{|f'(b)|}{|f'(a)|}\right)^{\frac{1}{2}}\right\}\left[\frac{|f'(a)|^q}{q(\ln|f'(b)| - \ln|f'(a)|)}\left(\left(\frac{|f'(b)|}{|f'(a)|}\right)^{\frac{q}{2}} - 1\right)\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. Using Lemma 1.1 via $f' \in L[a,b]$ and $|f'|$ is an AG(log)-convex function, we have

$$\begin{aligned}
&\Big|(1+\lambda(1-\alpha))f(b)+(1-\lambda\alpha)f(a)+\lambda(2\alpha-1)f((1-\alpha)b+\alpha a) \\
&-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\big({}_{RL}J^{\alpha}_{a^+}f(b)+{}_{RL}J^{\alpha}_{b^-}f(a)\big)\Big| \\
&\le (b-a)\Big\{\lambda\alpha\int_0^{1-\alpha}|f'(tb+(1-t)a)|dt+\lambda(1-\alpha)\int_{1-\alpha}^{1}|f'(tb+(1-t)a)|dt \\
&+\int_0^1|(t^{\alpha}-(1-t)^{\alpha})f'(tb+(1-t)a)|dt\Big\} \\
&\le (b-a)\Big\{Q'_1+Q'_2+Q'_3\Big\}. \qquad (3.1)
\end{aligned}$$

Integrating by parts and Using Lemma 3.1

$$\begin{aligned}
Q'_1 &:= \lambda\alpha\int_0^{1-\alpha}|f'(tb+(1-t)a)|dt \\
&\le \lambda\alpha\int_0^{1-\alpha}|f'(b)|^{t}|f'(a)|^{1-t}dt \\
&\le \frac{\lambda\alpha|f'(a)|}{\ln|f'(b)|-\ln|f'(a)|}\left\{\left(\frac{|f'(b)|}{|f'(a)|}\right)^{1-\alpha}-1\right\}, \qquad (3.2)
\end{aligned}$$

and similarly we get

$$Q'_2 \le \frac{\lambda(1-\alpha)|f'(a)|}{\ln|f'(b)|-\ln|f'(a)|}\left(\frac{|f'(b)|}{|f'(a)|}\right)^{1-\alpha}\left\{\left(\frac{|f'(b)|}{|f'(a)|}\right)^{\alpha}-1\right\}. \qquad (3.3)$$

Next, using Hölder inequality, we have

$$\begin{aligned}
Q'_3 &:= \int_0^1|(t^{\alpha}-(1-t)^{\alpha})f'(tb+(1-t)a)|dt \\
&\le \int_0^{\frac{1}{2}}((1-t)^{\alpha}-t^{\alpha})|f'(tb+(1-t)a)|dt \\
&\quad+\int_{\frac{1}{2}}^{1}(t^{\alpha}-(1-t)^{\alpha})|f'(tb+(1-t)a)|dt \\
&\le \left(\int_0^{\frac{1}{2}}((1-t)^{\alpha}-t^{\alpha})^{p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}(|f'(tb+(1-t)a)|^{q}dt\right)^{\frac{1}{q}} \\
&\quad+\left(\int_{\frac{1}{2}}^{1}((1-t)^{\alpha}-t^{\alpha})^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(|f'(tb+(1-t)a)|^{q}dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$\leq \left(\frac{1-(\frac{1}{2})^{\alpha p}}{\alpha p+1}\right)^{\frac{1}{p}}\left\{\left(\int_0^{\frac{1}{2}}(|f'(b)|^{qt}|f'(a)|^{q(1-t)}dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}(|f'(b)|^{qt}|f'(a)|^{q(1-t)}dt\right)^{\frac{1}{q}}\right\}$$

$$\leq \left(\frac{1-(\frac{1}{2})^{\alpha p}}{\alpha p+1}\right)^{\frac{1}{p}}\left\{1+\left(\frac{|f'(b)|}{|f'(a)|}\right)^{\frac{1}{2}}\right\}$$

$$\left[\frac{|f'(a)|^q}{q(\ln|f'(b)|-\ln|f'(a)|)}\left(\left(\frac{|f'(b)|}{|f'(a)|}\right)^{\frac{q}{2}}-1\right)\right]^{\frac{1}{q}}. \tag{3.4}$$

Submitting (3.2), (3.3) and (3.4) to (3.1), we have the result. □

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