FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 5 (2021), 1143-1153
https://doi.org/10.22190/FUMI210804083K
Original Scientific Paper

ON THE STRECH CURVATURE OF HOMOGENEOUS FINSLER METRICS

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Abstract. In this paper, we study the stretch curvature of homogeneous Finsler manifolds. First, we prove that every homogeneous Finsler metric has relatively isotropic stretch curvature if and only if it is a Landsberg metric. It follows that every weakly Berwald homogeneous metric has relatively isotropic stretch curvature if and only if it is a Berwald metric. We show that a homogeneous metric of non-zero scalar flag curvature has relatively isotropic stretch curvature if and only if it is a Riemannian metric of constant sectional curvature. It turns out that a homogeneous (α, β)-metric with relatively isotropic stretch curvature is a Berwald metric. Also, it follows that a homogeneous spherically symmetric metric with relatively isotropic stretch curvature reduces to a Riemannian metric. Finally, we prove that every homogeneous stretch-recurrent metric is a Landsberg metric.
Keywords: Strech metric, Landsberg metric, Berwald metric, (fi; fl)-metric, homogeneous metric.

1. Introduction

In [7], Deng-Hou proved that the group of isometries of a Finsler manifold $(M.F)$, denoted by $I(M, F)$, is a Lie transformation group of the underlying manifold that can be used to study homogeneous Finsler manifolds. This important result opens

Received August 4, 2021. accepted October 14, 2021.
Communicated by Uday Chand De
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2010 *Mathematics Subject Classification.* Primary 53B40; Secondary 53C60

an interesting window to generalize the concept of homogeneous Riemannian manifold to homogeneous Finsler manifold. An n-dimensional Finsler manifold (M, F) is called a homogeneous Finsler manifold if the group $I(M, F)$ acts transitively on the manifold M.

A Finsler metric F on a manifold M is called a Berwald metric if its spray coefficients G^i are quadratic in $y \in T_xM$ for all $x \in M$. The important described characteristic of a Berwald space is that all its tangent spaces are linearly isometric to a common Minkowski space. For a Landsberg space, all its tangent spaces are isometric to a common Minkowski space. Thus every Berwald space is a Landsberg space. However, it has been one of the longest-standing problems in Finsler geometry whether there exists a Landsberg space that is not a Berwald space. In [35], Xu-Deng conjectured that every homogeneous Landsberg space must be a Berwald space.

In 1924, at the annual meeting of the Mathematical Society of Germany in Innsbruck, Berwald defined of the stretch curvature as a generalization of Landsberg curvature and denoted it by $\mathbf{T}$ [3]. He published the stretch curvature in 1925 on the first of his main papers [5]. He showed that $\mathbf{T} = 0$ if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram. In his lecture at the International Congress of Mathematics, Bologna, 1928, he introduced a series of special classes of Finsler metrics, such as Landsberg metrics and stretch metrics [2]. He proved that for two-dimensional stretch metrics, the total curvature (curvature integral) $\int\int \mathbf{R}\sqrt{g}dx^1dx^2$ can be defined, which means the integrand is a function of position alone, where $\mathbf{R}$ is the Underhill curvature. Then, this curvature has been investigated by Shibata in [21] and Matsumoto in [10]. Matsumoto denoted this curvature by $\mathbf{\Sigma}$. We have the following big picture.

$$\{\text{Berwald metrics}\} \subseteq \{\text{Landsberg metrics}\} \subseteq \{\text{Stretch metrics}\}.$$

Let (M, F) be a Finsler manifold. Then F is called a relatively isotropic stretch metric if its stretch curvature is given by

$$\Sigma_{ijkl} = cF(C_{ijk|l} - C_{ijl|k}), \tag{1.1}$$

where $c = c(x)$ is a scalar function on M, and "|" denotes the horizontal covariant derivative with respect to the Berwald connection of F. In this case, (M, F) is called a relatively isotropic stretch manifold.

Example 1.1. A Finsler metric F satisfying $F_{x^k} = FF_{y^k}$ is called a Funk metric. The standard Funk metric on the Euclidean unit ball $\mathbb{B}^n(1)$ is defined by

$$F(x, y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x, y>^2)}}{1 - |x|^2} + \frac{<x, y>}{1 - |x|^2}, \quad y \in T_x\mathbb{B}^n(1) \simeq \mathbb{R}^n, \tag{1.2}$$

where $<,>$ and $|.|$ denote the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. It follows from $G^i = \frac{1}{2}Fy^i$ that F satisfies (1.1) with $c = -1$.

