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SOME NOTES ON THE PAPER "BANACH FIXED POINT THEOREM ON ORTHOGONAL CONE METRIC SPACES"

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Abstract. In 2020, Olia et al. [Olia, Z. E. D. D., Gordji, M. E. and Bagha, D. E. (2020). Banach fixed point theorem on orthogonal cone metric spaces. FACTA Universitatis (NIS) Ser. Math. Inform, 35, 1239-1250] examined orthogonal cone metric spaces. They assumed that P is a normal cone with normal constant K and that self mapping T is orthogonal continuous on the orthogonal cone metric space X in their study. This study now presentes certain required definitions on orthogonal cone metric spaces that were not previously given in [9]. The examples that show the link between existing and new definitions have also been included. The results are also generalized by eliminating the normalcy condition and utilizing point orthogonal continuity instead of general orthogonal continuity in the major results of [9]. The fundamental finding of the study is then generalized by removing the requirement of orthogonal continuity and introducing normality. In addition, certain outcomes of stated theorems are proven, and some examples are provided to demonstrate these theorems.
Keywords: fixed point theorem, metric spaces, orthogonal continuity.

1. Introduction and Preliminaries

The well-known theorem on the presence and uniqueness of a fixed point of exact self maps defined on certain metric spaces were stated by Stefan Banach [4] in 1992: Every self mapping h on a complete metric space (Ω, ρ) satisfying the condition

$$\rho(hx, hy) \leqslant \lambda\rho(x, y), \text{ for all } x, y \in \Omega, \lambda \in (0, 1) \tag{1.1}$$

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has a unique fixed point. This gracious theorem has been used to show the presence and uniqueness of the solution of differantial equation

$$y'(x) = F(x; y); y(x_0) = y_0 \tag{1.2}$$

where F is a continuously differantiable function. Consequently, after the Banach Contraction Principle on complete metric space, many researchers have investigated fixed point results and reported new fixed point theorems intended by the use of two very influential directions, together or apart. One of them is involved with the attempts to generalize the contractive conditions on the maps and thus, soften them; the other with to attempts to generalize the space on which these contractions are described. In addition, in recent studies, it is observed that some applications of fixed point theorems have come to the fore (see [7, 10, 20, 21, 22, 23]).

In 2007, Huang and Zhang [12] introduced cone metric spaces and proved some fixed point theorems of contractive mappings on cone metric spaces. Then, in 2008, Rezapour and Hamlbarani [18] obtained generalizations of some results in [12] by omitting the assumption of normality. Then many researchers are obtained fixed point theorems on cone metric spaces.(see [1, 2, 13, 14, 19, 24]) On the other hand, in 2017, Gordji et al [8] described the notion of orthogonal set and orthogonal metric spaces. Generalizations of theorems in this field have been considered in some research articles (see [11, 16, 3, 17, 5, 9]).

Recently, Olia et al. [15] examined orthogonal cone metric spaces in the year 2020. They assumed that P is a normal cone with normal constant K and that self mapping T is orthogonal continuous on the orthogonal cone metric space X in their study. Now, certain required definitions on orthogonal cone metric spaces are presented in this study, which are not given in [9]. The examples that show the link between existing and new definitions are also included. The results are also generalized by eliminating the normalcy condition and utilizing point orthogonal continuity instead of general orthogonal continuity in the major results of [9]. The fundamental finding of the study is then generalized by removing the requirement of orthogonal continuity and introducing normality.

Moreover, Bilgili Gungor and Turkoglu [6] gave some fixed point results of self mapping which is defined on orthogonal cone metric spaces are given by using extensions of orthogonal contractions in 2020. Also, in [6] the authors investigated the necessary conditions for self mappings on orthogonal cone metric space to have P property by taking advantage of these results.

In the sequel, respectively, $\mathbb{Q}, \mathbb{Q}^c, \mathbb{Z}, \mathbb{R}$ denote rational numbers, irrational numbers, integers and real numbers.

Definition 1.1. [8] Let $\Omega \neq \emptyset$ and $\perp \subseteq \Omega \times \Omega$ be a binary relation. If there exists a $k_0 \in \Omega$ and $\perp$satisfies the following condition

$$(\forall x \in \Omega, x \perp k_0) \vee (\forall x \in \Omega, k_0 \perp x), \tag{1.3}$$

then $(\Omega, \perp)$ is called an orthogonal set. And the element k_0 is called an orthogonal element.

Example 1.1. [11] Let $\Omega = \mathbb{Z}$. Define $k \perp l$ if there exists $a \in \mathbb{Z}$ such that $k = al$. It is easy to see that $0 \perp l$ for all $l \in \mathbb{Z}$. Hence $(\Omega, \perp)$ is an orthogonal set.

By the following example, we can see that k_0 is not necessarily unique.

Example 1.2. [11] Let $\Omega = [0, \infty)$, we define $k \perp l$ if $kl \in \{k, l\}$, then by setting $k_0 = 0$ or $k_0 = 1$, $(\Omega, \perp)$ is an orthogonal set.

Definition 1.2. [8] Let $(\Omega, \perp)$ be an orthogonal set. Any two elements $k, l \in \Omega$ are said to be orthogonally related if $k \perp l$.

Definition 1.3. [8] A sequence $\{k_n\}$ is called orthogonal sequence if

$$(\forall n \in \mathbb{N}; k_n \perp k_{n+1}) \vee (\forall n \in \mathbb{N}; k_{n+1} \perp k_n). \tag{1.4}$$

Similarly, a Cauchy sequence $\{k_n\}$ is said to be an orthogonal Cauchy sequence if

$$(\forall n \in \mathbb{N}; k_n \perp k_{n+1}) \vee (\forall n \in \mathbb{N}; k_{n+1} \perp k_n). \tag{1.5}$$

Definition 1.4. [8] Let $(\Omega, \perp)$ be an orthogonal set and d be an usual metric on Ω. Then $(\Omega, \perp, d)$ is called an orthogonal metric space.

