FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 37, No 5 (2022), 891–915
https://doi.org/10.22190/FUMI210920062C
**Original Scientific Paper**

# FIXED POINT SETS OF SELF-MAPPINGS WITH A GEOMETRIC VIEWPOINT

**Ufuk Çelik**[1] **and Nihal Özgür**[2]

[1] **Sındırgı Yağcıbedir Secondary School**
**Balıkesir, Türkiye**
[2] **Faculty of Arts and Sciences, Department of Mathematics**
**35140, Karabağlar, Izmir, Türkiye**

**Abstract.** In this paper, we obtain new fixed point results with the help of various techniques constructed using auxiliary numbers and some family of functions. In the context of the fixed-circle (resp. fixed-disc) problem, we consider the geometry of the fixed point set of a self-mapping on a metric space. Also, we discuss the effectiveness of our theoretical fixed point results by considering possible applications to the study of neural networks.
**Key words:** Fixed circle, fixed disc, implicit relation.

## 1. Introduction

The fixed point set $Fix\,(T)$ of a self-mapping $T$ on a metric space $(X, d)$ is defined as follows:

$$Fix\,(T) = \{x \in X : x = Tx\}\,.$$

In this paper, we give new fixed point results for self-mappings of a metric space using various techniques. Throughout the paper $\mathbb{R}_+$, $\mathbb{R}^+$ and $\mathbb{R}$ will denote the set of non-negative real numbers, positive real numbers and real numbers, respectively. Let $\alpha, \beta, \mu \in \mathbb{R}_+$ be numbers with $\alpha + \beta + \mu > 0$ and $0 \leq \theta < 1$. Our main tools are

Received September 20, 2021, accepted: August 27, 2022
Communicated by Marko Petković
Corresponding Author: Nihal Özgür, Faculty of Arts and Sciences, Department of Mathematics, 35140, Karabağlar, Izmir, Türkiye | E-mail: nihal.ozgur@idu.edu.tr
2010 *Mathematics Subject Classification.* Primary 54H25; Secondary 47H09, 47H10

the auxiliary numbers defined by

$$\begin{aligned} N_d(x,y) &= \alpha \max\{d(x,Tx), d(y,Ty)\} \\ &+\beta \max\left\{d(x,y), d(x,Tx), d(y,Ty), \theta\frac{d(x,Ty)+d(y,Tx)}{2}\right\} \\ &+\mu \max\left\{d(x,Tx), d(y,Ty), \frac{d(x,y)\,d(y,Ty)}{1+d(x,Tx)}, \frac{d(x,y)\,d(y,Ty)}{1+d(Tx,Ty)}\right\} \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} M_d(x,y) &= \alpha \max\{d(x,Tx), d(y,Ty)\} \\ &+\beta \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\} \\ &+\mu \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(y,Ty)\,(d(x,Ty)+d(y,Tx))}{1+d(x,Tx)+d(y,Ty)}\right\}. \end{aligned} \tag{1.2}$$

In [8], these numbers was used to study on the Rhoades' open problem on discontinuity at fixed point. Also, this problem was revised considering the geometry of fixed points, especially with the notion of a fixed circle (for more details see [8] and the references therein). Recall that a circle and a disc are defined by $C_{x_0,r} = \{x \in X : d(x,x_0) = r\}$ and $D_{x_0,r} = \{x \in X : d(x,x_0) \leq r\}$, respectively. In [26], the fixed-circle problem was introduced to study on the geometric properties of the set $Fix(T)$.

**Definition 1.1.** [26] Let $(X,d)$ be a metric space and $C_{x_0,r}$ be a circle. For a self-mapping $T : X \to X$, if $Tx = x$ for every $x \in C_{x_0,r}$ then the circle $C_{x_0,r}$ is a fixed circle of $T$.

In other words, a circle (resp. a disc) contained in the fixed point set $Fix(T)$ is called a fixed circle (resp. a fixed disc) of a self-mapping $T$. Then, the fixed-circle problem (resp. fixed-disc problem) can be described as the investigation of some appropriate conditions such that the set $Fix(T)$ contains a circle (resp. a disc). Recently, several fixed-circle (resp. fixed disc) results have been studied on metric or some generalized metric spaces with various techniques (see, for instance, [4, 7, 9, 14, 15, 20, 21, 22, 24, 25, 26, 27, 28, 32, 34] and the references therein). One of these techniques is constructed by means of the family of functions defined by Wardowski [35]. Now, we recall this family of functions.

**Definition 1.2.** [35] Let $\mathcal{F}$ be the family of all functions $F : (0,\infty) \to \mathbb{R}$ such that

$(F_1)$ $F$ is strictly increasing,

$(F_2)$ For each sequence $(a_n)$ in $(0,\infty)$ the following holds: $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$,

$(F_3)$ There exists $k \in (0,1)$ such that $\lim_{a\to 0^+} a^k F(a) = 0$.

The following functions are given in [35] as the examples of functions that satisfy the conditions $(F_1)$, $(F_2)$ and $(F_3)$ of Definition 1.2:

1. $F : (0, \infty) \to \mathbb{R}$ defined by $F(x) = \ln x$.
2. $F : (0, \infty) \to \mathbb{R}$ defined by $F(x) = \ln x + x$.
3. $F : (0, \infty) \to \mathbb{R}$ defined by $F(x) = \ln\left(x^2 + x\right)$.
4. $F : (0, \infty) \to \mathbb{R}$ defined by $F(x) = -\frac{1}{\sqrt{x}}$.

Using the family of functions defined in Definition 1.2, a new type of contraction is given as follows [16].

**Definition 1.3.** [16] If there exists $\tau > 0$, $F \in \mathcal{F}$ and $x_0 \in X$ such that for all $x \in X$ the following holds:

$$d(Tx, x) > 0 \Rightarrow \tau + F(d(Tx, x)) \leq F(d(x, x_0)),$$

then $T$ is said to be an $F_c-$contraction on $X$.

The importance of this kind contractions is the existence of a fixed circle, especially a fixed disc, contained in the set $Fix(T)$.

