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Δ^m-STATISTICAL CONVERGENCE OF ORDER α OF GENERALIZED DIFFERENCE SEQUENCES IN PROBABILISTIC NORMED SPACES

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Abstract. In this paper, we define the notion of Δ^m-statistical convergence of order α of generalized difference sequences in the probabilistic normed spaces and present their characterization. We also define the notion of Δ^m-statistical Cauchy of order α for these types of sequences in the probabilistic normed spaces. Also, we have given few examples which demonstrates that this notion is more generalized in the probabilistic normed spaces.
Keywords: statistical convergence, probabilistic space, normed space.

1. Introduction

The most interesting generalization of the concept of classical convergence of sequences was coined by Zygmund [21] and stated as statistical convergence in 1935. Steinhaus [19] and Fast [7] also presented the notion of statistical convergence simultaneously in the same year 1951. The notion was introduced to deal with the theory of series summation and has been studied by various researchers in different spaces such as intutionistic fuzzy normed spaces [13], random 2-normed spaces [15], probabilistic normed spaces [10] etc. It has also been studied for different sequences such as ordinary sequences [18], double sequences [4], triple sequences [17] and multiple sequences [14] by various authors see[16, 12, 2]

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In 1981, Kizmaz [11] introduced the difference sequence spaces $X(\Delta)$ for $X = l_\infty$,c, c_0, where $X(\Delta)$ is a Banach Space. Further, Et and Çolak[6] generalized the notion of difference sequences. In the present paper, we are introducing the notion of statistical convergence of order α for the generalized difference sequences in $PN - Spaces$. Now, lets recall some basic definitions and preliminaries. Using the definitions of continuous [1] t$-$norm and continuous t$-$conorm, the notion of probabilistic normed space ($PN - Space$) is given as:

Definition 1.1. [1] A triplet $(X; \mathcal{P}, *)$ is referred as probabilistic normed space with X as a real vector space, $\mathcal{P}$ as a map from X into D ($\mathcal{P}_x(t)$ is the value of $\mathcal{P}_x$ at $t \in \mathbb{R}$ where $\mathcal{P}_x(t)$ is the distribution function $\mathcal{P}$ for $x \in X$) and $*$ as a t-norm, if the following conditions holds:

1. $\mathcal{P}_x(0) = 0$,
2. $\mathcal{P}_x(t) = 1$ for all $t > 0$ if and only if $x = 0$,
3. $\mathcal{P}_{\alpha.x}(t) = \mathcal{P}_x(t/|\alpha)|$; where $\alpha \neq 0$;
4. $\mathcal{P}_{x+y}(s+t) \geq \mathcal{P}_x(s) * \mathcal{P}_x(t)$ for all $x, y \in X$ and $s, t \in \mathbb{R}_0^+ = [0, \infty)$.

The notion of the generalized difference sequence which is given by Et and Çolak[6] is defined in the following definition:

Definition 1.2. Let m be a non-negative integer, then the generalized difference operator $\Delta^m x_r$ is defined as

$$\Delta^m x_r = \Delta^{m-1} x_r - \Delta^{m-1} x_{r+1},$$

where $\Delta^0 x_r = x_r$ for all $r \in \mathbb{N}$.

Using this concept, the notions of Δ^m-convergence and Δ^m-Cauchy of generalized difference sequences in the probabilistic normed spaces are defined in the following definitions:

Definition 1.3. [20] Let $(X; \mathcal{P}, *)$ be a $PN - Space$ and $x = \{x_r\}$ be a generalized difference sequence in X. Then $x = \{x_r\}$ is said to be Δ^m-convergent to some $L \in X$ with respect to probabilistic norm $\mathcal{P}$ if, for every $\epsilon > 0$ and $\psi \in (0, 1)$ there exists $r_0 \in \mathbb{N}$ such that

$$\mathcal{P}(\Delta^m x_r - L, \epsilon) > 1 - \psi,$$

for all $r > r_0$.

Definition 1.4. [5] Let $(X; \mathcal{P}, *)$ be a $PN - Space$ and $x = \{x_r\}$ be a generalized difference sequence in X. Then $x = \{x_r\}$ is said to be Δ^m-Cauchy with respect to probabilistic norm $\mathcal{P}$ if, for every $\epsilon > 0$ and $\psi \in (0, 1)$ there exists $r_0 \in \mathbb{N}$ such that

$$\mathcal{P}(\Delta^m x_r - \Delta^m x_s, \epsilon) > 1 - \psi,$$

for all $r, s \geq r_0$.