Definition 1.5. [12] Let G be a real Banach space and K a subset of G. K is called a cone if and only if

(i) K is closed,nonempty, $K \neq \{\theta_G\}$,

(ii) $\alpha, \beta \in \mathbb{R}, \alpha, \beta \geqslant 0, k, l \in K \Rightarrow \alpha k + \beta l \in K$,

(iii) $k \in K$ and $-k \in K \Rightarrow k = \theta_G$.

Given a cone $K \subseteq G$, we define a partial ordering $\preceq$ with respect to K by $k \preceq l$ if and only if $l - k \in K$. We shall write $k \prec l$ to indicate that $k \preceq l$ but $k \neq l$ and $k \prec\prec l$ indicate that $l - k \in intK$, $intK$ denotes the interior of K.

The cone K is called normal if there is a number $L > 0$ such that for all $k, l \in G$, $0 \preceq k \preceq l$ implies $|| k ||_G \leqslant L || l ||_G$.

The least positive number satisfying above is called the normal constant of K.

The cone K is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{k_n\}$ is sequence such that

$$k_1 \preceq k_2 \preceq k_3 \preceq ... \preceq k_n \preceq ... \preceq l \tag{1.6}$$

for some $l \in G$, then there exists $k \in G$ such that $|| k_n - k ||_G \to 0 (n \to \infty)$. Equivalently, the cone K is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

In the following, we always suppose G is a Banach space, K is a cone in G with $intK \neq \emptyset$ and $\preceq$ is partial ordering with respect to K.

Definition 1.6. [12] Let Ω be a nonempty set. Suppose the mapping $d : \Omega \times \Omega \to G$ satisfies

(d_1) $\theta_G \preceq d(k,l)$ for all $k, l \in \Omega$ and $d(k,l) = \theta_G$ if and only if $k = l$.

(d_2) $d(k,l) = d(l,k)$ for all $k, l \in \Omega$,

(d_3) $d(k,l) \preceq d(k,t) + d(t,l)$ for all $k, l, t \in \Omega$.
Then d is called a cone metric on Ω and (Ω, d) is called a cone metric space.

Lemma 1.1. [19] *Let (Ω, d) be a cone metric space. Then for each $\theta \prec\prec g$, $g \in G$, there exists $\delta > 0$ such that $g - k \in intK$ whenever $\| k \| < \delta$, $k \in G$.*

Definition 1.7. [15] Let $(\Omega, \perp)$ be an orthogonal set and d be a cone metric on Ω. Then $(\Omega, \perp, d)$ is called orthogonal cone metric space.

Now, some examples of orthogonal cone metric spaces shall be given.

Example 1.3. Let $G = \mathbb{R}^2, K = \{(k,l) \in G : k, l \geqslant 0\} \subseteq \mathbb{R}^2$ and $\Omega = \mathbb{Z}$. And $d : \Omega \times \Omega \to G$, $d(k,l) = (| k - l |, \alpha | k - l |)$ is defined where $\alpha \geqslant 0, \alpha \in \mathbb{R}$. Assume that binary relation $\perp$ on $\Omega = \mathbb{Z}$ as Example 1.1 , then $(\Omega, d, \perp)$ is orthogonal cone metric space.

Example 1.4. Let $q, b \in \mathbb{R}$ where $q \geqslant 1, b > 1$, $G = \{\{k_n\} \mid k_n \in \mathbb{R}$ and $\sum_{n=1}^{\infty} (| k_n |)^q < \infty\}$ and $K = \{\{k_n\} \in G \mid k_n \geqslant 0, \forall n \in \mathbb{N}\}$. Assume that $(\Omega, \perp, \rho)$ is an orthogonal metric space, then the mapping

$$d : \Omega \times \Omega \to G, d(k,l) = (\frac{\rho}{b^n})^{\frac{1}{q}} \tag{1.7}$$

can be defined on Ω and this mapping is an orthogonal cone metric. So $(\Omega, \perp, d)$ is an orthogonal cone metric space.

Example 1.5. Let $G = (C_{\mathbb{R}}[0, \infty), \| \, . \, \|_\infty)$ and $K = \{f \in G \mid f(t) \geqslant 0\}$. Assume that $(\Omega, \perp, \rho)$ is an orthogonal metric space, then the mapping

$$d : \Omega \times \Omega \to G, d(k,l) = f_{k,l} \text{ where } f_{k,l}(x) = \rho(k,l)x \tag{1.8}$$

can be defined on Ω and this mapping is an orthogonal cone metric. So $(\Omega, \perp, d)$ is an orthogonal cone metric space.

Definition 1.8. Let $(\Omega, \perp, d)$ be an orthogonal cone metric space. Let $\{k_n\}$ be an orthogonal sequence in Ω and $k \in \Omega$. If for any $g \in G$ with $\theta \prec\prec g$ there is $N \in \mathbb{N}$ such that for all $n \geqslant N (n \in \mathbb{N})$, $d(k_n, k) \prec\prec g$, then orthogonal sequence k_n is said to be convergent and $\{k_n\}$ converges to k(or k is the limit of $\{k_n\}$). We denote this by

$$\lim_{n\to\infty} k_n = k \text{ or } k_n \to k (n \to \infty). \tag{1.9}$$

Definition 1.9. Let $(\Omega, \perp, d)$ be an orthogonal cone metric space. Let $\{k_n\}$ be an orthogonal sequence in Ω. If for any $g \in G$ with $\theta \prec\prec g$ there is $N \in \mathbb{N}$ such that for all $n, m \geqslant N (n, m \in \mathbb{N})$, $d(k_n, k_m) \prec\prec g$, then orthogonal sequence k_n is called an orthogonal Cauchy sequence in Ω.

Definition 1.10. [15] Let $(\Omega, \perp, d)$ be an orthogonal cone metric space, if every orthogonal Cauchy sequence in Ω is convergent in Ω, then $(\Omega, \perp, d)$ is called an orthogonal complete cone metric space.