Example 1.2. For $y \in T_x\mathbb{B}^n(1) \simeq \mathbb{R}^n$, let us define

$$F_a(x,y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - < x,y >^2)}}{1-|x|^2} + \frac{< x,y >}{1-|x|^2} + \frac{< a,y >}{1+ < a,x >}, \tag{1.3}$$

where $a \in \mathbb{R}^n$ is a constant vector with $|a| < 1$. For $a \neq 0$, it is easy to see that F_a is a locally projectively flat Finsler metric with negative constant flag curvature. It follows that F is a relatively isotropic stretch metric with $c = -1$.

In this paper, we prove the following.

Theorem 1.1. *Every homogeneous Finsler metric on a manifold M has relatively isotropic stretch curvature if and only if it is a Landsberg metric.*

In [27], Tayebi-Najafi proved that every homogeneous Landsberg surface is Riemannian or locally Minkowskian spaces. Then by Theorem 1.1, we conclude that every homogeneous Finsler surface of relatively isotropic stretch curvature is Riemannian or locally Minkowskian spaces.

There is another important quantity defined by the spray of a Finsler metric F. Taking a trace of Berwald curvature implies the mean Berwald curvature $\mathbf{E}$. A Finsler metric F is said to be weakly Berwaldian if $\mathbf{E} = 0$.

Corollary 1.1. *Every weakly Berwald homogeneous Finsler metric on a manifold M has relatively isotropic stretch curvature if and only if it is a Berwald metric.*

Douglas curvature is a non-Riemannian projectively invariant constructed from the Berwald curvature. The notion of Douglas curvature was proposed by Bácsó-Matsumoto as a generalization of Berwald curvature [1]. The Douglas curvature vanishes for Riemannian spaces; therefore, it plays a prominent role only outside the Riemannian world. Finsler metrics with $\mathbf{D} = 0$ are called Douglas metrics.

Corollary 1.2. *Every Douglas homogeneous Finsler metric on a manifold M has relatively isotropic stretch curvature if and only if it is a Berwald metric.*

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry, first introduced by L. Berwald [2][5]. For a Finsler manifold (M, F), the flag curvature is a function $\mathbf{K}(P, y)$ of tangent planes $P \subset T_xM$ and directions $y \in P$. A Finsler metric F is said to be of scalar flag curvature if the flag curvature $\mathbf{K}(P, y) = \mathbf{K}(x, y)$ is independent of flags P associated with any fixed flagpole y. Finsler metrics of scalar flag curvature are the natural extension of Riemannian metrics of isotropic sectional curvature (of constant sectional curvature in dimension $n \geq 3$ by the Schur Lemma). One of the central problems in Finsler geometry is to characterize Finsler manifolds of scalar flag curvature.

Corollary 1.3. *Let (M, F) be a homogeneous Finsler metric of dimension $n \geq 3$. Suppose that F has non-zero scalar flag curvature. Then F has relatively isotropic stretch curvature if and only if it is a Riemannian metric of constant sectional curvature.*

An (α, β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with a certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, and $\beta = b_i(x)y^i$ is a 1-form on M (see [26], [31] and [32]).

Corollary 1.4. *Every homogeneous (α, β)-metric on a manifold M has relatively isotropic stretch curvature if and only if it is a Berwald metric.*

A Finsler metric $F = F(x, y)$ on a domain $\Omega \subseteq \mathbb{R}^n$ is called spherically symmetric metric if it is invariant under any rotation in $\mathbb{R}^n$. Indeed, the class of spherically symmetric metrics in the Finsler setting was first introduced by S.F. Rutz, who studied the spherically symmetric Finsler metrics in 4-dimensional space-time and generalized the classic Birkhoff theorem in general relativity to the Finsler case [17]. According to the equation of Killing fields, there exists a positive function ϕ depending on two variables so that F can be written as

$$F = |y|\phi\Big(|x|, \frac{\langle x, y\rangle}{|y|}\Big),$$

where x is a point in the domain Ω, y is a tangent vector at the point x. There are classical Finsler metrics which are spherically symmetric, such as Funk metric, Berwald's metric, Bryant's metric, etc, (see [14] for more details). .