**Theorem 1.1.** *[16] Let $(X, d)$ be a metric space and $T$ be an $F_c$-contraction with $x_0 \in X$. Define the number $r$ by*

$$r = \inf\{d(x, Tx) : x \neq Tx\}.$$

*Then $C_{x_0,r}$ is a fixed circle of $T$. In particular, $T$ fixes every circle $C_{x_0,\rho}$ where $\rho < r$.*

Another technique in the fixed point theory is the usage of an implicit relation to obtain new fixed point results. Using this technique, various fixed point results have been given in metric and some generalized metric spaces, (see, for example, [2, 3, 12, 29, 30, 31, 33] and the references therein).

On the other hand, theoretical fixed point results have a wide range of applications in many aspects in the study of some applied areas such as neural networks and differential equations arising in the mathematical modeling of many real-world problems (see, for example, [5, 6, 10, 11, 17, 18, 19, 36, 37, 38] and the references therein).

The paper is organised as follows. In Section 2., we investigate new fixed-point results using an implicit relation and a modified version of the number $N_{\mathrm{d}}(x, y)$. In Section 3., we define new types of $F_c$-contractions and obtain new fixed circle (resp. fixed disc) results using the numbers $N_{\mathrm{d}}(x, y)$ and $M_{\mathrm{d}}(x, y)$. In Section 4., we consider geometric properties of the fixed point sets of some discontinuous and continuous activation functions used in the study of stability analysis of neural networks.

## 2. Fixed point results via implicit relations

In this section, we define an implicit relation to obtain new fixed point results on a metric space.

**Definition 2.1.** Let $\mathcal{M}$ be the family of all continuous functions of six variables $M : \mathbb{R}_+^6 \to \mathbb{R}_+$. We define the following conditions:

$(M1)$ For all $x, y, z \in \mathbb{R}_+$ and some $k_1 \in [0, 1)$, if $y \leq M(x, x, 0, z, y, y)$ with $z \leq x + y$, then $y \leq k_1 x$.

$(M2)$ For all $y \in \mathbb{R}_+$, if $y \leq M(y, 0, y, y, 0, y)$, then $y = 0$.

$(M3)$ If $x_i \leq y_i + z_i$ for all $x_i, y_i, z_i \in \mathbb{R}_+$, $i \leq 6$, then

$$M(x_1, x_2, \cdots, x_6) \leq M(y_1, y_2, \cdots, y_6) + M(z_1, z_2, \cdots, z_6).$$

Moreover, for all $y \in \mathbb{R}_+$ and some $k_3 \in [0, 1)$, we have $M(0, 0, 0, y, y, y) \leq k_3 y$.

We note that the coefficients $k_1$ and $k_3$ in the conditions $(M1)$ and $(M3)$ can be assumed equal by taking $k = \max\{k_1, k_3\}$.

Now we give two examples of functions $M \in \mathcal{M}$.

**Example 2.1.** Define $M : \mathbb{R}_+^6 \to \mathbb{R}_+$ as

$$M(x_1, x_2, \cdots, x_6) = \frac{[x_1 + ax_2 + b(x_3 + x_4) + c(x_5 + x_6)]}{d},$$

with $a, b, c, d \in \mathbb{R}^+$, $a, b, c \leq 1$ and $d > 6$. Then

$(M1)$ Let $y \leq M(x, x, 0, z, y, y)$ with $z \leq x + y$ for all $x, y, z \in \mathbb{R}_+$. We get

$$\begin{aligned} y \quad &\leq \quad M(x, x, 0, z, y, y) = \frac{[x + ax + b(0 + z) + c(2y)]}{d} \\ &= \frac{[x + ax + bz + 2cy]}{d} \leq \frac{[x + ax + b(x + y) + 2cy]}{d} \\ &= \frac{[x + ax + bx + by + 2cy]}{d} \end{aligned}$$

and hence

$$\begin{aligned} y(d - b - 2c) \quad &\leq \quad x(1 + a + b) \\ &\to \quad y \leq \frac{(1 + a + b)}{(d - b - 2c)} x. \end{aligned}$$

Since $\frac{1+a+b}{d-b-2c} < 1$, if we take $k = \frac{1+a+b}{d-b-2c}$ then we have $y \leq kx$.

$(M2)$ Let $y \leq M(y, 0, y, y, 0, y) = \frac{y+2by+cy}{d}$ for all $y \in \mathbb{R}_+$. We have

$$y \leq \frac{1 + 2b + c}{d} y$$

and so $y = 0$ since $\frac{1+2b+c}{d} < 1$.

$(M3)$ Let $x_i \leq y_i + z_i$, $i \leq 6$, for all $x_i, y_i, z_i \in \mathbb{R}_+$. Then we obtain

$$\begin{aligned} M(x_1, x_2, \cdots, x_6) &= \frac{[x_1 + ax_2 + b(x_3 + x_4) + c(x_5 + x_6)]}{d} \\ &\leq \frac{[y_1 + z_1 + a(y_2 + z_2) + b(y_3 + z_3 + y_4 + z_4) + c(y_5 + z_5 + y_6 + z_6)]}{d} \\ &= \frac{[y_1 + ay_2 + b(y_3 + y_4) + c(y_5 + y_6)]}{d} + \frac{[z_1 + az_2 + b(z_3 + z_4) + c(z_5 + z_6)]}{d} \\ &= M(y_1, y_2, \cdots, y_6) + M(z_1, z_2, \cdots, z_6). \end{aligned}$$

Additionally, we get $M(0,0,0,y,y,y) = \frac{by+2cy}{d} = \left(\frac{b+2c}{d}\right) y$ for all $y \in \mathbb{R}_+$. Since $\frac{b+2c}{d} < 1$ if $k$ is chosen such that $k \in \left[\frac{b+2c}{d}, 1\right)$ then we have $M(0,0,0,y,y,y) \leq ky$.

**Example 2.2.** Define $M : \mathbb{R}_+^6 \to \mathbb{R}_+$ as

$$M(x_1, x_2, \cdots, x_6) = \frac{x_6}{a} + \frac{x_5}{b} + \frac{\max\{x_1 + x_2, x_3 + x_4, x_5 + x_6\}}{c}$$

with $a, b, c \in \mathbb{R}^+$, $a, b, c \geq 6$.