Lemma 1.2. *Let $(\Omega, \perp, d)$ be an orthogonal cone metric space, $\{k_n\}$ be an orthogonal sequence in Ω. $\{k_n\}$ converges to $k \in \Omega$, then $\{k_n\}$ is orthogonal Cauchy sequence.*

Definition 1.11. Let $(\Omega, \perp, d)$ be an orthogonal cone metric space. If for any orthogonal sequence $\{k_n\}$ in Ω, there is an orthogonal subsequence $\{k_{n_i}\}$ of $\{k_n\}$ such that $\{k_{n_i}\}$ is convergent in Ω. Then $(\Omega, \perp, d)$ is called a sequently compact orthogonal cone metric space.

Definition 1.12. Let $(\Omega, \perp, d)$ be an orthogonal cone metric space and $\gamma \in \mathbb{R}, 0 < \gamma < 1$. A mapping $h : \Omega \to \Omega$ is said to be orthogonal contraction with Lipschitz constant γ when

$$d(hk, hl) \preceq \gamma d(k, l) \text{ if } k \perp l. \tag{1.10}$$

Definition 1.13. Let $(\Omega, \perp, d)$ be an orthogonal cone metric space. A mapping $h : \Omega \to \Omega$ is called orthogonal preserving when

$$hk \perp hl \text{ if } k \perp l. \tag{1.11}$$

Definition 1.14. Let $(\Omega, \perp, d)$ be an orthogonal cone metric space. A mapping $h : \Omega \to \Omega$ is called orthogonal continuous at $k \in \Omega$ if for each orthogonal sequence $\{k_n\}$ in Ω such that $k_n \to k$ then $h(k_n) \to h(k)$. Also h is orthogonal continuous on Ω if h is orthogonal continuous in each $k \in \Omega$.

Now, the following remarkable notes can be given.

Remark 1.1. It is easy to see that every Lipschitz contraction is orthogonal Lipschitz contraction. The following example shows that the converse of the statement is not true in general.

Example 1.6. Let $\Omega = [0, 1), G = \mathbb{R}^2, K = \{(k, l) \in G : k \geqslant 0, l \geqslant 0\} \subseteq \mathbb{R}^2, d : \Omega \times \Omega \to G, d(k, l) = (| k - l |, \alpha | k - l |), \alpha \geqslant 0$. Assume that $k \perp l$ if and only if $kl \in \{k, l\}$. Then $(\Omega, \perp, d)$ be an orthogonal cone metric space. Define $h : \Omega \to \Omega$,

$$h(k) = \begin{cases} \frac{k}{3} & \text{if } k \in \mathbb{Q} \cap \mathbb{R}, \\ 0 & \text{if } k \in \mathbb{Q}^c \cap \mathbb{R}. \end{cases} \tag{1.12}$$

In this case, h is orthogonal Lipschitz contraction. In fact,

$$\begin{aligned} k \perp l \quad &\Rightarrow kl \in \{k, l\} \\ &\Rightarrow kl = k \text{ or } kl = l \\ &\Rightarrow k = 0, l \in \Omega \text{ or } l = 0, k \in \Omega. \end{aligned} \tag{1.13}$$

We can choose $k = 0, l \in X$(The other case is similar to this case. So it can be ignored.) In this case,

$$h(k) = \frac{k}{3} = 0 \text{ and } (h(l) = \frac{l}{3}(\text{ when } l \in \mathbb{Q} \cap \mathbb{R}) \text{ or } h(l) = 0(\text{ when } l \in \mathbb{Q}^c \cap \mathbb{R})) \tag{1.14}$$

Case I: $k = 0$ and $l \in \mathbb{Q} \cap \mathbb{R}$ then

$$\begin{aligned} d(hk, hl) \quad &= (\mid \tfrac{l}{3} \mid, \alpha \mid \tfrac{l}{3} \mid) \\ &\preceq \gamma(\mid l \mid, \alpha \mid l \mid), \text{ for } \gamma = \tfrac{1}{3} \in (0, 1) \end{aligned} \tag{1.15}$$

Case II: $k = 0$ and $l \in \mathbb{Q}^c \cap \mathbb{R}$ then

$$d(hk, hl) = (0, 0) \preceq \gamma d(k, l), \forall \gamma \in (0, 1). \tag{1.16}$$

But, h is not a Lipschitz contraction. Otherwise, for two points $k = \frac{5}{6}$, $l = \sqrt{\frac{5}{6}}$, there exists $\gamma \in \mathbb{R}$, $0 < \gamma < 1$ and we have $d(hk, hl) \preceq \gamma d(k, l)$. One can conclude that, it is a contradiction.Indeed,

$$\begin{aligned} &h(k) = \tfrac{5}{18} \text{ and } h(l) = 0 \\ &d(hk, hl) = (\mid \tfrac{5}{18} \mid, \alpha \mid \tfrac{5}{18} \mid) \text{ and } d(k, l) = (\mid \tfrac{5}{6} - \sqrt{\tfrac{5}{6}} \mid, \alpha \mid \tfrac{5}{6} - \sqrt{\tfrac{5}{6}} \mid) \end{aligned} \tag{1.17}$$

Assume that there exists $\gamma \in \mathbb{R}$, $0 < \gamma < 1$ and we have $d(hk, hl) \preceq \gamma d(k, l)$. In this case, we obtain

$$(\mid \frac{5}{18} \mid, \alpha \mid \frac{5}{18} \mid) \preceq \gamma(\mid \frac{5}{6} - \sqrt{\frac{5}{6}} \mid, \alpha \mid \frac{5}{6} - \sqrt{\frac{5}{6}} \mid) \tag{1.18}$$

and so the definition of $\preceq$ with respect to K,

$$\gamma[\mid \frac{5}{6} - \sqrt{\frac{5}{6}} \mid - \mid \frac{5}{18} \mid] \geqslant 0. \tag{1.19}$$

This contradicts with $\gamma \in \mathbb{R}$, $0 < \gamma < 1$.

Now, we shall give examples for orthogonal preserving or not orthogonal preserving mappings.