Corollary 1.5. *Every homogeneous spherically symmetric Finsler metric on a manifold M has relatively isotropic stretch curvature if and only if it is a Riemannian metric.*

A homogeneous Finsler manifold (M, F) is said to be stretch-recurrent or $\mathbf{\Sigma}$-recurrent if its stretch curvature satisfies following

$$\Sigma_{ijkl|s}y^s = \Psi\Sigma_{ijkl}, \tag{1.4}$$

where Ψ is a non-zero smooth function on TM_0 satisfying $\Psi(x, ty) = t\Psi(x, y)$ for all positive real number t and $(x, y) \in TM_0$. It is easy to see that every stretch metric and then Landsberg metric is a Σ-recurrent metric. However, the converse is not valid in general. Here, we prove that every homogeneous Σ-recurrent Finsler metric is a Landsberg metric.

Theorem 1.2. *Any homogeneous Σ-recurrent metric is a Landsberg metric.*

2. Preliminaries

Let (M, F) be an n-dimensional Finsler manifold. The fundamental tensor $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ of F is defined by following

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are scalar functions on TM_0 given by

$$G^i := \frac{1}{4} g^{ij}\left\{\frac{\partial^2[F^2]}{\partial x^k \partial y^j} y^k - \frac{\partial[F^2]}{\partial x^j}\right\}, \quad y \in T_xM. \tag{2.1}$$

The $\mathbf{G}$ is called the spray associated to (M, F).

For a non-zero vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

The quantity $\mathbf{B}$ is called the Berwald curvature. F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

Define the mean of Berwald curvature by $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$, where

$$\mathbf{E}_y(u, v) := \frac{1}{2}\sum_{i=1}^{n} g^{ij}(y) g_y\Big(\mathbf{B}_y(u, v, e_i), e_j\Big). \tag{2.2}$$

The family $\mathbf{E} = \{\mathbf{E}_y\}_{y \in TM\setminus\{0\}}$ is called the *mean Berwald curvature* or *E-curvature*. In local coordinates, $\mathbf{E}_y(u, v) := E_{ij}(y) u^i v^j$, where

$$E_{ij} := \frac{1}{2} B^m{}_{mij}.$$

By definition, $\mathbf{E}_y(u,v)$ is symmetric in u and v and we have $\mathbf{E}_y(y,v)=0$. The quantity $\mathbf{E}$ is called the mean Berwald curvature. F is called a weakly Berwald metric if $\mathbf{E}=\mathbf{0}$.

For non-zero vector $y\in T_xM_0$, define $\mathbf{D}_y: T_xM\times T_xM\times T_xM\to T_xM$ by $\mathbf{D}_y(u,v,w):=D^i{}_{jkl}(y)u^iv^jw^k\frac{\partial}{\partial x^i}|_x$, where

$$D^i{}_{jkl} := \frac{\partial^3}{\partial y^j\partial y^k\partial y^l}\Big[G^i-\frac{2}{n+1}\frac{\partial G^m}{\partial y^m}y^i\Big]. \tag{2.3}$$

$\mathbf{D}$ is called the Douglas curvature. F is called a Douglas metric if $\mathbf{D}=\mathbf{0}$ [1]. By definition, it follows that the Douglas tensor $\mathbf{D}_y$ is symmetric trilinear form and has the following properties

$$\mathbf{D}_y(y,u,v)=0,\qquad trace(\mathbf{D}_y)=0.$$

According to (2.3), the Douglas tensor can be written as follows

$$D^i{}_{jkl}=B^i{}_{jkl}-\frac{2}{n+1}\Big\{E_{jk}\delta^i{}_l+E_{kl}\delta^i{}_j+E_{lj}\delta^i{}_k+E_{jk,l}y^i\Big\}.$$

For $y\in T_xM$, define the Landsberg curvature $\mathbf{L}_y: T_xM\times T_xM\times T_xM\to\mathbb{R}$ by

$$\mathbf{L}_y(u,v,w):=-\frac{1}{2}\mathbf{g}_y\big(\mathbf{B}_y(u,v,w),y\big).$$

F is called a Landsberg metric if $\mathbf{L}_y=0$. By definition, every Berwald metric is a Landsberg metric.

For $y\in T_xM_0$, define the stretch curvature $\mathbf{\Sigma}_y: T_xM\times T_xM\times T_xM\times T_xM\to\mathbb{R}$ by $\mathbf{\Sigma}_y(u,v,w,z):=\Sigma_{ijkl}(y)u^iv^jw^kz^l$, where

$$\Sigma_{ijkl}:=2(L_{ijk|l}-L_{ijl|k}), \tag{2.4}$$

and "|" denotes the horizontal derivation with respect to the Berwald connection of F. A Finsler metric is said to be a stretch metric if $\mathbf{\Sigma}=0$.