$(M1)$ Let $y \leq M(x, x, 0, z, y, y) = \frac{y}{a} + \frac{y}{b} + \frac{\max\{2x, z, 2y\}}{c}$ with $z \leq x+y$ for all $x, y, z \in \mathbb{R}_+$.

If $\max\{2x, z, 2y\} = z$ then we get $x + y < z$, which is a contradiction since $z \leq x + y$.

If $\max\{2x, z, 2y\} = 2y$ then we get

$$y \leq \frac{y}{a} + \frac{y}{b} + \frac{2y}{c} = \left(\frac{1}{a} + \frac{1}{b} + \frac{2}{c}\right) y.$$

Since $\frac{1}{a} + \frac{1}{b} + \frac{2}{c} < 1$ then we have $y = 0$ and so the inequality $y \leq kx$ is satisfied.

If $\max\{2x, z, 2y\} = 2x$ then we get

$$\begin{aligned} y &\leq \frac{a+b}{ab} y + \frac{2}{c} x \Rightarrow y - \frac{a+b}{ab} y \leq \frac{2}{c} x \\ &\Rightarrow y\left(1 - \frac{a+b}{ab}\right) \leq \frac{2}{c} x \\ &\Rightarrow y\left(\frac{ab-(a+b)}{ab}\right) \leq \frac{2}{c} x \\ &\Rightarrow y \leq \left(\frac{2ab}{c(ab-(a+b))}\right) x, \end{aligned}$$

$$\begin{aligned} 6 &\leq a \Rightarrow 6bc \leq abc \Rightarrow bc \leq \frac{abc}{6} \\ 6 &\leq b \Rightarrow 6ac \leq abc \Rightarrow ac \leq \frac{abc}{6} \\ 6 &\leq c \Rightarrow 6ab \leq abc \Rightarrow ab \leq \frac{abc}{6} \end{aligned}$$

and

$$\begin{aligned} 2ab + ac + bc &\leq \frac{4}{6} abc < abc \\ &\Rightarrow 2ab < abc - ac - bc \\ &\Rightarrow \frac{2ab}{abc - ac - bc} = \frac{2ab}{c(ab-(a+b))} < 1. \end{aligned}$$

If we take $k=\frac{2ab}{c(ab-(a+b))}$ then we have $y\leq kx$.

$(M2)$ Let $y\leq M\left(y,0,y,y,0,y\right)=\frac{y}{a}+\frac{\max\{y,2y,y\}}{c}=\frac{y}{a}+\frac{2y}{c}=\left(\frac{c+2a}{ac}\right)y$ for all $y\in\mathbb{R}_{+}$. Then, we get $y=0$ since $\frac{c+2a}{ac}<1$ ($\frac{1}{a}\leq\frac{1}{6}$ and $\frac{2}{c}\leq\frac{2}{6}$ implies $\frac{c+2a}{ac}<1$).

$(M3)$ Let $x_i\leq y_i+z_i$, $i\leq 6$, for all $x_i,y_i,z_i\in\mathbb{R}_{+}$. Then we get

$$\begin{aligned}
M\left(x_1,x_2,\cdots,x_6\right) &= \frac{x_6}{a}+\frac{x_5}{b}+\frac{\max\left\{x_1+x_2,x_3+x_4,x_5+x_6\right\}}{c}\\
&\leq \frac{y_6+z_6}{a}+\frac{y_5+z_5}{b}\\
&\quad+\frac{\max\left\{y_1+z_1+y_2+z_2,y_3+z_3+y_4+z_4,y_5+z_5+y_6+z_6\right\}}{c}\\
&\leq \frac{y_6}{a}+\frac{y_5}{b}+\frac{\max\left\{y_1+y_2,y_3+y_4,y_5+y_6\right\}}{c}\\
&\quad+\frac{z_6}{a}+\frac{z_5}{b}+\frac{\max\left\{z_1+z_2,z_3+z_4,z_5+z_6\right\}}{c}\\
&= M\left(y_1,y_2,\cdots,y_6\right)+M\left(z_1,z_2,\cdots,z_6\right).
\end{aligned}$$

Furthermore, we get

$$M\left(0,0,0,y,y,y\right)=\frac{y}{a}+\frac{y}{b}+\frac{\max\left\{0,y,2y\right\}}{c}=\left(\frac{1}{a}+\frac{1}{b}+\frac{2}{c}\right)y\leq\frac{4y}{6},$$

for all $y\in\mathbb{R}_{+}$ since $\frac{1}{a}\leq\frac{1}{6}$, $\frac{1}{b}\leq\frac{1}{6}$ and $\frac{2}{c}\leq\frac{2}{6}$. If $k$ is chosen such that $k\in\left[\frac{4}{6},1\right)$ then we have $M\left(0,0,0,y,y,y\right)\leq ky$.

We give a general fixed point theorem for self-mappings of a complete metric space using the functions belonging in the family $\mathcal{M}$.

**Theorem 2.1.** *Let $T$ be a self-mapping on a complete metric space $(X,d)$ and*

$$d\left(Tx,Ty\right)\leq M\left(d\left(x,y\right),d\left(x,Tx\right),d\left(y,Tx\right),d\left(x,Ty\right),d\left(y,Ty\right),d\left(Tx,Ty\right)\right) \tag{2.1}$$

*for all $x,y,z\in X$ and some $M\in\mathcal{M}$. Then we have*

(1) *If $M$ satisfies the condition $(M1)$, then $T$ has a fixed point $x$. Furthermore, for any $x_0\in X$ with $x_0\notin Fix(T)$ and the fixed point $x$, we have*

$$d\left(x,Tx_n\right)\leq\frac{k^{n+1}}{1-k}d\left(x_0,Tx_0\right).$$

(2) *If $M$ satisfies the condition $(M2)$ and $T$ has a fixed point $x$, then the fixed point is unique.*

(3) *If $M$ satisfies the condition $(M3)$ and $T$ has a fixed point $x$, then $T$ is continuous at $x$.*

*Proof.* For the first part of the proof, assume that $M$ satisfies the condition $(M1)$. We show that $T$ has a fixed point $x$. To do this, let $x_0\in X$ with $x_0\notin Fix(T)$

and define the sequence $(x_n)$ in $X$ recursively by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. From the inequality (2.1), we get