Example 1.7. $\Omega = \mathbb{R}, G = \mathbb{R}^2, K = \{(k, l) \in G : k \geqslant 0, l \geqslant 0\} \subseteq \mathbb{R}^2, d : \Omega \times \Omega \to G, d(k, l) = (\mid k - l \mid, \alpha \mid k - l \mid), \alpha \geqslant 0$. Assume that $k \perp l$ if and only if $k = 0$ or $0 \neq l \in \mathbb{Q}$. Then $(\Omega, \perp, d)$ be an orthogonal cone metric space. Define $h : \Omega \to \Omega$,

$$h(k) = \begin{cases} 5 & \text{if } k \in \mathbb{Q}, \\ 0 & \text{if } k \in \mathbb{Q}^c. \end{cases} \tag{1.20}$$

Then h is not orthogonal preserving, since $0 \perp \sqrt{2}$ but $h(0) = 5$ is not orthogonal to $h(\sqrt{2}) = 0$.

Example 1.8. $\Omega = [0,1), G = \mathbb{R}^2, K = \{(k,l) \in G : k \geqslant 0, l \geqslant 0\} \subseteq \mathbb{R}^2, d : \Omega \times \Omega \to G, d(k,l) = (\mid k-l \mid, \alpha \mid k-l \mid), \alpha \geqslant 0$. Assume that $k \perp l$ if and only if $kl \in \{k,l\}$. Then $(\Omega, \perp, d)$ be an orthogonal cone metric space. Define $h : \Omega \to \Omega$,

$$h(k) = \begin{cases} \frac{k}{3} & \text{if } k \in \mathbb{Q} \cap \mathbb{R}, \\ 0 & \text{if } k \in \mathbb{Q}^c \cap \mathbb{R}. \end{cases} \tag{1.21}$$

In this case, h is orthogonal preserving. Indeed,

$$\begin{aligned} k \perp l \quad & \Rightarrow kl \in \{k,l\} \\ & \Rightarrow kl = k \text{ or } kl = l \\ & \Rightarrow k = 0, l \in \Omega \text{ or } l = 0, x \in \Omega. \end{aligned} \tag{1.22}$$

We can choose $k = 0, l \in X$(The other case is similar to this case. So it can be ignored.) In this case,

$$\begin{aligned} & h(k) = \tfrac{k}{3} = 0 \text{ and } h(l) \in \Omega \\ \Rightarrow\ & h(k)h(l) \in \{h(k), h(l)\} \\ \Rightarrow\ & h(k) \perp h(l) \end{aligned} \tag{1.23}$$

Remark 1.2. It is easy to see that every continuous mapping is orthogonal continuous. The following examples show that the converse of the statement is not true in general.

Example 1.9. $\Omega = \mathbb{R}, G = \mathbb{R}^2, K = \{(k,l) \in G : k \geqslant 0, l \geqslant 0\} \subseteq \mathbb{R}^2, d : \Omega \times \Omega \to G, d(k,l) = (\mid k-l \mid, \alpha \mid k-l \mid), \alpha \geqslant 0$. Assume that $k \perp l$ if and only if $k = 0$ or $0 \neq l \in \mathbb{Q}$. Then $(\Omega, \perp, d)$ be an orthogonal cone metric space. Define $h : \Omega \to \Omega$,

$$h(k) = \begin{cases} 5 & \text{if } k \in \mathbb{Q}, \\ 0 & \text{if } k \in \mathbb{Q}^c. \end{cases} \tag{1.24}$$

Then the mapping h is orthogonal continuous at all rational numbers. But h is not continuous on real numbers.

Example 1.10. $\Omega = \mathbb{R}, G = \mathbb{R}^2, K = \{(k,l) \in G : k \geqslant 0, l \geqslant 0\} \subseteq \mathbb{R}^2, d : \Omega \times \Omega \to G, d(k,l) = (\mid k-l \mid, \alpha \mid k-l \mid), \alpha \geqslant 0$. Assume that $k \perp l$ if and only if $k \geqslant 0$. Then $(\Omega, \perp, d)$ be an orthogonal cone metric space. Define $h : \Omega \to \Omega$,

$$h(k) = \begin{cases} 1 & \text{if } k \in \{k \in \mathbb{R} : k \geqslant 0\}, \\ 0 & \text{if } k \in \{k \in \mathbb{R} : k < 0\}. \end{cases} \tag{1.25}$$

Then the mapping h is orthogonal continuous at all nonnegative real numbers. But h is not continuous on real numbers.

Remark 1.3. It is easy to see that every complete cone metric space is orthogonal complete cone metric space. The following example shows that the converse of the statement is not true in general. Firstly, we shall give the following Lemma in cone metric spaces.

Lemma 1.3. *Every Cauchy sequence in cone metric space which has convergent subsequence is convergent.*

Proof. Let (Ω, d) be a cone metric space and K is a cone in G Banach space. Then there are two cases:

Case I: If K is normal cone, since $\{k_n\}$ is Cauchy sequence and $\{k_{n_i}\}$ is convergent subsequence of $\{k_n\}$, respectively we have $d(k_n, k_m) \to \theta(n, m \to \infty)$ and $d(k_{n_i}, k) \to \theta(n_i \to \infty$ and $k \in \Omega)$. Thus $d(k_n, k) \preceq d(k_n, k_{n_i}) + d(k_{n_i}, k) \to \theta(n, n_i \to \infty)$.