Definition 1.5. [9] Let $(X; \mathcal{P}, *)$ be a $PN-Space$ and $x = \{x_r\}$ be a generalized difference sequence in X. Then $x = \{x_r\}$ is said to be Δ^m-bounded with respect to probabilistic norm $\mathcal{P}$ if for every $\epsilon > 0$ and $\psi \in (0,1)$,

$$\mathcal{P}(\Delta^m x_r, \epsilon) > 1 - \psi.$$

In the following definition, the notion of statistical convergence is mentioned using natural density.

Definition 1.6. [8] Let A be the subset of $\mathbb{N}$. The natural density of A, denoted by $\delta(A) = \lim\limits_{n\to\infty} |A_n|$, where $A_n = \{r \leq n : r \in A\}$ and $|\,.\,|$ indicates the order of the enclosed set.
A sequence $x = \{x_r\}$ is said to be statistically convergent to number L if, for every $\epsilon > 0$, we have $\delta(\{r \leq n : |x_r - L| \geq \epsilon\}) = 0$. It can be written as $St - \lim x_r = L$ and St is the collection of all the statistically convergent sequences.

For $\alpha \in (0,1]$, the concept of statistical convergence of order α is given as:

Definition 1.7. [3] Let $\alpha \in (0,1]$ and a sequence $x = \{x_r\}$ in X is said to be statistically convergent of order α to number L if for given $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{r \leq n : |x_r - L| \geq \epsilon\}| = 0.$$

It can be written as $St^\alpha - \lim x_r = L$ and St^α is the set of all the statistically convergent sequences of order α.

Definition 1.8. [5] Let $(X; \mathcal{P}, *)$ be a $PN - Space$ and $\alpha \in (0,1]$. A sequence $x = \{x_r\}$ in X is said to be statistically convergent of order α with respect to probabilistic norm $\mathcal{P}$ to some number L in X if for every $\epsilon > 0$ and $\psi \in (0,1)$,

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{r \leq n : \mathcal{P}(x_r - L; \epsilon) \leq 1 - \psi\}| = 0.$$

It can be written as $St^\alpha_{\mathcal{P}} - \lim x_r = L$ and $St^\alpha_{\mathcal{P}}$ is the set of all the statistically convergent sequences of order α in probabilistic normed space.

Definition 1.9. [5] Let $(X; \mathcal{P}, *)$ be a $PN-Space$ and $\alpha \in (0,1]$. A sequence $x = \{x_r\}$ in X is said to be statistically Cauchy of order α with respect to probabilistic norm $\mathcal{P}$ if for every $\epsilon > 0$ and $\psi \in (0,1)$,

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{r \leq n : \mathcal{P}(x_r - x_s; \epsilon) \leq 1 - \psi\}| = 0.$$

2. Main Results

First we consider some definitions to present our findings for generalized difference sequences in the $PN - Spaces$:

Definition 2.1. Let $(X;\mathcal{P},*)$ be a $PN-Space$ and $x=\{x_r\}$ be a sequence in X. Then $x=\{x_r\}$ is said to be Δ^m-statistically convergent to some $L\in X$ with respect to probabilistic norm $\mathcal{P}$ if for every $\psi\in(0,1)$ and $\epsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\left|\{r\leq n:\mathcal{P}(\Delta^m x_r-L;\epsilon)\leq 1-\psi\}\right|=0.$$

It can be written as $St_{\mathcal{P}}-\lim\Delta^m x_r=L$ and $St_{\mathcal{P}}^{\Delta}$ represents the collection of all the statistically convergent generalized difference sequences in probabilistic normed space $(X;\mathcal{P},*)$.

Definition 2.2. Let $(X;\mathcal{P},*)$ be a $PN-Space$ and $x=\{x_r\}$ be a sequence in X. Then for $\alpha\in(0,1]$, $x=\{x_r\}$ is said to be Δ^m-statistically convergent of order α to some $L\in X$ with respect to probabilistic norm $\mathcal{P}$ if for every $\psi\in(0,1)$ and $\epsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\{r\leq n:\mathcal{P}(\Delta^m x_r-L;\epsilon)\leq 1-\psi\}\right|=0.$$

Represented as $St_{\mathcal{P}}^{\alpha}-\lim\Delta^m x_r=L$ where $St_{\mathcal{P}}^{\alpha}$ represents the collection of all the statistically convergent difference sequences of order α in the probabilistic normed spaces.