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\leq M\left( \begin{array}{c} d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+1}), \\ d(x_n, x_{n+2}), d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_{n+2}) \end{array} \right). \end{aligned}$$

By the triangle inequality, we have

$$d(x_n, x_{n+2}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}).$$

Since $M$ satisfies the condition $(M1)$, there exists $k \in [0, 1)$ such that

$$d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1}) \leq k^{n+1} d(x_0, x_1). \tag{2.2}$$

Hence for all $n < m$, by using triangle inequality and (2.2), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_m) \\ &\leq \left[k^n + k^{n+1} + \cdots + k^{m-1}\right] d(x_0, x_1) \\ &\leq \frac{k^n}{1-k} d(x_0, x_1). \end{aligned}$$

Taking the limit as $n, m \to \infty$, we get $d(x_n, x_m) \to 0$. This proves that $\{x_n\}$ is a Cauchy sequence in the complete metric space $(X, d)$.

Then we have $x_n \to x \in X$. Furthermore, taking the limit as $m \to \infty$ we get

$$d(x_n, x) \leq \frac{k^n}{1-k} d(x_0, x_1).$$

This implies that

$$d(Tx_n, x) \leq \frac{k^{n+1}}{1-k} d(x_0, Tx_0).$$

Now we prove that $x = Tx$. Using (2.1), we find

$$\begin{aligned} d(x_{n+1}, Tx) &= d(Tx_n, Tx) \\ &\leq M(d(x_n, x), d(x_n, Tx_n), d(x, Tx_n), d(x_n, Tx), d(x, Tx), d(Tx_n, Tx)) \\ &= M(d(x_n, x), d(x_n, x_{n+1}), d(x, x_{n+1}), d(x_n, Tx), d(x, Tx), d(x_{n+1}, Tx)). \end{aligned}$$

Since $M \in \mathcal{M}$, taking the limit as $n \to \infty$ we get

$$d(x, Tx) \leq M(0, 0, 0, d(x, Tx), d(x, Tx), d(x, Tx)).$$

Using the condition $(M1)$, we have $d(x, Tx) \leq k.0 = 0$. This shows that $x = Tx$.

For the second part of the proof, assume that $M$ satisfies the condition $(M2)$. Let $x$ and $y$ be fixed points of $T$. We show that $x = y$. Using (2.1), we obtain

$$\begin{aligned} d(x,y) &= d(Tx,Ty) \\ &\leq M(d(x,y), d(x,Tx), d(y,Tx), d(x,Ty), d(y,Ty), d(Tx,Ty)) \\ &= M(d(x,y), 0, d(x,y), d(x,y), 0, d(x,y)). \end{aligned}$$

Since $M$ satisfies the condition $(M2)$, then we have $d(x,y) = 0$ and hence $x = y$.

For the last part of the proof, assume that $M$ satisfies the condition $(M3)$ and $T$ has a fixed point $x$. Let $(y_n)$ be any sequence in $X$ such that $y_n \to x \in X$. We prove that $Ty_n \to Tx$. By (2.1), we have

$$\begin{aligned} d(x,Ty_n) &= d(Tx,Ty_n) \\ &\leq M(d(x,y_n), d(x,Tx), d(y_n,Tx), d(x,Ty_n), d(y_n,Ty_n), d(Tx,Ty_n)) \\ &= M(d(x,y_n), 0, d(y_n,x), d(x,Ty_n), d(y_n,Ty_n), d(x,Ty_n)). \end{aligned}$$

Since $M$ satisfies the condition $(M3)$ and by the triangle inequality

$$d(y_n,Ty_n) \leq d(y_n,x) + d(x,Ty_n),$$

we obtain

$$\begin{aligned} d(x,Ty_n) &\leq M(d(x,y_n), 0, d(x,y_n), 0, d(x,y_n), d(x,y_n)) \\ &\quad + M(0,0,0, d(x,Ty_n), d(x,Ty_n), d(x,Ty_n)) \\ &\leq M(d(x,y_n), 0, d(x,y_n), 0, d(x,y_n), d(x,y_n)) + kd(x,Ty_n) \end{aligned}$$

and therefore

$$d(x,Ty_n) \leq \frac{1}{1-k} M(d(x,y_n), 0, d(x,y_n), 0, d(x,y_n), d(x,y_n)).$$

Since $M \in \mathcal{M}$, taking the limit as $n \to \infty$ we get $d(x,Ty_n) \to 0$. This proves that $Ty_n \to x = Tx$, that is, $T$ is continuous at $x$. $\square$

**Remark 2.1.** In Theorem 2.1, the existence of a fixed point of the self-mapping $T$ depends on the function $M$ satisfying the condition $(M1)$ in the Definition 2.1. If $T$ has a fixed point and the function $M$ satisfies the condition $(M2)$, then $T$ has a unique fixed point. If $T$ has a fixed point and the function $M$ satisfies the condition $(M3)$, then $T$ is continuous at the fixed point.

**Corollary 2.1.** *Let $T$ be a self-mapping on a complete metric space $(X,d)$ satisfying*

$$d(Tx,Ty) \leq \frac{d(x,y) + ad(x,Tx) + b[d(y,Tx) + d(x,Ty)] + c[d(y,Ty) + d(Tx,Ty)]}{d},$$

*for some $a,b,c,d \in \mathbb{R}^+$, $a,b,c \leq 1$, $d > 6$ and all $x,y \in X$. Then $T$ has a unique fixed point in $X$. In addition, $T$ is continuous at the fixed point.*

**Corollary 2.2.** *Let $T$ be a self-mapping on a complete metric space $(X,d)$ satisfying*

$$\begin{aligned} d(Tx,Ty) &\leq \frac{d(Tx,Ty)}{a}+\frac{d(y,Ty)}{b} \\ &+\frac{\max\{d(x,y)+d(x,Tx),d(y,Tx)+d(x,Ty),d(y,Ty)+d(Tx,Ty)\}}{c}, \end{aligned}$$

*for some $a,b,c\in\mathbb{R}^{+}$, $a,b,c\geq 6$ and all $x,y\in X$. Then $T$ has a unique fixed point in $X$. In addition, $T$ is continuous at the fixed point.*

Now we give another fixed point theorem using a modified version of the number $N_d(x,y)$ given in (1.1).