Case II: If K is not normal cone, since $\{k_n\}$ is Cauchy sequence and $\{k_{n_i}\}$ is convergent subsequence of $\{k_n\}$, for all $g \in G$ which satisfies $\theta \prec\prec g$, respectively there is $N_1 \in \mathbb{N}$

$$d(k_n, k_m) \prec\prec \frac{g}{2}, \forall n, m > N_1 \tag{1.26}$$

and there is $N_2 \in \mathbb{N}, k \in \Omega$

$$d(k_{n_i}, k) \prec\prec \frac{g}{2}, \forall n_i > N_2. \tag{1.27}$$

And so, assume that $\max\{N_1, N_2\} = N$, then for $\forall n, n_i > N$,

$$\theta \preceq d(k_n, k) \preceq d(k_n, k_{n_i}) + d(k_{n_i}, k). \tag{1.28}$$

Using inequalities 1.26 and 1.27, for $\forall n, n_i > N$ we get

$$\frac{g}{2} - d(k_n, k_{n_i}) \in intK \text{ and } \frac{g}{2} - d(k_{n_i}, k) \in intK. \tag{1.29}$$

And so, using inequality 1.28

$$\begin{aligned} & g - (d(k_n, k_{n_i}) + d(k_{n_i}, k)) \in intK \\ \Rightarrow \quad & g - d(k_n, k) \in intK. \end{aligned} \tag{1.30}$$

□

Example 1.11. $\Omega = \mathbb{R} - \mathbb{Q} = \mathbb{Q}^c, G = \mathbb{R}^2, K = \{(k, l) \in G : k \geqslant 0, l \geqslant 0\} \subseteq \mathbb{R}^2, d : \Omega \times \Omega \to G, d(k, l) = \{| k - l |, \alpha \mid k - l |\}, \alpha \geqslant 0$. Assume that $k \perp l$ if and only if $k = \sqrt{2}$ or $l = \sqrt{2}$. Then $(\Omega, \perp, d)$ be an orthogonal cone metric space. Clearly, $(\mathbb{Q}^c, d)$ is not a complete cone metric space. Indeed, if we take the general term $k_n = 1 + \frac{\sqrt{2}}{n}$ of the sequence $\{k_n\} \subset \mathbb{Q}^c$, then $\{k_n\}$ is a Cauchy sequence but $k_n \to 1 \notin \mathbb{Q}^c$. Otherwise $(\mathbb{Q}^c, \perp, d)$ is an orthogonal complete cone metric space. Actually, assume that (k_n) is an arbitrary orthogonal Cauchy sequence in $\mathbb{Q}^c$. Then there exists a subsequence $\{k_{n_i}\}$ of $\{k_n\}$ for which $k_{n_i} = \sqrt{2}$ for all n_i. It follows that $k_{n_i} \to \sqrt{2} \in \mathbb{Q}^c$. On the other hand, using the Lemma 1.3, the sequence $\{k_n\}$ is convergent in $\mathbb{Q}^c$.

2. Main Results

Now, we are ready to give and prove our main result by omitting the normality assumption and using point orthogonal continuity instead of general orthogonal continuity in main results of [15].

Theorem 2.1. *Let $(\Omega, \perp, d)$ is an orthogonal complete cone metric space (it is not necessarily complete cone metric space) and $\gamma \in \mathbb{R}, 0 < \gamma < 1$. Let $h : (\Omega, \perp, d) \to (\Omega, \perp, d)$ is orthogonal contraction with Lipschitz constant γ and orthogonal preserving. In this case, there exists a point $k^* \in \Omega$ such that for any orthogonal element $k_0 \in X$, the iteration sequence $\{h^n(k_0)\}$ converges to this point. Also, if h is orthogonal continuous at $k^* \in \Omega$, then $k^* \in \Omega$ is a unique fixed point of h. In addition h is a Picard operator.*

Proof. Because of $(\Omega, \perp)$ is an orthogonal set, there exists $k_0 \in \Omega$:

$$(\forall x \in \Omega, x \perp k_0) \vee (\forall x \in \Omega, k_0 \perp x). \tag{2.1}$$

And from h is a self mapping on Ω, for any orthogonal element $k_0 \in \Omega$, $k_1 \in \Omega$ can be chosen as $k_1 = h(k_0)$. Thus,

$$\begin{aligned} & k_0 \perp h(k_0) \vee h(k_0) \perp k_0 \\ \Rightarrow\ & k_0 \perp k_1 \vee k_1 \perp k_0. \end{aligned} \tag{2.2}$$

Then, if we continue in the same way

$$k_1 = h(k_0), k_2 = h(k_1) = h^2(k_0), ..., k_n = h(k_{n-1}) = h^n(k_0) \tag{2.3}$$

so $\{h^n(k_0)\}$ is an iteration sequence. Since h is orthogonal preserving and orthogonal contraction with Lipschitz constant γ, respectively $\{h^n(k_0)\}$ is an orthogonal sequence and

$$\begin{aligned} d(k_{n+1}, k_n) &= d(h(k_n), h(k_{n-1})) \\ &\preceq \gamma d(k_n, k_{n-1}) \\ &\preceq ... \\ &\preceq \gamma^n d(k_1, k_0). \end{aligned} \tag{2.4}$$

If any $n \in \mathbb{N}, k_n = k_{n+1}$ then we get $k_n = h(k_n)$ and so h has a fixed point. Assume that $\forall n, n+1 \in \mathbb{N}, k_n \neq k_{n+1}$. In this case, $\forall n, m \in \mathbb{N}, n > m$,

$$\begin{aligned} \theta \preceq d(k_n, k_m) &\preceq d(k_n, k_{n-1}) + d(k_{n-1}, k_{n-2}) + ... + d(k_{m+1}, k_m) \\ &\preceq \gamma^{n-1} d(x_1, x_0) + \gamma^{n-2} d(x_1, x_0) + ... + \gamma^m d(x_1, x_0) \\ &\preceq \frac{\gamma^m}{1-\gamma} d(x_1, x_0). \end{aligned} \tag{2.5}$$

In the sequel there are two cases:
Case I: If K is normal cone with normal constant L, from the inequality 2.5,

$$\begin{aligned} \| d(k_n, k_m) \| &\leqslant L \| \frac{\gamma^m}{1-\gamma} d(k_1, k_0) \| \\ &\leqslant \frac{\gamma^m}{1-\gamma} L \| d(k_1, k_0) \| \end{aligned} \tag{2.6}$$

