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SOME RESULTS ON MIXED SUPER QUASI-EINSTEIN MANIFOLDS SATISFYING CERTAIN VECTOR FIELDS

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Abstract. The objective of this paper is to discuss various properties of mixed super quasi-Einstein manifolds admitting certain vector fields. We analyze the behaviour of $MS\left(QE\right)_n$ satisfying Codazzi type of Ricci tensor. We have also constructed a non-trivial example related to mixed super quasi-Einstein manifolds.
Keywords: Mixed super quasi-Einstein manifolds, pseudo quasi-Einstein manifold, Codazzi type of Ricci tensor, cyclic parallel Ricci tensor, Killing vector field, concurrent vector field.

1. Introduction

An n-dimensional semi-Riemannian or Riemannian manifold (M^n, g) $(n > 2)$, is called an Einstein manifold if its Ricci tensor S satisfies the criteria

$$S = \frac{r}{n}g, \tag{1.1}$$

where r denotes the scalar curvature of (M^n, g). We can also say an Einstein manifold is a Riemannian or pseudo Riemannian manifold whose Ricci tensor is proportional to the metric. The notion of quasi-Einstein manifold was introduced by M.C. Chaki and R.K. Maity [5]. A non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is a quasi-Einstein manifold if its Ricci tensor S satisfies the criteria

$$S\left(X, Y\right) = ag\left(X, Y\right) + bA\left(X\right)A\left(Y\right) \tag{1.2}$$

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and is not identically zero, where a, b are scalars, $b \neq 0$ and A is a non-zero 1-form such that

$$g(X,U) = A(X),$$

for all vector field X. U being a unit vector field.
Here a and b are called the associated scalars, A is called the associated 1-form and U is called the generator of the manifold. Such an n-dimensional manifold is denoted by $(QE)_n$. The quasi-Einstein manifolds have also been studied by De and Ghosh [7], Bejan [1], De and De [6], Han, De and Zhao [15] and many others. Quasi-Einstein manifolds have been generalized by many authors in several ways such as generalized quasi-Einstein manifolds [3, 9, 11, 23], $N(K)$-quasi Einstein manifolds [17, 24], super quasi-Einstein manifolds [4, 10, 19] etc.

Chaki [4] introduced the notion of a super quasi-Einstein manifold. His work suggested a non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$ is called a super quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) &= ag(X,Y) + bA(X)A(Y) \\ &+ c[A(X)B(Y) + A(Y)B(X)] + dD(X,Y), \end{aligned} \tag{1.3}$$

where a, b, c, d are scalars in which $b \neq 0$, $c \neq 0$ $d \neq 0$ and A, B are non-zero 1-forms such that

$$g(X,U) = A(X), \quad g(X,V) = B(X),$$

where U, V are mutually orthogonal unit vector fields, D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,U) = 0,$$

for all X. In that case a, b, c, d are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, U, V are called the main and auxiliary generators of the manifold and D is called the associated tensor of the manifold. Such an n-dimensional manifold is denoted by $S(QE)_n$.

In [2], A. Bhattacharyya, M. Tarafdar and D. Debnath introduced the notion of mixed super quasi-Einstein manifolds. Their work suggested that a non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is said to be mixed super quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) &= ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ &+ d[A(X)B(Y) + A(Y)B(X)] + eD(X,Y), \end{aligned} \tag{1.4}$$

where a, b, c, d, e are scalars on (M^n, g) of which $b \neq 0$, $c \neq 0$, $d \neq 0$, $e \neq 0$ and A, B are two non-zero 1-forms such that

$$g(X,U) = A(X), \quad g(X,V) = B(X), \tag{1.5}$$

U, V being unit vector fields which are orthogonal, D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,U) = 0, \tag{1.6}$$

for all X. Here a, b, c, d, e are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, U, V are called the main and auxiliary generators of the manifold and D is called the associated tensor of the manifold. If $c = 0$, then the manifold becomes $S(QE)_n$. This type of manifold is denoted by the symbol $MS(QE)_n$. If $c = d = 0$, then the manifold is reduced to a pseudo quasi-Einstein manifold which was studied by Shaikh [22].