**Theorem 2.2.** *Let $(X,d)$ be a complete metric space, $T$ be a self-mapping of $X$ and the number $N_d^*(x,y)$ be defined as follows:*

$$\begin{aligned} N_d^*(x,y) &= \alpha\max\{d(x,Tx),d(y,Ty)\} \\ &+\beta\max\left\{d(x,y),d(x,Tx),d(y,Ty),\theta\frac{d(x,Ty)+d(y,Tx)}{2}\right\} \\ &+\mu\max\left\{d(x,Tx),d(y,Ty),\frac{d(x,y)}{1+d(x,y)},\frac{d(y,Tx)}{1+d(y,Tx)},\frac{d(Tx,Ty)}{1+d(Tx,Ty)}\right\}, \end{aligned}$$

*with the coefficients $\alpha,\beta,\mu\in\mathbb{R}_{+}$, $\alpha+\beta+\mu\in(0,1)$ and $0\leq\theta<1$. If the following inequality holds for all $x,y\in X$*

$$d(Tx,Ty)\leq N_d^*(x,y), \tag{2.3}$$

*then $T$ has a unique fixed point in $X$. Furthermore, $T$ is continuous at the fixed point.*

*Proof.* The proof follows from Theorem 2.1 with the function

$$\begin{aligned} M(x,y,z,s,t,u) &= \alpha\max\{y,t\}+\beta\max\left\{x,y,t,\theta\frac{s+z}{2}\right\} \\ &+\mu\max\left\{y,t,\frac{x}{1+x},\frac{z}{1+z},\frac{u}{1+u}\right\}. \end{aligned}$$

Indeed, $M$ is continuous. First, we have

$$\begin{aligned} M(x,x,0,z,y,y) &= \alpha\max\{x,y\}+\beta\max\left\{x,x,y,\theta\frac{z+0}{2}\right\} \\ &+\mu\max\left\{x,y,\frac{x}{1+x},\frac{0}{0+1},\frac{y}{1+y}\right\} \\ &= \alpha\max\{x,y\}+\beta\max\left\{x,y,\theta\frac{z}{2}\right\}+\mu\max\{x,y\} \\ &= (\alpha+\beta+\mu)\max\{x,y\}, \end{aligned}$$

with $z \leq x+y$. So, if $y \leq M(x,x,0,z,y,y)$ with $z \leq x+y$, then $y \leq (\alpha+\beta+\mu)x$ or $y \leq (\alpha+\beta+\mu)y$. Therefore, $T$ satisfies the condition $(M1)$. Next, if

$$\begin{aligned} y \quad \leq \quad & M(y,0,y,y,0,y) = \alpha \max\{0,0\} + \beta \max\left\{y,0,0,\theta\frac{y+y}{2}\right\} \\ & +\mu \max\left\{0,0,\frac{y}{1+y},\frac{y}{1+y},\frac{y}{1+y}\right\} \\ & = \beta y + \mu \frac{y}{1+y}, \end{aligned}$$

then $y=0$ since $\beta y + \mu \frac{y}{1+y} < y$. Therefore, $T$ satisfies the condition $(M2)$. Finally, if $x_i \leq y_i + z_i$ for $i \leq 6$, then

$$\begin{aligned} M(x_1,x_2,\cdots,x_6) &= \alpha \max\{x_2,x_5\} + \beta \max\left\{x_1,x_2,x_5,\theta\frac{x_3+x_4}{2}\right\} \\ &+ \mu \max\left\{x_2,x_5,\frac{x_1}{1+x_1},\frac{x_3}{1+x_3},\frac{x_6}{1+x_6}\right\} \\ &\leq \alpha \max\{y_2+z_2,y_5+z_5\} \\ &+ \beta \max\left\{y_1+z_1,y_2+z_2,y_5+z_5,\theta\frac{y_3+z_3+y_4+z_4}{2}\right\} \\ &+ \mu \max\left\{y_2+z_2,y_5+z_5,\frac{(y_1+z_1)}{1+(y_1+z_1)},\frac{(y_3+z_3)}{1+(y_3+z_3)},\frac{(y_6+z_6)}{1+(y_6+z_6)}\right\} \\ &\leq \alpha \max\{y_2,y_5\} + \alpha \max\{z_2,z_5\} \\ &+ \beta \max\left\{y_1,y_2,y_5,\theta\frac{y_3+y_4}{2}\right\} + \beta \max\left\{z_1,z_2,z_5,\theta\frac{z_3+z_4}{2}\right\} \\ &+ \mu \max\left\{y_2,y_5,\frac{y_1}{1+y_1},\frac{y_3}{1+y_3},\frac{y_6}{1+y_6}\right\} \\ &+ \mu \max\left\{z_2,z_5,\frac{z_1}{1+z_1},\frac{z_3}{1+z_3},\frac{z_6}{1+z_6}\right\} \\ &= M(y_1,y_2,\cdots,y_6) + M(z_1,z_2,\cdots,z_6). \end{aligned}$$

Moreover we get

$$\begin{aligned} M(0,0,0,y,y,y) &= \alpha \max\{0,y\} + \beta \max\left\{0,0,y,\theta\frac{y+0}{2}\right\} \\ &+ \mu \max\left\{0,y,\frac{0}{1+0},\frac{0}{1+0},\frac{y}{1+y}\right\} \\ &= (\alpha+\beta+\mu)y. \end{aligned}$$

Therefore, $T$ satisfies the condition $(M3)$ since $\alpha+\beta+\mu<1$. $\square$

**Example 2.3.** Let $X=[0,1] \subset \mathbb{R}$ be the usual metric space and consider the number $N_d^*(x,y)$ with the coefficients $\alpha=\frac{1}{3}, \beta=\mu=0$. Define the self-mapping $Tx=\frac{x}{4}$ for all

$x \in [0,1]$. Then we have

$$\frac{|x-y|}{4} \leq \frac{1}{3}\max\left\{\frac{3|x|}{4}, \frac{3|y|}{4}\right\} = \max\left\{\frac{|x|}{4}, \frac{|y|}{4}\right\}.$$

Therefore, $T$ satisfies the condition of Theorem 2.2 and $x = 0$ is the unique fixed point of $T$.