Using the above equation, since $0 < \gamma < 1$, $d(k_n, k_m) \to \theta (n, m \to \infty)$ and so $\{k_n\} = \{h^n(k_0)\}$ is an orthogonal Cauchy sequence.
Case II: If K is not normal cone, let $g \in G$ such that $\theta \prec\prec g$. Then $g \in intK$. Also

$\delta > 0$ can be chosen such that $g + N_\delta(\theta) \subset K$ where $N_\delta(\theta) = \{x \in G : \| x - \theta \| < \delta\}$. Since $0 < \gamma < 1$,

$$\| \frac{\gamma^m}{1-\gamma} d(k_1, k_0) \| = \frac{\gamma^m}{1-\gamma} \| d(k_1, k_0) \| \to 0 (m \to \infty). \tag{2.7}$$

From the choosing of δ, $\| \frac{\gamma^m}{1-\gamma} d(k_1, k_0) \| < \delta$ and using the Lemma 1.1 we get

$$g - \frac{\gamma^m}{1-\gamma} d(k_1, k_0) \in intK \text{ that is } \frac{\gamma^m}{1-\gamma} d(k_1, k_0) \prec\prec g (m \to \infty). \tag{2.8}$$

Thus, for all $n, m \in \mathbb{N}$ such that $n \geqslant m$, we obtain that $d(k_n, k_m) \leqslant \frac{\gamma^m}{1-\gamma} d(k_1, k_0) \prec\prec g$ so $\{k_n\} = \{h^n(k_0)\}$ is an orthogonal Cauchy sequence.

In both cases, since $(\Omega, \perp, d)$ is an orthogonal complete cone metric space, there exists $k^* \in \Omega$ such that $\{k_n\} = \{h^n(k_0)\}$ converges to this point. Now, assume that h is orthogonal continuous at $k^* \in \Omega$ and let $g \in G$ such that $\theta \prec\prec g$. Because of $\{k_n\} = \{h^n(k_0)\}$ converges to $k^* \in \Omega$ and h is orthogonal continuous at $k^* \in \Omega$, there exists $n_0 \in \mathbb{N}$ and for all $n \in \mathbb{N}$ such that $n \geqslant n_0$,

$$d(k_{n+1}, k^*) \prec\prec \frac{g}{2} \text{ and } d(hk_n, k^*) \prec\prec \frac{g}{2}. \tag{2.9}$$

And so for all $n \in \mathbb{N}$ such that $n \geqslant n_0$, $d(hk^*, k^*) \preceq d(hk^*, hk_n) + d(hk_n, k^*) \prec\prec g$. On the other hand, for $m \in \mathbb{N}, m \geqslant 1$ we obtain $0 < \frac{1}{m} \leqslant 1$. Using $g \in intK$ and $\gamma intK \subseteq intK (\gamma \in \mathbb{R}, \gamma > 0)$ we get $\frac{g}{m} \in intK$. Thus, for all $n \in \mathbb{N}$ such that $n \geqslant n_0$ and for $m \in \mathbb{N}, m \geqslant 1$ we hold $d(hk^*, k^*) \prec\prec \frac{g}{m}$, then $\frac{g}{m} - d(hk^*, k^*) \in K$. Using the cone K is closed set, where taking limit $m \to \infty$ we get $\lim_{m\to\infty}(\frac{g}{m} - d(hk^*, k^*) = -d(hk^*, k^*) \in K$. Besides $\theta \preceq d(hk^*, k^*)$ that is $d(hk^*, k^*) \in K$. So, because of K is cone $d(hk^*, k^*) = \theta$ that is $hk^* = k^*$, so $k^* \in \Omega$ is a fixed point of h.
Now we can show the uniqueness of the fixed point. Suppose that there exist two distinct fixed points k^* and l^*.Then,

(i) If $k^* \perp l^* \vee l^* \perp k^*$,

$$d(k^*, l^*) = d(fk^*, fl^*) \preceq \gamma d(k^*, l^*) \tag{2.10}$$

So $\gamma d(k^*, l^*) - d(k^*, l^*) = (1-\gamma) d(k^*, l^*) \in K$. Because of K is cone $\frac{1}{1-\gamma}(\gamma - 1) d(k^*, l^*) = -d(k^*, l^*) \in K$ and so $d(k^*, l^*) = 0$. That is $k^* \in \Omega$ is an unique fixed point of h.

(ii) If not $k^* \perp l^* \vee l^* \perp k^*$, for the chosen orthogonal element $k_0 \in \Omega$,

$$[(k_0 \perp k^*) \wedge (k_0 \perp l^*)] \vee [(k^* \perp k_0) \wedge (l^* \perp k_0)] \tag{2.11}$$

and since h is orthogonal preserving,

$$[(h(k_n) \perp k^*) \wedge (h(k_n) \perp l^*)] \vee [(k^* \perp h(k_n)) \wedge (l^* \perp h(k_n))] \tag{2.12}$$

is obtained. So,

$$
\begin{aligned}
d(k^*, l^*) &\preceq d(k^*, hk_{n+1}) + d(hk_{n+1}, l^*) \\
&= d(hk^*, h(hk_n)) + d(h(hk_n), hl^*) \\
&\preceq \gamma[d(k^*, hk_n) + d(hk_n, l^*)] \\
&= \gamma[d(k^*, k_{n+1}) + d(k_{n+1}, l^*)]
\end{aligned} \tag{2.13}
$$

and taking limit $n \to \infty$, we get that $-d(k^*, l^*) \in K$. and so $d(k^*, l^*) = 0$. That is $k^* \in \Omega$ is an unique fixed point of h.
Finally we show that h is a Picard operator. Let k be a arbitrary point. Then

$$k_0 \perp k \vee k \perp k_0. \tag{2.14}$$

Since h is orthogonal preserving, for all $n \in \mathbb{N}$,

$$h^n(k_0) \perp h^n(k) \vee h^n(k) \perp h^n(k_0). \tag{2.15}$$

Hence for all $n \in \mathbb{N}$

$$d(h^n(k_0), h^n(k)) \preceq \gamma d(h^{n-1}(k_0), h^{n-1}(k)) \preceq ... \preceq \gamma^n d(k_0, k). \tag{2.16}$$

In the sequel there are two cases:
Case I: If K is normal cone with normal constant L, from the inequality 2.5,

$$|| d(h^n(k_0), h^n(k)) || \quad \leqslant L || \gamma^n d(k_0, k) || \tag{2.17}$$

.