The second variation of geodesics gives rise to a family of linear maps $\mathbf{R}_y: T_xM\to T_xM$ with homogeneity $\mathbf{R}_{\lambda y}=\lambda^2\mathbf{R}_y$, $\forall\lambda>0$ which is defined by

$$\mathbf{R}_y(u):=R^i{}_k(y)u^k\frac{\partial}{\partial x^i},$$

where

$$R^i{}_k(y)=2\frac{\partial G^i}{\partial x^k}-\frac{\partial^2G^i}{\partial x^j\partial y^k}y^j+2G^j\frac{\partial^2G^i}{\partial y^j\partial y^k}-\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

$\mathbf{R}_y$ is called the Riemann curvature in the direction y.

For a flag $P := \mathrm{span}\{y, u\} \subset T_xM$ with flagpole y, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(x, y, P) := \frac{\mathbf{g}_y\big(u, \mathbf{R}_y(u)\big)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}. \tag{2.5}$$

The flag curvature $\mathbf{K}(x, y, P)$ is a function of tangent planes $P = \mathrm{span}\{y, v\} \subset T_xM$. A Finsler metric F is of scalar flag curvature if $\mathbf{K} = \mathbf{K}(x, y)$ is independent of flag P (see [23], [24] and [25]).

3. Proof of Theorem 1.1

Every two points of a homogeneous Finsler manifold map to each other by an isometry. Then, the norm of arbitrary tensor of a homogeneous Finsler manifold is a constant function on the underlying manifold. Thus the norm of an arbitrary tensor of a homogeneous Finsler space is bounded. This fact is proved in [28].

Lemma 3.1. ([28]) *Let (M, F) be a homogeneous Finsler manifold. Then the norm of an arbitrary tensor of F which is invariant under every isometry of F is bounded.*

We define the norm of the Landsberg curvature at $x \in M$ by

$$\|\mathbf{L}\|_x := \sup_{y,u,v,w \in T_xM\setminus\{0\}} \frac{F(y)|\mathbf{L}_y(u, v, w)|}{\sqrt{\mathbf{g}_y(u, u)\mathbf{g}_y(v, v)\mathbf{g}_y(w, w)}}.$$

We showed that the Landsberg curvature of homogeneous Finsler metric F is bounded.

Lemma 3.2. ([28]) *Let (M, F) be a homogeneous Finsler manifold. Then the Landsberg curvature of F is bounded.*

In order to prove Theorem 1.1, we need the following.

Theorem 3.1. ([29]) Homogeneous Finsler manifolds are complete.

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1: Let p be an arbitrary point of manifold M, and $y, u, v, w \in T_pM$. Let $c : (-\infty, \infty) \to M$ is the unit speed geodesic passing from p and

$$\frac{dc}{dt}(0) = y.$$

If $U(t), V(t)$ and $W(t)$ are the parallel vector fields along c with

$$U(0) = u, \quad V(0) = v \quad W(0) = w.$$

Let us put

$$\mathbf{L}(t) = \mathbf{L}\big(U(t), V(t), W(t)\big),$$
$$\mathbf{L}'(t) = \mathbf{L}'_{\dot{c}}\big(U(t), V(t), W(t)\big).$$

Contracting (1.1) with y^l implies that

$$L_{ijk|l}y^l = cFC_{ijl|k}y^l. \tag{3.1}$$

By definition, we have

$$L_{ijk} = C_{ijl|k}y^l. \tag{3.2}$$

From (3.1) and (3.2), we have

$$L_{ijk|l}y^l = cFL_{ijk}. \tag{3.3}$$

According to the definition, (3.3) yields the following ODE

$$\mathbf{L}'(t) = c\mathbf{L}(t), \tag{3.4}$$

which its general solution is

$$\mathbf{L}(t) = e^{ct}\mathbf{L}(0). \tag{3.5}$$

Using $||\mathbf{L}|| < \infty$, and letting $t \to +\infty$ or $t \to -\infty$, we get

$$\mathbf{L}(0) = \mathbf{L}(u, v, w) = 0.$$

So $\mathbf{L} = 0$, i.e., (M, F) is a Landsberg manifold. □

Proof of Corollary 1.1: In [6], Crampin showed that every Landsberg metric with vanishing mean Berwald curvature is a Berwald metric. Then by Theorem 1.1, we get the proof. □