Definition 2.3. Let $(X;\mathcal{P},*)$ be a $PN-Space$ and $x=\{x_r\}$ be a sequence in X. Then for $\alpha\in(0,1]$, $x=\{x_r\}$ is said to be Δ^m-statistically Cauchy of order α to some $L\in X$ with respect to probabilistic norm $\mathcal{P}$ if, for every $\epsilon>0$ and $\psi\in(0,1)$ there exists $s\in\mathbb{N}$ such that,

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\{r\leq n:\mathcal{P}(\Delta^m x_r-\Delta^m x_s;\epsilon)\leq 1-\psi\}\right|=0.$$

Now, we are giving an example to show that the notion of $St_{\mathcal{P}}^{\alpha}-\lim\Delta^m x_r=L$ is not defined for $\alpha>1$.

Example 2.1. Consider a space $(\mathbb{R},|.|)$, where $|.|$ is the usual norm for the set of real numbers. Let $a*b=ab$ and $\mathcal{P}(\Delta^m x_r,t)=\frac{t}{t+|\Delta^m x_r|}$, for $x\in X$ and $t>0$. Then $(\mathbb{R},\mathcal{P};*)$ is a $PN-Space$. Define a sequence $x=\{x_r\}$ as

$$\Delta^m x_r=\begin{cases}1 & \text{if } r=k^2\\ \frac{1}{n} & \text{if } r\neq k^2\end{cases}$$

Then, for every $\epsilon>0$, $t>0$ and $\psi\in(0,1)$. Consider

$$\begin{aligned}\{r\leq n:\mathcal{P}(\Delta^m x_r-0;t)\leq 1-\psi\} &= \{r\leq n:\mathcal{P}(\Delta^m x_r;t)\leq 1-\psi\}\\ &= \left\{r\leq n:\frac{t}{t+|\Delta^m x_r|}\leq 1-\psi\right\}\\ &= \left\{r\leq n:|\Delta^m x_r|\geq\frac{t\psi}{1-\psi}>0\right\}\\ &= \{r\leq n:r=k^2\}.\end{aligned}$$

Then $\frac{1}{n^{\alpha}}|\{r\leq n:\mathcal{P}(\Delta^m x_r,t)\leq 1-\psi\}|\leq\frac{n^{1}/2}{n^{\alpha}}\to 0$ if $\alpha\in(\frac{1}{2},1]$. This implies $St_{\mathcal{P}}^{\alpha}-\lim\Delta^m x_r=0$ for $\alpha\in(\frac{1}{2},1]$. But $St_{\mathcal{P}}^{\alpha}-\lim\Delta^m x_r\neq 0$ for $0<\alpha\leq\frac{1}{2}$.

Remark 2.1. The notion of Δ^m-statistical convergence of order α for $\alpha > 1$ is not well defined because if $\Delta^m x_r$ is a difference sequence in $(X, \mathcal{P}; *)$ and $\alpha > 1$, then for every $L \in X$, we have $St^{\alpha}_{\mathcal{P}} - \lim \Delta^m x_r = L$.

Theorem 2.1. *Let $(X, \mathcal{P}; *)$ be PN$-$Space and $0 < \alpha \leq \beta \leq 1$. Then $St^{\alpha}_{\mathcal{P}}(\Delta^m x_r) \subseteq St^{\beta}_{\mathcal{P}}(\Delta^m x_r)$ and this inclusion is strict for $\alpha < \beta$.*

Proof. Let $x = \{x_r\}$ be any generalized difference sequence in X such that $St^{\alpha}_{\mathcal{P}} - \lim \Delta^m x_r = L$. Then for every $\epsilon > 0$ and $\psi \in (0,1)$, we have

$$\frac{|\{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - L, \epsilon) \leq 1 - \psi\}|}{n^{\beta}} \leq \frac{|\{r \in \mathbb{N} : \mathbb{P}(\Delta^m x_r - L, \epsilon) \leq 1 - \psi\}|}{n^{\alpha}}$$