Using the above equation, since $0 < \gamma < 1$, $d(h^n(k_0), h^n(k)) \to \theta (n \to \infty)$ and so $\lim_{n\to\infty} h^n(k) = k^*$, that is h is Picard operator.
Case II: If K is not normal cone, let $g \in G$ such that $\theta \prec\prec g$. Then $g \in intK$. Also $\delta > 0$ can be chosen such that $g + N_\delta(\theta) \subset K$ where $N_\delta(\theta) = \{x \in G : || x - \theta || < \delta\}$. Since $0 < \gamma < 1$,

$$|| \gamma^n d(k_0, k) || = \gamma^n \, || d(k_0, k) || \to 0 (n \to \infty). \tag{2.18}$$

From the choosing of δ, $|| \gamma^n d(k_0, k) || < \delta$ and using the Lemma 1.1 we get

$$g - \gamma^n d(k_0, k) \in intK \text{ that is } \gamma^n d(k_0, k) \prec\prec g (n \to \infty). \tag{2.19}$$

Thus, for all $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ and for all $n \geqslant N$ we obtain that $d(h^n(k_0), h^n(k) \preceq \gamma^n d(k_0, k) \prec\prec g$ and so $\lim_{n\to\infty} h^n(k) = k^*$, that is h is Picard operator. □

Now, omitting the assumption of orthogonal continuity of h and adding the normality of K, we can give the following theorem.

Theorem 2.2. *Let $(\Omega, \perp, d)$ is an orthogonal complete cone metric space, K be a normal cone with normal constant L and $\gamma \in \mathbb{R}, 0 < \gamma < 1$. Let $h : (\Omega, \perp, d) \to (\Omega, \perp, d)$ is orthogonal contraction with Lipschitz constant γ and orthogonal preserving. In this case, there exists a point $k^* \in \Omega$ such that for any orthogonal element $k_0 \in \Omega$, the iteration sequence $\{h^n(k_0)\}$ converges to this point. Also, for all $n \in \mathbb{N}, k_n \perp k^*$, then $k^* \in \Omega$ is a unique fixed point of h. In addition h is a Picard operator.*

Proof. Because of $(\Omega, \perp)$ is an orthogonal set, there exists $k_0 \in \Omega$:

$$(\forall x \in \Omega, x \perp k_0) \vee (\forall x \in \Omega, k_0 \perp x). \tag{2.20}$$

And from h is a self mapping on Ω, for any orthogonal element $k_0 \in \Omega$, $k_1 \in \Omega$ can be chosen as $k_1 = h(k_0)$. Thus,

$$\begin{aligned} & k_0 \perp h(k_0) \vee h(k_0) \perp k_0 \\ \Rightarrow\ & k_0 \perp k_1 \vee k_1 \perp k_0. \end{aligned} \tag{2.21}$$

Then, if we continue in the same way

$$k_1 = h(k_0), k_2 = h(k_1) = h^2(k_0), ..., k_n = h(k_{n-1}) = h^n(k_0) \tag{2.22}$$

so $\{h^n(k_0)\}$ is an iteration sequence. Since h is orthogonal preserving and orthogonal contraction with Lipschitz constant γ, respectively $\{h^n(k_0)\}$ is an orthogonal sequence and

$$\begin{aligned} d(k_{n+1}, k_n) & = d(h(k_n), h(k_{n-1})) \\ & \preceq \gamma d(k_n, k_{n-1}) \\ & \preceq ... \\ & \preceq \gamma^n d(k_1, k_0). \end{aligned} \tag{2.23}$$

If any $n \in \mathbb{N}, k_n = k_{n+1}$ then we get $k_n = h(k_n)$ and so h has a fixed point. Assume that $\forall n, n+1 \in \mathbb{N}, k_n \neq k_{n+1}$. In this case, $\forall n, m \in \mathbb{N}, n > m$,

$$\begin{aligned} \theta \preceq d(x_n, x_m) & \preceq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m) \\ & \preceq \gamma^{n-1} d(x_1, x_0) + \gamma^{n-2} d(x_1, x_0) + ... + \gamma^m d(x_1, x_0) \\ & \preceq \tfrac{\gamma^m}{1-\gamma} d(x_1, x_0). \end{aligned} \tag{2.24}$$

Since K is normal cone with normal constant L, from the inequality 2.24,

$$\begin{aligned} \| d(k_n, k_m) \| & \leqslant L \| \tfrac{\gamma^m}{1-\gamma} d(k_1, k_0) \| \\ & \leqslant \tfrac{\gamma^m}{1-\gamma} L \| d(k_1, k_0) \| \end{aligned} \tag{2.25}$$

Using the above equation, since $0 < \gamma < 1$, $d(k_n, k_m) \to \theta (n, m \to \infty)$ and so $\{k_n\} = \{h^n(k_0)\}$ is an orthogonal Cauchy sequence.
Since $(\Omega, \perp, d)$ is an orthogonal complete cone metric space, there exists $k^* \in \Omega$ such that $\{k_n\} = \{h^n(k_0)\}$ converges to this point. Also, assume that for all $n \in \mathbb{N}, k_n \perp k^*$, then from h is orthogonal contraction with Lipschitz constant γ and K is normal cone with normal constant L ,

$$\begin{aligned} & \theta \preceq d(hk_n, hk^*) \preceq \gamma d(k_n, k^*) \\ \Rightarrow\ & \| d(hk_n, hk^*) \| \leqslant \lambda L \| d(k_n, k^*) \| \end{aligned} \tag{2.26}$$

so taking limit $n \to \infty$, $\| d(hk_n, hk^*) \| \to 0 \Rightarrow h(k_n) \to h(k^*)$ be related to d that is h is orthogonal continuous at $k^* \in \Omega$.
Thus, all the conditions of the Theorem 2.1 are provided and $k^* \in \Omega$ is a unique fixed point of h. In addition h is a Picard operator. $\square$