**Example 2.4.** Let us consider the set $X = [0,1]$ with the usual metric and the number $N_d^*(x,y)$ with the coefficients $\beta = \frac{1}{2}, a = \mu = 0$ and $\theta = \frac{1}{2}$. Define the self-mapping $Tx = \frac{x}{3}$ for all $x \in [0,1]$. Then we have

$$\frac{|x-y|}{3} \leq \frac{1}{2}\max\left\{|x-y|, \frac{2|x|}{3}, \frac{2|y|}{3}, \frac{1}{2}\left[\frac{\left|x-\frac{y}{3}\right| + \left|y-\frac{x}{3}\right|}{2}\right]\right\}.$$

Therefore, $T$ satisfies the condition of Theorem 2.2 and $x = 0$ is the unique fixed point of $T$.

**Remark 2.2.** Let us consider the number $N_d^*(x,y)$ with the coefficients $a = \frac{1}{3}, \beta = \mu = 0$ and the self-mapping $T$ defined in Example 2.4. Then we have

$$\frac{|x-y|}{3} \leq \frac{1}{3}\max\left\{\frac{2|x|}{3}, \frac{2|y|}{3}\right\} = \frac{2}{9}\max\{|x|, |y|\}.$$

This shows that $T$ does not satisfy the condition of Theorem 2.2 for $x = 0$ and $y = 1$ and hence we deduce that the converse statement of Theorem 2.2 is not true everywhen.

**Example 2.5.** Let us consider the set $X = [0,1]$ with the usual metric and the number $N_d^*(x,y)$ with the coefficients $\mu = \frac{1}{6}$ and $a = \beta = 0$. Define the self-mapping $Tx = \frac{x}{8}$ for all $x \in [0,1]$. Then we have

$$\frac{|x-y|}{8} \leq \frac{1}{6}\max\left\{\frac{7|x|}{8}, \frac{7|y|}{8}, \frac{|x-y|}{1+|x-y|}, \frac{\left|y-\frac{x}{8}\right|}{1+\left|y-\frac{x}{8}\right|}, \frac{\frac{|x-y|}{8}}{1+\frac{|x-y|}{8}}\right\}.$$

Therefore, $T$ satisfies the condition of Theorem 2.2 and $x = 0$ is the unique fixed point of $T$.

**Remark 2.3.** Let us consider the number $N_d^*(x,y)$ with the coefficients $\beta = \frac{1}{2}, a = \mu = 0, \theta = \frac{1}{2}$ and the self-mapping $T$ defined by

$$Tx = \begin{cases} \frac{3}{4} & ; \quad x \in (0,1] \\ \frac{1}{4} & ; \quad x = 0 \end{cases}. \tag{2.4}$$

Then we get $Tx = \frac{1}{4}$ for $x = 0$ and $Ty = \frac{3}{4}$ for $y = \frac{1}{8}$. Hence we obtain

$$\begin{aligned} \left|\frac{1}{4} - \frac{3}{4}\right| &\geq \frac{1}{2}\max\left\{\left|0-\frac{1}{8}\right|, \left|0-\frac{1}{4}\right|, \left|\frac{1}{8}-\frac{3}{4}\right|, \frac{1}{2}\left[\frac{\left|0-\frac{3}{4}\right| + \left|\frac{1}{8}-\frac{1}{4}\right|}{2}\right]\right\} \\ &= \frac{1}{2}\max\left\{\frac{1}{8}, \frac{1}{4}, \frac{5}{8}, \frac{7}{32}\right\} = \frac{5}{16}. \end{aligned}$$

This shows that the condition of Theorem 2.2 is not satisfied. Similarly, let $N_d^*(x,y)$ has coefficients of $\mu = \frac{1}{6}, a = \beta = 0$ and $T$ be defined as in (2.4). Then we get $Tx = \frac{1}{4}$ for $x = 0$ and $Ty = \frac{3}{4}$ for $y = \frac{1}{8}$. Hence we get

$$\begin{aligned} \left|\frac{1}{4} - \frac{3}{4}\right| &\geq \frac{1}{6}\max\left\{\left|0 - \frac{1}{4}\right|, \left|\frac{1}{8} - \frac{3}{4}\right|, \frac{\frac{1}{8}}{1+\frac{1}{8}}, \frac{\frac{1}{8}}{1+\frac{1}{8}}, \frac{\frac{1}{2}}{1+\frac{1}{2}}\right\} \\ &= \frac{1}{6}\max\left\{\frac{1}{4}, \frac{5}{8}, \frac{1}{9}, \frac{1}{9}, \frac{1}{3}\right\} = \frac{5}{48}. \end{aligned}$$

This shows that the condition of Theorem 2.2 is not satisfied and hence we deduce that the converse statement of Theorem 2.2 is not true everywhen.

## 3. New types of $F_c$-contractions

In this section, we obtain new fixed-circle results using the numbers $N_d(x,y)$ and $M_d(x,y)$ given in (1.1) and (1.2), respectively. We define new types of $F_c$-contractions.

**Definition 3.1.** Let $(X,d)$ be a metric space, $T$ be a self-mapping on $X$ and the coefficients of the number $N_d(x,y)$ given in (1.1) be chosen such that $0 \leq \theta < 1, \alpha, \beta, \mu \in \mathbb{R}_+$ and $\alpha + \beta + \mu \in (0,1]$. If there exists $F \in \mathcal{F}$, $\tau > 0$ and $x_0 \in X$ such that for all $x \in X$ the following holds

$$d(Tx,x) > 0 \Rightarrow \tau + F(d(Tx,x)) \leq F(N_d(x,x_0)),$$

then the self-mapping $T$ is called an $F_c^N$-contraction on $X$.

From now on, we will use the number $r$ defined below:

$$r = \inf\{d(Tx,x) : x \in X, x \neq Tx\}. \tag{3.1}$$

Using the notion of an $F_c^N$-contraction, we give the following fixed-circle theorem.