On the other hand, Gray [14] introduced two classes of Riemannian manifolds determined by the covariant differentiation of Ricci tensor. The class A consists of all Riemannian manifolds whose Ricci tensor S is a Codazzi type tensor, i.e.,

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z).$$

The class B contains all Riemannian manifolds whose Ricci tensor is cyclic parallel, i.e.,

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0.$$

A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$ is called a generalized Ricci recurrent manifold [8] if its Ricci tensor S of type $(0,2)$ satisfies the condition

$$(\nabla_X S)(Y,Z) = \gamma(X) S(Y,Z) + \delta(X) g(Y,Z),$$

where $\gamma(X)$ and $\delta(X)$ are non-zero 1-forms such that $\gamma(X) = g(X,\rho)$ and $\delta(X) = g(X,\mu)$; ρ and μ being associated vector fields of the 1-forms γ and δ, respectively. If $\delta = 0$, then the manifold reduces to a Ricci recurrent manifold [20].

After studying and analyzing various papers [12, 13, 18], we got motivation to work in this area. Recently in the paper [16], we have studied generalized Quasi-Einstein manifolds satisfying certain vector fields. In the present work we have tried to develop a new concept. This paper is organized as follows: After introduction in Section 2, we have studied that if the generators U and V of a $MS(QE)_n$ are Killing vector fields, then the manifold satisfies cyclic parallel Ricci tensor if and only if the associated tensor D is cyclic parallel. Section 3 is concerned with $MS(QE)_n$ satisfying Codazzi type of Ricci tensor. In the next two sections, we have studied $MS(QE)_n$ with generators U and V both as concurrent and recurrent vector fields. Finally the existence of $MS(QE)_n$ is shown by constructing non-trivial example.

2. The generators U and V as Killing vector fields

In this section we consider the generators U and V of the manifold are Killing vector fields.

Theorem 2.1. *If the generators of a $MS\left(QE\right)_n$ are Killing vector fields and the associated scalars are constants, then the manifold satisfies cyclic parallel Ricci tensor if and only if the associated tensor D is cyclic parallel.*

Proof. Let us assume that the generators U and V of the manifold are Killing vector fields. Then we have

$$(\pounds_U g)(X,Y) = 0 \tag{2.1}$$

and

$$(\pounds_V g)(X,Y) = 0, \tag{2.2}$$

where $\pounds$ denotes the Lie derivative.
From (2.1) and (2.2), we get

$$g(\nabla_X U, Y) + g(X, \nabla_Y U) = 0 \tag{2.3}$$

and

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) = 0. \tag{2.4}$$

Since $g(\nabla_X U, Y) = (\nabla_X A)(Y)$ and $g(\nabla_X V, Y) = (\nabla_X B)(Y)$.
Thus from (2.3) and (2.4) we obtain

$$(\nabla_X A)(Y) + (\nabla_Y A)(X) = 0 \tag{2.5}$$

and

$$(\nabla_X B)(Y) + (\nabla_Y B)(X) = 0, \tag{2.6}$$

for all X, Y.
Similarly, we have

$$(\nabla_X A)(Z) + (\nabla_Z A)(X) = 0, \tag{2.7}$$

$$(\nabla_Z A)(Y) + (\nabla_Y A)(Z) = 0, \tag{2.8}$$

$$(\nabla_X B)(Z) + (\nabla_Z B)(X) = 0, \tag{2.9}$$

$$(\nabla_Z B)(Y) + (\nabla_Y B)(Z) = 0, \tag{2.10}$$

for all X, Y, Z.
We assume that the associated scalars are constants. Then from (1.4) we have

$$\begin{aligned}(\nabla_Z S)(X,Y) \;=\; & b\left[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)\right] \\ & + c\left[(\nabla_Z B)(X)B(Y) + B(X)(\nabla_Z B)(Y)\right] \\ & + d\left[(\nabla_Z A)(X)B(Y) + A(X)(\nabla_Z B)(Y)\right. \\ & + \left.(\nabla_Z A)(Y)B(X) + A(Y)(\nabla_Z B)(X)\right] \\ & + e(\nabla_Z D)(X,Y).\end{aligned} \tag{2.11}$$