This gives

$$St^{\alpha}_{\mathcal{P}}(\Delta^m x_r) \subseteq St^{\beta}_{\mathcal{P}}(\Delta^m x_r).$$

Now, for strict inclusion $St^{\alpha}_{\mathcal{P}}(\Delta^m x_r) \subset St^{\beta}_{\mathcal{P}}(\Delta^m x_r)$ for $\alpha < \beta$. Consider a sequence $x = \{x_r\}$ in $PN - Space$ $(X, \mathcal{P}; *)$, for $k > 0$ which is defined as:

$$\Delta^m x_r = \begin{cases} 1 & \text{if } r = [n^k] \\ 0 & \text{if } r \neq [n^k] \end{cases}$$

where $n \in \mathbb{N}$. Now, for given $\epsilon > 0$ and $\psi \in (0,1)$ we have

$$\{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - L, \epsilon) \leq 1 - \psi\} \subseteq \{[1^k], [2^k], , , , , , , \}.$$

So that

$$\frac{|\{r \leq \mathbb{N} : \mathcal{P}(\Delta^m x_r - L, \epsilon) \leq 1 - \psi\}|}{n^{\beta}} \leq \frac{n^{\frac{1}{k}}}{n^{\beta}}.$$

Therefore the above inequality implies that $St^{\beta}_{\mathcal{P}} - lim \Delta^m x_r = 0$ for $\frac{1}{k} < \beta \leq 1$. But $St^{\alpha}_{\mathcal{P}} - lim \Delta^m x_r \neq 0$ for $0 < \alpha \leq \frac{1}{k}$. $\square$

Remark 2.2. For $\alpha \in (0,1]$, $St^{\alpha}_{\mathcal{P}} - \lim \Delta^m x_r = L$. This implies that $St_{\mathcal{P}} - \lim \Delta^m x_r = L$, *i.e.* $St^{\alpha}_{\mathcal{P}}(\Delta^m x_r) \subseteq St_{\mathcal{P}}(\Delta^m x_r)$ and this inclusion is strict if $\alpha \in (0,1)$.

Theorem 2.2. *Let $(X, \mathcal{P}; *)$ be a $PN - Space$ and $x = \{x_r\}$ be a sequence in X, then for $\alpha \in (0,1]$, if $St^{\alpha}_{\mathcal{P}} - \lim \Delta^m x_r = L$, then L is unique.*

Proof. Let if possible, $St^{\alpha}_{\mathcal{P}} - \lim \Delta^m x_r = L_1$ and $St^{\alpha}_{\mathcal{P}} - \lim \Delta^m x_r = L_2$, where $L_1 \neq L_2$. For every $\epsilon > 0$ and $\psi \in (0,1)$, take $\theta \in (0,1)$ such that $(1-\theta) * (1-\theta) > 1 - \psi$. Define

$$\mathcal{K}_1(\theta, \epsilon) = \{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - L_1; \epsilon) \leq 1 - \theta\}$$

and

$$\mathcal{K}_2(\theta, \epsilon) = \{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - L_2; \epsilon) \leq 1 - \theta\}.$$

Since $St^{\alpha}_{\mathcal{P}} - \lim \Delta^m x_r = L_1$, then

$$\lim_{n \to \infty} \frac{1}{n^{\alpha}} |\{r \leq n : \mathcal{P}(\Delta^m x_r - L_1; t) \leq 1 - \theta\}| = 0.$$

Similarly, for $St_{\mathcal{P}}^{\alpha} - \lim \Delta^m x_r = L_2$. We obtain,

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \left|\{r \le n : \mathcal{P}(\Delta^m x_r - L_2; t) \le 1-\theta\}\right| = 0.$$

Now let

$$\mathcal{K}(\theta, \epsilon) = \mathcal{K}_1(\theta, \epsilon) \cup \mathcal{K}_2(\theta, \epsilon).$$

Clearly,

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |K(\theta, \epsilon)| = 0.$$

This implies,

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |K^c(\theta, \epsilon)| = 1.$$

For $r \in \mathbb{N} - K(\theta, \epsilon)$, we get

$$\mathcal{P}(L_1 - L_2, \epsilon) \ge \mathcal{P}\left(\Delta^m x_r - L_1; \frac{\epsilon}{2}\right) * \mathcal{P}\left(\Delta^m x_r - L_2; \frac{\epsilon}{2}\right) \ge (1-\theta)*(1-\theta) > 1-\psi,$$

as $\psi > 0$ is arbitrary, which gives $\mathcal{P}(L_1 - L_2, \epsilon) = 1$, for all $\epsilon > 0$. This implies, $L_1 = L_2$. Therefore, $St_{\mathcal{P}}^{\alpha} - \lim \Delta^m x_r = L$ is unique. □