**Theorem 3.1.** *Let $(X,d)$ be a metric space, $T$ be an $F_c^N$-contraction with $x_0 \in X$ and $r$ be defined as in (3.1). If $d(Tx,x_0) \leq r$ for all $x \in C_{x_0,r}$ then, the set $Fix(T)$ contains the circle $C_{x_0,r}$. Furthermore, if $d(Tx,x_0) \leq r$ for all $D_{x_0,r}$ then we have $D_{x_0,r} \subset Fix(T)$.*

*Proof.* Assume that $Tx_0 \neq x_0$. From the definition of an $F_c^N$-contraction, we find

$$\begin{aligned} d(Tx_0,x_0) &> 0 \Rightarrow \tau + F(d(Tx_0,x_0)) \leq F(N_d(x_0,x_0)) = F((\alpha+\beta+\mu)\, d(x_0,Tx_0)) \\ &\leq F(d(x_0,Tx_0)). \end{aligned}$$

This is a contradiction since $\tau > 0$ and so, it should be $Tx_0 = x_0$. Let $x \in C_{x_0,r}$ be any point. If $Tx \neq x$ then $d(x,Tx) \neq 0$ and by the definition of $r$, we have

$d(x,Tx) \geq r$. Hence, using $F_c^N$-contractive property, the hypothesis $d(Tx,x_0) \leq r$ and the fact $x_0 \in Fix(T)$, we get

$$\begin{aligned}
F(r) &\leq F(d(x,Tx)) \leq F(N_d(x,x_0)) - \tau < F(N_d(x,x_0)) \\
&= F\left(\begin{array}{c} \alpha \max\{d(x,Tx), d(x_0,Tx_0)\} \\ +\beta \max\left\{d(x,x_0), d(x,Tx), d(x_0,Tx_0), \theta \frac{d(x,Tx_0)+d(x_0,Tx)}{2}\right\} \\ +\mu \max\left\{d(x,Tx), d(x_0,Tx_0), \frac{d(x,x_0)d(x_0,Tx_0)}{1+d(x,Tx)}, \frac{d(x,x_0)d(x_0,Tx_0)}{1+d(Tx,Tx_0)}\right\} \end{array}\right) \\
&\leq F\left(\begin{array}{c} \alpha \max\{d(x,Tx), 0\} + \beta \max\left\{r, d(x,Tx), 0, \theta \frac{2r}{2}\right\} \\ +\mu \max\{d(x,Tx), 0, 0, 0\} \end{array}\right) \\
&\leq F((\alpha+\beta+\mu) d(x,Tx)) \\
&\leq F(d(x,Tx)).
\end{aligned}$$

This is a contradiction. Therefore, we find $d(x,Tx) = 0$ and so $Tx = x$. Since the point $x \in C_{x_0,r}$ is arbitrary, we deduce that the set $Fix(T)$ contains the circle $C_{x_0,r}$.

Now, we prove that the set $Fix(T)$ contains the disc $D_{x_0,r}$ under the hypothesis $d(Tx,x_0) \leq r$ for all $x \in D_{x_0,r}$. Similarly, for any $x \in D_{x_0,r}$ with $Tx \neq x$, by the $F_c^N$-contractive property and the hypothesis $d(Tx,x_0) \leq r$, we get

$$F(d(x,Tx)) \leq F(N_d(x,x_0)) - \tau < F(N_d(x,x_0)) < F(d(x,Tx)).$$

Again, this contradiction requires $d(x,Tx) = 0$ and so $Tx = x$. Consequently, we have $D_{x_0,r} \subset Fix(T)$. $\square$

Now we give an example for Theorem 3.1.

**Example 3.1.** Let the set $X$ be defined as follows and the metric $d$ be the usual metric on $X$ :

$$X = \left\{0, 4, e, e^2, e^4, e^8, e^8 - 4, e^8 + 4, e^{16}, e^{16} - 4, e^{16} + 4, e^{16} - e^8 + 4, e^{16} + e^8 + 4\right\}$$

Let the self-mapping $T$ be defined as follows

$$Tx = \left\{\begin{array}{ccc} e^8 + 4 & ; & x = 4 \\ x & ; & x \neq 4 \end{array}\right.,$$

for all $x \in X$. Then, the self-mapping $T$ is an $F_c^N$-contractive self-mapping with $F = \ln x + x$, $\tau = \beta$, $x_0 = e^{16} + 4$ and a number $N_d(x,y)$ with the coefficients be chosen such that $\alpha + \beta + \mu = 1$. Indeed, we get $d(Tx,x) = e^8$ and $d(x,x_0) = e^{16}$ for $x = 4$. Hence, we obtain

$$\begin{aligned}
\beta + 8 + e^8 &< \beta\left(e^{16} - e^8\right) + (\alpha+\beta+\mu)e^8 + \ln\left(\alpha e^8 + \beta e^8 + \mu e^8\right) \\
&< \alpha e^8 + \beta e^{16} + \mu e^8 + \ln\left(\alpha e^8 + \beta e^{16} + \mu e^8\right) \\
&= F\left(\alpha e^8 + \beta e^{16} + \mu e^8\right) \\
&= F\left(\begin{array}{c} \alpha \max\left\{e^8, 0\right\} + \beta \max\left\{e^{16}, e^8, 0, \theta \frac{\left(2e^{16} - e^8\right)}{2}\right\} \\ +\mu \max\left\{e^8, 0, \frac{e^{16}.0}{1+e^8}, \frac{e^{16}.0}{1+e^{16}-e^8}\right\} \end{array}\right)
\end{aligned}$$

and hence

$$\tau + F(d(Tx,x)) \leq F(N_d(x,x_0)).$$

By the definition of $r$, we get

$$r = \min\{d(Tx,x) : x \in X, x \neq Tx\} = e^8.$$

Clearly, $Fix(T)$ contains the circle

$$C_{e^{16}+4,e^8} = \left\{e^{16} - e^8 + 4, e^{16} + e^8 + 4\right\}$$

and the disc

$$D_{e^{16}+4,e^8} = \left\{e^{16}, e^{16} - 4, e^{16} + 4, e^{16} - e^8 + 4, e^{16} + e^8 + 4\right\}.$$

Now we give an example which shows that the converse statement of Theorem 3.1 is not always true.