Using (2.11), we get

$$\begin{aligned}(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = b\,[\{(\nabla_X A)(Y) \\ + (\nabla_Y A)(X)\}\,A(Z) + \{(\nabla_X A)(Z) + (\nabla_Z A)(X)\}\,A(Y) \\ + \{(\nabla_Y A)(Z) + (\nabla_Z A)(Y)\}\,A(X)] + c\,[\{(\nabla_X B)(Y) \\ + (\nabla_Y B)(X)\}\,B(Z) + \{(\nabla_X B)(Z) + (\nabla_Z B)(X)\}\,B(Y) \\ + \{(\nabla_Y B)(Z) + (\nabla_Z B)(Y)\}\,B(X)] + d\,[\{(\nabla_X B)(Y) \\ + (\nabla_Y B)(X)\}\,A(Z) + \{(\nabla_X B)(Z) + (\nabla_Z B)(X)\}\,A(Y) \\ + \{(\nabla_Y B)(Z) + (\nabla_Z B)(Y)\}\,A(X) + \{(\nabla_X A)(Y) \\ + (\nabla_Y A)(X)\}\,B(Z) + \{(\nabla_X A)(Z) + (\nabla_Z A)(X)\}\,B(Y) \\ + \{(\nabla_Y A)(Z) + (\nabla_Z A)(Y)\}\,B(X)] + e\,[(\nabla_X D)(Y,Z) \\ + (\nabla_Y D)(Z,X) + (\nabla_Z D)(X,Y)]\,. \end{aligned} \tag{2.12}$$

Using the equations (2.5) - (2.10) in (2.12), we get

$$\begin{aligned}(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = e\,[(\nabla_X D)(Y,Z) \\ + (\nabla_Y D)(Z,X) + (\nabla_Z D)(X,Y)]\,.\end{aligned}$$

Thus the proof of theorem is completed. □

3. $MS\,(QE)_n$ admits Codazzi type of Ricci tensor

We know that a Riemannian or semi-Riemannian manifold satisfies Codazzi type of Ricci tensor if its Ricci tensor S satisfies the following condition

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)\,, \tag{3.1}$$

for all X, Y, Z.

Theorem 3.1. *If a $MS\,(QE)_n$ admits the Codazzi type of Ricci tensor with the associated tensor D satisfying the relation $(\nabla_X D)(Y,V) = (\nabla_Y D)(V,X)$, then either $d = \pm\sqrt{bc}$ or the associated 1-forms A and B are closed.*

Proof. Using (2.11) and (3.1), we obtain

$$\begin{aligned} b\,[(\nabla_X A)(Y)\,A(Z) + A(Y)\,(\nabla_X A)\,(Z)] + c\,[(\nabla_X B)(Y)\,B(Z) \\ + B(Y)\,(\nabla_X B)(Z)] + d\,[(\nabla_X A)(Y)\,B(Z) + A(Y)\,(\nabla_X B)(Z) \\ + (\nabla_X A)(Z)\,B(Y) + A(Z)\,(\nabla_X B)(Y)] + e\,(\nabla_X D)(Y,Z) \\ - b\,[(\nabla_Y A)(Z)\,A(X) + A(Z)\,(\nabla_Y A)(X)] - c\,[(\nabla_Y B)(Z)\,B(X) \\ + B(Z)\,(\nabla_Y B)(X)] - d\,[(\nabla_Y A)(Z)\,B(X) + A(Z)\,(\nabla_Y B)(X) \\ + (\nabla_Y A)(X)\,B(Z) + A(X)\,(\nabla_Y B)(Z)] - e\,(\nabla_Y D)(Z,X) = 0. \end{aligned} \tag{3.2}$$