Theorem 2.3. *Let $(X; \mathcal{P}, *)$ be a $PN-Space$ and $0 < \alpha \le 1$. The Δ^m-convergence of a sequence in X with respect to probabilistic norm $\mathcal{P}$ implies the Δ^m-statistically convergence of order α with respect to probabilistic norm $\mathcal{P}$.*

Proof. Let $x = \{x_r\}$ be any sequence in X such that $\mathcal{P} - \lim \Delta^m x_r = L$. For every $\epsilon > 0$, $\psi \in (0, 1)$, there exists $r_0 \in \mathbb{N}$ such that,

$$\mathcal{P}(\Delta^m x_r - L, \epsilon) > 1 - \psi,$$

for $r \ge r_0$. This implies the set $\{r \le n : \mathcal{P}(\Delta^m x_r - L; \epsilon) \le 1 - \psi\}$ has almost finite many terms. We know that the α-density of every finite set is zero. Therefore,

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \left|\{r \le n : \mathcal{P}(\Delta^m x_r - L; \epsilon) \le 1 - \psi\}\right| = 0.$$

Thus,

$$St_{\mathcal{P}}^{\alpha} - \lim \Delta^m x_r = L.$$

□

The converse of the above theorem is not true which can be justified in the next example:

Example 2.2. Let $(\mathbb{R}, |\,.\,|)$ be real normed space under the usual norm. Let $u * v = uv$ and $\mathcal{P}(\Delta x_r, t) = \frac{t}{t+|\Delta^m x_r|}$, where $x \in X$ and $t \ge 0$ Here $(X; \mathcal{P}, *)$ is $PN - Space$. Take a difference sequence $x = \{x_r\}$ as

$$\Delta^m x_r = \begin{cases} 1 & \text{if r is a cube} \\ 0 & \text{otherwise} \end{cases}$$

Then for every $\psi \in (0,1)$ and $\epsilon > 0$, let

$$R_r(\psi,\epsilon) = \{r \leq n : \mathcal{P}(\Delta^m x_r;\epsilon) \leq 1-\psi\}.$$

Since

$$\begin{aligned} R_r(\psi,\epsilon) &= \left\{r \leq n : \frac{t}{t+|\Delta^m x_r|} \leq 1-\psi\right\} = \left\{r \leq n : |\Delta^m x_r| \geq \frac{\psi t}{1-\psi} > 0\right\} \\ &= \{r \leq n : |\Delta^m x_r| = 1\} \\ &= \{r \leq n : r \text{ is a cube}\} \end{aligned}$$

We have,

$$\frac{1}{n^\alpha}|R_r(\psi,\epsilon)| = \frac{1}{n^\alpha}|\{r \leq n : r \text{ is a cube}\}| \leq \frac{n^{1/3}}{n^\alpha}.$$

This implies

$$\lim_{n\to\infty} \frac{1}{n^\alpha}|R_r(\psi,\epsilon)| = 0 \text{ where } \alpha \in \left(\frac{1}{3},1\right),$$

which gives that

$$St^\alpha_{\mathcal{P}} - \lim \Delta^m x_r = 0.$$

i.e. $\Delta^m x_r$ is $St^\alpha_{\mathcal{P}}$-convergent to zero. But, $\Delta^m x_r$ is not convergent with respect to $\mathcal{P}$ in the space $(\mathbb{R}, |\,.\,|)$.

Theorem 2.4. *Let $(X;\mathcal{P},*)$ be a $PN-Space$ and $0 < \alpha \leq 1$. Then*
(i) If $St^\alpha_{\mathcal{P}} - \lim \Delta^m x_r = L_1$, $St^\alpha_{\mathcal{P}} - \lim \Delta^m y_r = L_2$ then $St^\alpha_{\mathcal{P}} - \lim \Delta^m(x_r + y_r) = L_1 + L_2$.
(ii) If $St^\alpha_{\mathcal{P}} - \lim \Delta^m x_r = L$, $c \in \mathbb{R}$, then $St^\alpha_{\mathcal{P}} - \lim \Delta^m(cx_r) = cL$.