Example 2.1. Let $G=\mathbb{R}^2$ be the Euclidean plane, $K=\{(k,l)\in G: k,l\geqslant 0\}$ be a cone in G and $\Omega=\{(k,0)\in G: 0\leqslant k<1\}$. Define the binary relation $\perp$ on G such that

$$(k_1,l_1)\perp(k_2,l_2) \iff \langle (k_1,l_1),(k_2,l_2)\rangle_e \in \{\|(k_1,l_1)\|_e, \|(k_2,l_2)\|_e\}. \tag{2.27}$$

(In here $\langle .,.\rangle_e$ denotes Euclide inner product and $\|\ .\ \|_e$ denotes Euclide norm.) In this case, $(\Omega,\perp)$ is an orthogonal set. The mapping $d:\Omega\times\Omega\to G$ is defined by

$$d((k,0),(l,0)) = (\frac{5}{4}\mid k-l\mid, \mid k-l\mid). \tag{2.28}$$

Then,$(\Omega,\perp,d)$ is an orthogonal complete cone metric space. Let mapping $h:(\Omega,\perp,d)\to(\Omega,\perp,d)$ with

$$h(k,0) = (\frac{k}{2},0). \tag{2.29}$$

Then, h is orthogonal contraction with Lipschitz constant $\gamma=\frac{1}{2}$ and orthogonal preserving. Also h is orthogonal continuous on Ω. All hypothesis of Theorem 2.1 satisfy and so, it is obvious that h has a unique fixed point $(0,0)\in\Omega$.

Corollary 2.1. *Let $(\Omega,\perp,d)$ is an orthogonal complete cone metric space, K be a normal cone with normal constant L and $\gamma\in\mathbb{R}, 0<\gamma<1$. For $g\in G$ with $0\prec\prec g$ and any $k_0\in\Omega$, define $B(k_0,g)=\{k\in\Omega: d(k_0,k)\leqslant g\}$. Let $h:(\Omega,\perp,d)\to(\Omega,\perp,d)$ is orthogonal contraction with Lipschitz constant γ for all $k,l\in B(k_0,g)$, orthogonal preserving on $B(k_0,g)$ and $d(hk_0,k_0)\preceq(1-\gamma)g$. In this case, there exists a point $k^*\in B(k_0,g)$ such that for any orthogonal element $k_0\in\Omega$, the iteration sequence $\{h^n(k_0)\}$ converges to this point. Also, if h is orthogonal continuous on $B(k_0,g)$, then $k^*\in B(k_0,g)$ is a unique fixed point of h.*

Proof. We only need to prove that $B(k_0,g)$ is complete and $hk\in B(k_0,g)$ for all $k\in B(k_0,g)$. Thus, when Ω replaced with to $B(k_0,g)$ then $h:(B(k_0,g),\perp,d)\to(B(k_0,g),\perp,d)$ is satisfy all conditions of Theorem 2.1. So h has an unique fixed point in $B(k_0,g)$. Suppose k_n is a Cauchy sequence in $B(k_0,g)$. Then k_n is also a Cauchy sequence in Ω. By the completeness of Ω, there is $k\in\Omega$ such that $k_n\to k(n\to\infty)$. Then, from K is a normal cone with normal constant L, $d(k_n,k)\to\theta(n\to\infty)$ and taking limit n to infty,

$$d(k_0,k)\preceq d(k_0,k_n)+d(k_n,k)\preceq c \tag{2.30}$$

is obtained. Hence $k\in B(k_0,g)$. Therefore $B(k_0,g)$ is complete.
For every $k\in B(k_0,g)$,

$$d(k_0,hk)\preceq d(k_0,hk_0)+d(hk_0,hk)\preceq(1-\gamma)g+\gamma d(k_0,k)\preceq(1-\gamma)g+\gamma g=g \tag{2.31}$$

Hence $hk\in B(k_0,g)$. □

Corollary 2.2. *Let $(\Omega,\perp,d)$ is an orthogonal complete cone metric space (it is not necessarily complete cone metric space) and $\gamma\in\mathbb{R}, 0<\gamma<1$. Let $h:(\Omega,\perp,d)\to(\Omega,\perp,d)$ is orthogonal preserving and h^n is orthogonal contraction with*

Lipschitz constant γ. In this case, there exists a point $k^* \in \Omega$ such that for any orthogonal element $k_0 \in \Omega$, the iteration sequence $\{h^n(k_0)\}$ converges to this point. Also, if h is orthogonal continuous at $k^* \in \Omega$, then $k^* \in \Omega$ is a unique fixed point of h.

Proof. From Theorem 2.1, h^n has a unique fixed point k^*.

$$h^n(hk^*) = h(h^n k^*) = hk^*, \tag{2.32}$$

so hk^* is also a fixed point of h^n. Hence $hk^* = k^*$, k^* is a fixed point of h. Since the fixed point of h is also fixed point of h^n, the fixed point of h is unique. □

3. Conclusion

In this study, as a result of a comprehensive literature review, the developments related to the existence of fixed points for mappings that provide the appropriate contraction conditions from the beginning of the fixed point theory studies are mentioned, and then the general subject of this study is emphasized.

Also, certain required definitions on orthogonal cone metric spaces are presented in this study, which are not given in [9]. The examples that show the link between existing and new definitions are also included. The results are also generalized by eliminating the normalcy condition and utilizing point orthogonal continuity insteadof general orthogonal continuity in the major results of [9]. The fundamental finding of the study is then generalized by removing the requirement of orthogonal continuity and introducing normality. In addition, certain outcomes of stated theorems are proven, and some examples are provided to demonstrate these theorems.

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