Putting $Z = U$ in (3.2) and using $(\nabla_X A)(U) = 0$, we have

$$b\,[(\nabla_X A)(Y) - (\nabla_Y A)(X)] + d\,[(\nabla_X B)(Y) - (\nabla_Y B)(X)] = 0,$$

i.e.,

$$bdA(X,Y) = -d\mathbf{d}B(X,Y). \tag{3.3}$$

Similarly, putting $Z = V$ in (3.2) and using $(\nabla_X B)(V) = 0$, we have

$$c[(\nabla_X B)(Y) - (\nabla_Y B)(X)] + d[(\nabla_X A)(Y) - (\nabla_Y A)(X)] \\ +e[(\nabla_X D)(Y,V) - (\nabla_Y D)(V,X)] = 0,$$

i.e.,

$$c\mathbf{d}B(X,Y) + d\mathbf{d}A(X,Y) + e[(\nabla_X D)(Y,V) - (\nabla_Y D)(V,X)] = 0. \tag{3.4}$$

If $(\nabla_X D)(Y,V) = (\nabla_Y D)(V,X)$, then from the equations (3.3) and (3.4) we get either

$$d = \pm\sqrt{bc}$$

or

$$\mathbf{d}A(X,Y) = 0$$

and

$$\mathbf{d}B(X,Y) = 0.$$

Thus, we complete the proof. □

Theorem 3.2. *If a $MS(QE)_n$ admits the Codazzi type of Ricci tensor with the associated tensor D satisfying the condition $(\nabla_V D)(Y,V) = (\nabla_Y D)(V,V)$, then the integral curves of the parallel vector fields U and V are geodesics.*

Proof. Putting $X = Z = U$ in (3.2), we get

$$b(\nabla_U A)(Y) + d(\nabla_U B)(Y) = 0,$$

which means that

$$bg(\nabla_U U, Y) + dg(\nabla_U V, Y) = 0. \tag{3.5}$$

Similarly, putting $X = Z = V$ in (3.2), we get

$$c(\nabla_V B)(Y) + d(\nabla_V A)(Y) + e[(\nabla_V D)(Y,V) - (\nabla_Y D)(V,V)] = 0,$$

i.e.,

$$cg(\nabla_V V, Y) + dg(\nabla_V U, Y) + e[(\nabla_V D)(Y,V) - (\nabla_Y D)(V,V)] = 0. \tag{3.6}$$

If U, V are parallel vector fields, then $\nabla_U V = 0 = \nabla_V U$.
We assume that $(\nabla_V D)(Y,V) = (\nabla_Y D)(V,V)$. So from (3.5) and (3.6), we obtain

$$g(\nabla_U U, Y) = 0, \text{ for all Y, i.e., } \nabla_{\mathrm{U}}\mathrm{U} = 0$$

and

$$g(\nabla_V V, Y) = 0, \text{ for all Y, i.e., } \nabla_{\mathrm{V}}\mathrm{V} = 0.$$

Thus the theorem is proved. □

4. The generators U and V as concurrent vector fields

A vector field ξ is called concurrent if [21]

$$\nabla_X \xi = \rho X, \tag{4.1}$$

where ρ is a non-zero constant. If $\rho = 0$, then the vector field reduces to a parallel vector field.

Theorem 4.1. *If the associated vector fields of a $MS(QE)_n$ are concurrent vector fields and the associated scalars are constants, then the manifold reduces to a pseudo quasi-Einstein manifold.*

Proof. We consider the vector fields U and V corresponding to the associated 1-forms A and B respectively are concurrent. Then

$$(\nabla_X A)(Y) = \alpha g(X,Y) \tag{4.2}$$

and

$$(\nabla_X B)(Y) = \beta g(X,Y), \tag{4.3}$$

where α and β are non-zero constants.
Using (4.2) and (4.3) in (2.11), we get

$$\begin{aligned}(\nabla_Z S)(X,Y) &= b[\alpha g(Z,X)A(Y) + \alpha g(Z,Y)A(X)] + c[\beta g(Z,X)B(Y) \\ &\quad + \beta g(Z,Y)B(X)] + d[\alpha g(Z,X)B(Y) + \beta g(Z,Y)A(X) \\ &\quad + \alpha g(Z,Y)B(X) + \beta g(Z,X)A(Y)] + e(\nabla_Z D)(X,Y).\end{aligned} \tag{4.4}$$