Proof. (i) Let $St^\alpha_{\mathcal{P}} - \lim \Delta^m x_r = L_1$, $St^\alpha_{\mathcal{P}} - \lim \Delta^m y_r = L_2$, $\psi \in (0,1)$ and take $\theta \in (0,1)$ such that $(1-\theta)*(1-\theta) > 1-\psi$ then for $\epsilon > 0$,

$$\mathcal{K}_1(\theta,\epsilon) = \{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - L_1,\epsilon) \leq 1-\theta\},$$

$$\mathcal{K}_2(\theta,\epsilon) = \{r \in \mathbb{N} : \mathcal{P}(\Delta^m y_r - L_2,\epsilon) \leq 1-\theta\}.$$

Since $St^\alpha_{\mathcal{P}} - \lim \Delta^m x_r = L_1$ therefore, α-density of the set $\mathcal{K}_1(\theta,\epsilon) = 0$. Also for $St^\alpha_{\mathcal{P}} - \lim \Delta^m y_r = L_2$ we get α-density of the set $\mathcal{K}_2(\theta,\epsilon) = 0$. For all $\epsilon > 0$, Let $\mathcal{K}(\theta,\epsilon) = \mathcal{K}_1(\theta,\epsilon) \cap \mathcal{K}_2(\theta,\epsilon)$. Then α-density of the set $\mathcal{K}(\theta,\epsilon)$ is zero, which implies $\lim\limits_{n\to\infty} \frac{1}{n^\alpha}|\{\mathbb{N} - \mathcal{K}(\theta,\epsilon)\}| = 1$. Let $r \in \mathbb{N} - \mathcal{K}(\theta,\epsilon)$, then

$$\begin{aligned} \mathcal{P}(\Delta^m x_r - (L_1+L_2);\epsilon) &\geq \mathcal{P}(\Delta^m x_r - L_1,\epsilon/2) * \mathcal{P}(\Delta^m x_r - L_2,\epsilon/2) \\ &> (1-\theta)*(1-\theta) \\ &> 1-\phi. \end{aligned}$$

This shows that

$$\lim_{n\to\infty} \frac{1}{n^\alpha}|r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - L_1 + \Delta^m y_r - L_2, \epsilon \leq 1-\psi| = 0.$$

Hence

$$St_{\mathcal{P}}^{\alpha} - \lim \Delta^m (x_r + y_r) = L_1 + L_2.$$

(ii) Let

$$St_{\mathcal{P}}^{\alpha} - \lim \Delta^m x_r = L.$$

First we take $c = 0$. For $\epsilon > 0$ and $\psi \in (0, 1)$

$$\mathcal{P}(\Delta^m 0x_r - 0L; \epsilon) = \mathcal{P}(0; \epsilon) = 1 > 1 - \psi.$$

So

$$\mathcal{P}(\Delta^m 0x_r, \epsilon) = 0.$$

Now let $c \in \mathbb{R}$ $(c \neq 0)$. Since $St_{\mathcal{P}}^{\alpha} - \lim \Delta^m x_r = L$, so for $\epsilon > 0$ and $\psi \in (0, 1)$ take

$$\mathcal{K}(\theta, \epsilon) = \{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - L, \epsilon) \leq 1 - \theta\}.$$

We have, $\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{\mathcal{K}(\theta, \epsilon)\}| = 0$ *i.e.* $\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{\mathcal{N} - \mathcal{K}(\theta, \epsilon)\}| = 1$. If $r \in \mathbb{N} - \mathcal{K}(\theta, \epsilon)$, then

$$\begin{aligned}
\mathcal{P}(\Delta^m cx_r - L, \epsilon) &\geq \mathcal{P}\left(\Delta^m x_r - L, \frac{\epsilon}{|c|}\right) \\
&\geq \mathcal{P}(\Delta^m x_r - L, \epsilon) * \mathcal{P}\left(0; \frac{\epsilon}{|c|} - \epsilon\right) \\
&> 1 - \psi * 1 \\
&= 1 - \psi.
\end{aligned}$$

For $c \in \mathbb{R}(c \neq 0)$ this shows that

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - cL, \epsilon) \leq 1 - \psi\}| = 0.$$