Proof of Corollary 1.2: Let (M, F) be a Douglas manifold of dimension n. Suppose that F has vanishing Landsberg curvature. In [4], Berwald proved that every 2-dimensional Douglas metric with vanishing Landsberg curvature is a Berwald metric. In 1984, Izumi pointed out that the Berwald theorem must be true for the higher dimensions [8]. In [1], Bácsó-Matsumoto proved that every Douglas metric with vanishing Landsberg curvature is a Berwald metric. Then by Theorem 1.1, we get the proof. □

Proof of Corollary 1.3: According to by Theorem 1.1, F is a Landsberg metric. In [16], Numata proved that every Landsberg metric of non-zero scalar flag curvature is a Riemannian metric of constant sectional curvature. This completes the proof. □

Proof of Corollary 1.4: In [19], Shen proved that an (α, β)-metric with vanishing Landsberg curvature is a Berwald metric. Then by Theorem 1.1, we get the proof. □

Proof of Corollary 1.5: In [14], Mo-Zhou classified the spherically symmetric Finsler metrics in $\mathbb{R}^n$ with Landsberg type and found some exceptional almost regular metrics which do not belong to Berwald type. They proved that every regular spherically symmetric Finsler metric in $\mathbb{R}^n$ is a Berwald metric. Then they proved that all of Berwaldian spherically symmetric Finsler metrics are Riemannian. Then by Theorem 1.1, we get the proof. □

4. Stretch-Recurrent Homogeneous Metrics

In this section, we are going to prove Theorem 1.2.

Proof of Theorem 1.2: We know that (M, F) is homogeneous, and the scalar function Ψ is invariant under the isometries of F. In general, if a continuous function $f : TM_0 \to \mathbb{R}$ is invariant under isometries of (M, F) and also is positively homogeneous of degree zero with respect to directions, then f is a bounded function. Thus, $f := \Psi/F$ is bounded and, by definition, is everywhere non-zero. Since M is connected, the range of Ψ/F is an interval, say $(c_1, c_2) \subseteq \mathbb{R}$, which does not contain zero. Without loss of generality, suppose that $c_1 > 0$. Thus, we have

$$c_1 F(x, y) \leq \Psi(x, y) \leq c_2 F(x, y), \ \ \forall (x, y) \in TM_0. \tag{4.1}$$

For $y \in T_xM$, let $c = c(t)$ be the unit speed geodesic of (M, F) with $\dot{c}(0) = y$ and $c(0) = x$. Suppose $X = X(t)$, $Y = Y(t)$, $Z = Z(t)$ and $W = W(t)$ are parallel vector fields along the geodesic c. Define $\mathbf{\Sigma}(t)$ as follows

$$\mathbf{\Sigma}(t) = \mathbf{\Sigma}_{\dot{c}}\Big(X(t), Y(t), Z(t), W(t)\Big). \tag{4.2}$$

Thus, the restriction of (1.4) to the canonical lift of c, i.e., $(c, \dot{c})$ becomes

$$\mathbf{\Sigma}'(t) = \Psi(t)\mathbf{\Sigma}(t). \tag{4.3}$$

For simplicity, we have used the following nomination:

$$\Psi(t) := \Psi\big(c(t), \dot{c}(t)\big).$$

By (4.3), we get

$$\mathbf{\Sigma}(t) = e^{\int_0^t \Psi(s)\ ds}\,\mathbf{\Sigma}(0) \tag{4.4}$$

It follows from (4.1) and $F(c(t), \dot{c}(t)) = 1$ that

$$e^{c_1 t} \leq e^{\int_0^t \Psi(s)\ ds} \leq e^{c_2 t}, \quad \forall t > 0. \tag{4.5}$$

The stretch tensor of any homogeneous metric is a bounded tensor. Let $\mathbf{\Sigma}(0) \neq 0$. By Theorem 3.1, M is complete, and the parameter t takes all the values in $(-\infty, +\infty)$. Letting $t \to \infty$, we conclude that the norm of $\mathbf{\Sigma}(t)$ is unbounded which arises a contradiction. Therefore, we get

$$\mathbf{\Sigma}(0) = 0,$$

and F reduces to a stretch metric. On the other hand, in [28] it is proved that every homogenous stretch metric is a Landsberg metric. This completes the proof. □

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