Contracting (4.4) over X and Y, we obtain

$$dr(Z) = 2[(b\alpha + d\beta)A(Z) + (c\beta + d\alpha)B(Z)], \tag{4.5}$$

where r is the scalar curvature of the manifold.
In a $MS(QE)_n$ if the associated scalars a, b, c, d and e are constants, then contracting (1.4) over X and Y we get

$$r = an + b + c,$$

which implies that the scalar curvature r is constant, i.e., $dr(X) = 0$, for all X.
Thus equation (4.5) gives

$$(b\alpha + d\beta)A(Z) + (c\beta + d\alpha)B(Z) = 0. \tag{4.6}$$

Since α and β are non-zero constants, using (4.6) in (1.4), we finally get

$$S(X,Y) = ag(X,Y) + \left[b + c\left(\frac{b\alpha + d\beta}{c\beta + d\alpha}\right)^2 - 2d\left(\frac{b\alpha + d\beta}{c\beta + d\alpha}\right)\right]A(X)A(Y) + eD(X,Y).$$

Thus the manifold reduces to a pseudo quasi-Einstein manifold. □

5. The generators U and V as recurrent vector fields

Definition 5.1. A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$ will be called a pseudo generalized Ricci recurrent manifold if its Ricci tensor S of type $(0,2)$ satisfies the condition

$$(\nabla_X S)(Y,Z) = \beta(X) S(Y,Z) + \gamma(X) g(Y,Z) + \delta(X) D(Y,Z),$$

where $\beta(X)$, $\gamma(X)$ and $\delta(X)$ are non-zero 1-forms such that

$$\beta(X) = g(X,\xi_1), \quad \gamma(X) = g(X,\xi_2), \quad \delta(X) = g(X,\xi_3);$$

ξ_1, ξ_2 and ξ_3 are associated vector fields of the 1-forms β, γ and δ respectively, D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,\xi_1) = 0,$$

for all X.

Theorem 5.1. *If the generators of a $MS(QE)_n$ corresponding to the associated 1-forms are recurrent with the same vector of recurrence and the associated scalars are constants with an additional condition that D is covariant constant, then the manifold is a pseudo generalized Ricci recurrent manifold.*

Proof. A vector field ξ corresponding to the associated 1-form η is said to be recurrent if [21]

$$(\nabla_X \eta)(Y) = \psi(X)\eta(Y), \tag{5.1}$$

where ψ is a non-zero 1-form.

Here, we consider the generators U and V corresponding to the associated 1-forms A and B as recurrent. Then we have

$$(\nabla_X A)(Y) = \lambda(X) A(Y) \tag{5.2}$$

and

$$(\nabla_X B)(Y) = \mu(X) B(Y), \tag{5.3}$$

where λ and μ are non-zero 1-forms.
Using (5.2) and (5.3) in (2.11), we obtain

$$\begin{aligned} (\nabla_Z S)(X,Y) &= 2b\lambda(Z) A(X) A(Y) + 2c\mu(Z) B(X) B(Y) \\ &+ d[\lambda(Z) + \mu(Z)][A(X) B(Y) + A(Y) B(X)] \\ &+ e(\nabla_Z D)(X,Y). \end{aligned} \tag{5.4}$$

We assume that the 1-forms λ and μ are equal, i.e.,

$$\lambda(Z) = \mu(Z), \tag{5.5}$$

for all Z. From the equations (5.4) and (5.5), we get

$$\begin{aligned}(\nabla_Z S)(X,Y) &= 2\lambda(Z)\left[bA(X)A(Y)+cB(X)B(Y)\right.\\ &\quad \left.+d\{A(X)B(Y)+A(Y)B(X)\}\right]\\ &\quad +e(\nabla_Z D)(X,Y). \end{aligned} \tag{5.6}$$

Using (1.4) and (5.6), we obtain

$$(\nabla_Z S)(X,Y) = \alpha_1(Z)S(X,Y)+\alpha_2(Z)g(X,Y)+\alpha_3(Z)D(X,Y)+e(\nabla_Z D)(X,Y),$$

where $\alpha_1(Z) = 2\lambda(Z)$, $\alpha_2(Z) = -2a\lambda(Z)$ and $\alpha_3(Z) = -2e\lambda(Z)$.
So the proof is complete. □

6. Example of $MS(QE)_4$

In this section, we prove the existence of $MS(QE)_4$ by constructing a non-trivial concrete example.