Hence

$$St_{\mathcal{P}}^{\alpha} - \lim \Delta^m cx_r = cL.$$

□

Theorem 2.5. *Let* $(X; \mathcal{P}, *)$ *be a* $PN - Space$ *and* $\alpha \in (0, 1]$. *A generalized difference sequence* $x = \{x_r\}$ *is* Δ^m*-statistically convergent with respect to the probabilistic norm* $\mathcal{P}$ *i.e.* $St_{\mathcal{P}}^{\alpha} - \lim \Delta^m x_r = L$ *if and only if there exists an increasing index sequence* $R = \{r_i\}$ *of natural numbers such that* $\lim_{n\to\infty} \frac{1}{n^{\alpha}} |R| = 1$ *and* $\mathcal{P} - \lim \Delta^m x_r = L$.

Proof. For the necessary part, first we assume that

$$St_{\mathcal{P}}^{\alpha} - \lim \Delta^m x_r = L.$$

For every $\epsilon > 0$ and $\theta \in \mathbb{N}$, define

$$R(\theta,\epsilon) = \left\{ r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - L, \epsilon) \leq 1 - \frac{1}{\theta} \right\},$$

$$S(\theta,\epsilon) = \left\{ r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - L, \epsilon) > 1 - \frac{1}{\theta} \right\}.$$

Then

$$\lim_{n\to\infty} \frac{1}{n^\alpha}|R(\theta,\epsilon)| = 0 \text{ and } \lim_{n\to\infty} \frac{1}{n^\alpha}|S(\theta,\epsilon)| = 1. \tag{2.1}$$

$$S(1,\epsilon) \supset S(2,\epsilon) \supset S(3,\epsilon) \supset \ldots \supset S(i,\epsilon) \supset S(i+1,\epsilon)\ldots \tag{2.2}$$

Now, we prove that, for $r \in S(\theta,\epsilon)$, the sequence $x = \{x_r\}$ is $St^\alpha_\mathcal{P} - \lim \Delta^m x_r = L$. Suppose, on the contrary that $St^\alpha_\mathcal{P} - \lim \Delta^m x_r \neq L$. Then, there exists $\psi > 0$ such that the set $\{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - L, \epsilon) \leq 1 - \psi\}$ has infinitely many terms. Let $S(\psi,\epsilon) = \{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - L, \epsilon) \leq 1 - \psi\}$, where $\psi > \frac{1}{\theta}$ $(\theta \in \mathbb{N})$. Then $\lim_{n\to\infty} \frac{1}{n^\alpha}|S(\psi,\epsilon)| = 0$, which contradicts with equation (2.1) as $S(\theta,\epsilon) \subset S(\psi,\epsilon)$. Consequently, the sequence $x = \{x_r\}$ is $St^\alpha_\mathcal{P} - \lim \Delta^m x_r = L$. Next, to prove the sufficient part, we assume that there is a set

$$R = r_1 < r_2 < r_3 < \ldots < r_r < \ldots \subseteq \mathbb{N}$$

such that $\lim_{n\to\infty} \frac{1}{n^\alpha}|R| = 1$, and $\mathcal{P} - \lim \Delta^m x_r = L$. Then, for every $\psi \in (0,1)$ and $t > 0$ we have $\mathcal{P}(\Delta^m x_r - L, \epsilon) > 1 - \psi$. Also

$$\begin{aligned} S(\theta,\epsilon) &= \{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - L, \epsilon) \leq 1 - \psi\} \\ &\subseteq \mathbb{N} - \{S_\mathbb{N} + 1, S_\mathbb{N} + 2, \ldots\} \end{aligned}$$

Therefore $\lim_{n\to\infty} \frac{1}{n^\alpha}|S(\psi,\epsilon)| \leq 1 - 1 = 0$. Hence $\mathcal{P} - \lim \Delta^m x_r = L$. $\square$

Theorem 2.6. *A sequence $x = \{x_r\}$ in $PN-$ Space $(X;\mathcal{P},*)$ is Δ^m-statistically convergent of order-α if and only if it is Δ^m-statistically Cauchy of order α; for all $\alpha \in (0,1]$.*