Let $(x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$, where $\mathbb{R}^n$ is an n-dimensional real number space. We consider a Riemannian metric g on $\mathbb{R}^4 = (x^1, x^2, x^3, x^4)$, by

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 + (dx^4)^2, \tag{6.1}$$

where $i, j = 1, 2, 3, 4$. Using (6.1), we see the non-vanishing components of Riemannian metric are

$$g_{11} = 1,\ \ g_{22} = (x^1)^2,\ \ g_{33} = (x^2)^2,\ \ g_{44} = 1 \tag{6.2}$$

and its associated components are

$$g^{11} = 1,\ \ g^{22} = \frac{1}{(x^1)^2},\ \ g^{33} = \frac{1}{(x^2)^2},\ \ g^{44} = 1. \tag{6.3}$$

Using (6.2) and (6.3), we can calculate that the non-vanishing components of Christoffel symbols, curvature tensor and Ricci tensor are given by

$$\Gamma^1_{22} = -x^1,\ \ \Gamma^2_{33} = -\frac{x^2}{(x^1)^2},\ \ \Gamma^2_{12} = \frac{1}{x^1},\ \ \Gamma^3_{23} = \frac{1}{x^2},\ \ R_{1332} = -\frac{x^2}{x^1},\ \ S_{12} = -\frac{1}{x^1x^2}$$

and the other components are obtained by the symmetric properties. It can be easily shown that the scalar curvature r of the resulting manifold $(\mathbb{R}^4, g)$ is zero. We shall show that $(\mathbb{R}^4, g)$ is a $MS(QE)_4$.
Let us consider the associated scalars as follows:

$$a = \frac{1}{x^1(x^2)^2},\ \ b = \frac{1}{(x^2)^3},\ \ c = -\frac{1}{x^2},\ \ d = \frac{1}{x^1},\ \ e = -\frac{1}{(x^1)^2x^2}. \tag{6.4}$$

We choose the 1-form as follows:

$$A_i(x) = \begin{cases} x^1, & \text{when } i = 2\\ 0, & \text{otherwise}\end{cases} \tag{6.5}$$

and

$$B_i(x) = \begin{cases} x^2, & \text{when } i = 3 \\ 0, & \text{otherwise} \end{cases} \tag{6.6}$$

at any point $x \in \mathbb{R}^4$.
We take the associated tensor as follows:

$$D_{ij}(x) = \begin{cases} 1, & \text{when } i = j = 1, 3 \\ -2, & \text{when } i = j = 2 \\ x^1, & \text{when } i = 1, j = 2 \\ 0, & \text{otherwise} \end{cases} \tag{6.7}$$

at any point $x \in \mathbb{R}^4$. Now the equation (1.4) reduces to the equation

$$S_{12} = ag_{12} + bA_1A_2 + cB_1B_2 + d\left[A_1B_2 + A_2B_1\right] + eD_{12}, \tag{6.8}$$

since, for the other cases (1.4) holds trivially.
From the equations (6.4), (6.5), (6.6), (6.7) and (6.8) we get

$$\begin{aligned} \text{Right hand side of (6.8)} &= ag_{12} + bA_1A_2 + cB_1B_2 + d\left[A_1B_2 + A_2B_1\right] + eD_{12} \\ &= \frac{1}{x^1\left(x^2\right)^2} \cdot 0 + \frac{1}{\left(x^2\right)^3} \cdot 0 \cdot x^1 + \left(-\frac{1}{x^2}\right) \cdot 0 \cdot 0 \\ &+ \frac{1}{x^1}\left[0 + x^1 \cdot 0\right] + \left(-\frac{1}{\left(x^1\right)^2 x^2}\right) \cdot x^1 \\ &= -\frac{1}{x^1x^2} = S_{12}. \end{aligned}$$

Clearly, the trace of the $(0,2)$ tensor D is zero.
We shall now show that the 1-forms A_i and B_i are unit and also they are orthogonal. Here,

$$g^{ij}A_iA_j = 1, \quad g^{ij}B_iB_j = 1, \quad g^{ij}A_iB_j = 0.$$

So, $\left(\mathbb{R}^4, g\right)$ is a $MS\left(QE\right)_4$.

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