Proof. Let the sequence $x = \{x_r\}$ is Δ^m-statistically convergent of order-α in $PN-$ *Space* $(X;\mathcal{P},*)$ *i.e.* $St^\alpha_\mathcal{P} - \lim \Delta^m x_r = L$. Then for every $\epsilon > 0$ and $\psi \in (0,1)$ take $\theta > 0$ such that $(1-\theta) * (1-\theta) > 1 - \psi$. Define

$$A(\theta,\epsilon) = \{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - L, \epsilon/2) \leq 1 - \theta\},$$

Since $St^\alpha_\mathcal{P} - \lim \Delta^m x_r = L$ then $\lim_{n\to\infty}, \frac{1}{n^\alpha}|A(\theta,\epsilon)| = 0$. Let $s \in A^c(\theta,\epsilon)$ then $\mathcal{P}(\Delta^m x_s - L, \epsilon/2) > 1 - \theta$. If $B(\psi,\epsilon) = \{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - \Delta^m x_s, \epsilon) \leq 1 - \psi\}$, then to prove result we show that $B(\psi,\epsilon) \subseteq A(\theta,\epsilon)$. Let $r \in B(\psi,\epsilon)$ then $\mathcal{P}(\Delta^m x_r - \Delta^m x_s, \epsilon) \leq 1 - \psi$. Either $\mathcal{P}(\Delta^m x_r - L, \epsilon/2) \leq 1 - \theta$ or $\mathcal{P}(\Delta^m x_r - L, \epsilon/2) > 1 - \theta$.

(a) If $\mathcal{P}(\Delta^m x_r - L, \epsilon/2) \leq 1-\theta$ then $r \in A(\theta,\epsilon)$.
(b) If $\mathcal{P}(\Delta^m x_r - L, \epsilon/2) > 1-\theta$ then

$$\begin{aligned} 1-\psi \geq \mathcal{P}(\Delta^m x_r - \Delta^m x_s, \epsilon) &\geq \mathcal{P}(\Delta^m x_r - L, \epsilon/2) * \mathcal{P}(\Delta^m x_s - L, \epsilon/2) \\ &> (1-\theta) * (1-\theta) \\ &> 1-\psi. \end{aligned}$$

which is not possible. Therefore, $B(\psi,\epsilon) \subseteq A(\theta,\epsilon)$. Hence, $x = \{x_r\}$ is Δ^m-statistically Cauchy of order α in $PN - Space$.
Conversely, let $x = \{x_r\}$ be Δ^m-statistically Cauchy of order-α in $PN - Space$ but not Δ^m-statistically convergent of order α. Then, we get $s \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |C^c(\psi,\epsilon)| = 1 \text{ and } \lim_{n\to\infty} \frac{1}{n^\alpha} |D^c(\psi,\epsilon)| = 0,$$

where

$$C(\psi,\epsilon) = \{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - \Delta^m x_s, \epsilon) \leq 1-\psi\},$$

$$D(\psi,\epsilon) = \{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - L, \epsilon/2) \leq 1-\psi\}.$$

As $\mathcal{P}(\Delta^m x_r - \Delta^m x_s, \epsilon) \geq 2\mathcal{P}\left(\Delta^m x_r - L, \frac{\epsilon}{2}\right) > 1-\psi$ if $\mathcal{P}(\Delta^m x_r - L, \epsilon) > \frac{1-\psi}{2}$. Then, we get $\lim_{n\to\infty} \frac{1}{n^\alpha} |\{r \in \mathbb{N} : \mathcal{P}(\Delta^m x_r - \Delta^m x_s, \epsilon) > 1-\psi\}| = 0$ i.e. $\lim_{n\to\infty} \frac{1}{n^\alpha} |C^c(\psi,\epsilon)| = 0$ which contradicts that the set $C^c(\psi,\epsilon)$ has α-natural density 1. Consequently, we get that sequence $x = \{x_r\}$ is Δ^m-statistically Cauchy of order α in $PN - Space$. $\square$

3. Conclusion

In the paper, we have presented the notion of Δ^m-statistical convergence and Δ^m-statistically Cauchy of generalized difference sequences of order α in probabilistic normed spaces. The notion Δ^m-statistical convergence and Δ^m-statistically Cauchy of generalized difference sequences of order α in probabilistic normed spaces is more generalized than the notion of Δ^m-statistical convergence and Δ^m-statistically Cauchy in probabilistic normed spaces. We have proved some useful results of order α for these notions in the probabilistic normed spaces. We have discussed some examples which are the proof that this notion is more generalized than the corresponding results of normed spaces.

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