**Example 3.2.** Let $(\mathbb{C}, d)$ be the usual metric space and the self-mapping $T_\xi$ be defined as follows:

$$T_\xi z = \begin{cases} z & ; \quad |z-1| \leq \xi \\ 1 & ; \quad |z-1| > \xi \end{cases},$$

for all complex numbers $z \in \mathbb{C}$ and the number $\xi > 0$ .

We show that the self-mapping $T_\xi$ is not an $F_c^N$-contractive self-mapping for the point $z_0 = 1$. If $|z-1| > \xi$ for $z \in \mathbb{C}$, by the $F_c^N$-contraction definition, we get

$$\begin{aligned}
d(z, T_\xi z) &= d(z,1) > 0 \Rightarrow \tau + F(d(z,1)) \leq F(N_d(z,1)) \\
&= F\left(\begin{array}{c} \alpha \max\{d(z,T_\xi z), d(1,T_\xi 1)\} \\ +\beta \max\left\{d(z,1), d(z,T_\xi z), d(1,T_\xi 1), \theta\frac{\left(d\left(z,T_\xi 1\right)+d\left(1,T_\xi z\right)\right)}{2}\right\} \\ +\mu \max\left\{d(z,T_\xi z), d(1,T_\xi 1), \frac{d(z,1)d\left(1,T_\xi 1\right)}{1+d\left(z,T_\xi z\right)}, \frac{d(z,1)d\left(1,T_\xi 1\right)}{1+d\left(T_\xi z,T_\xi 1\right)}\right\} \end{array}\right) \\
&= F((\alpha+\beta+\mu)\, d(z,1)) \\
&\leq F(d(z,1)).
\end{aligned}$$

This is a contradiction since $\tau > 0$. Consequently, the self-mapping $T_\xi$ is not an $F_c^N$-contractive self-mapping but $Fix(T_\xi)$ contains all circles $C_{1,\rho}$ for $\rho \leq \xi$.

Now we give the following definition.

**Definition 3.2.** Let $(X,d)$ be a metric space, $T$ be a self-mapping on $X$ and the coefficients of the number $M_d(x,y)$ given in (1.2) be chosen such that $\alpha, \beta, \mu \in \mathbb{R}_+$ and $\alpha + \beta + \mu \in (0,1]$. If there exists $F \in \mathcal{F}$, $\tau > 0$ and $x_0 \in X$ such that for all $x \in X$ the following holds

$$d(Tx,x) > 0 \Rightarrow \tau + F(d(Tx,x)) \leq F(M_d(x,x_0)),$$

then the self-mapping $T$ is called $F_c^M$-contraction on $X$.

We give the following theorem using the $F_c^M$-contractive property.

**Theorem 3.2.** *Let $(X,d)$ be a metric space, $T$ be an $F_c^M$-contractive self-mapping with $x_0 \in X$ and $r$ be defined as in (3.1). If $d(Tx,x_0) \leq r$ for all $x \in C_{x_0,r}$, the set $Fix(T)$ contains the circle $C_{x_0,r}$. Furthermore, if $d(Tx,x_0) \leq r$ for all $x \in D_{x_0,r}$ then we have $D_{x_0,r} \subset Fix(T)$.*

*Proof.* First, assume that $Tx_0 \neq x_0$. By the definition of an $F_c^M$-contraction, we find

$$\begin{aligned} d(Tx_0,x_0) &> 0 \Rightarrow \tau + F(d(Tx_0,x_0)) \leq F(M_d(x_0,x_0)) \\ &= F\left(\begin{array}{c} \alpha \max\{d(x_0,Tx_0), d(x_0,Tx_0)\} \\ +\beta \max\left\{\begin{array}{c} d(x_0,x_0), d(x_0,Tx_0), d(x_0,Tx_0), \\ \frac{d(x_0,Tx_0)+d(x_0,Tx_0)}{2} \end{array}\right\} \\ +\mu \max\left\{\begin{array}{c} d(x_0,x_0), d(x_0,Tx_0), d(x_0,Tx_0), \\ \frac{d(x_0,Tx_0)(d(x_0,Tx_0)+d(x_0,Tx_0))}{1+d(x_0,Tx_0)+d(x_0,Tx_0)} \end{array}\right\} \end{array}\right) \\ &= F((\alpha+\beta+\mu)d(x_0,Tx_0)) \\ &\leq F(d(x_0,Tx_0)). \end{aligned}$$

This is a contradiction since $\tau > 0$ and so it should be $Tx_0 = x_0$.

Let $x \in C_{x_0,r}$ be any point. If $Tx \neq x$, by the definition of $r$, we have $d(x,Tx) \geq r$. Hence, using the $F_c^M$-contractive property, the hypothesis $d(Tx,x_0) \leq r$ and the fact $Tx_0 = x_0$, we get

$$\begin{aligned} F(r) &\leq F(d(x,Tx)) \leq F(M_d(x,x_0)) - \tau < F(M_d(x,x_0)) \\ &= F\left(\begin{array}{c} \alpha \max\{d(x,Tx), d(x_0,Tx_0)\} \\ +\beta \max\left\{d(x,x_0), d(x,Tx), d(x_0,Tx_0), \frac{d(x,Tx_0)+d(x_0,Tx)}{2}\right\} \\ +\mu \max\left\{d(x,x_0), d(x,Tx), d(x_0,Tx_0), \frac{d(x_0,Tx_0)(d(x,Tx_0)+d(x_0,Tx))}{1+d(x,Tx)+d(x_0,Tx_0)}\right\} \end{array}\right) \\ &\leq F\left(\begin{array}{c} \alpha \max\{d(x,Tx),0\} + \beta \max\{r, d(x,Tx), 0, r\} \\ +\mu \max\{r, d(x,Tx), 0, 0\} \end{array}\right) \\ &\leq F((\alpha+\beta+\mu)d(x,Tx)) \\ &\leq F(d(x,Tx)). \end{aligned}$$

This is a contradiction. Therefore, we find $d(x,Tx) = 0$ and so $Tx = x$. Consequently, $Fix(T)$ contains the circle $C_{x_0,r}$.

Now, we prove that $Fix(T)$ contains the disc $D_{x_0,r}$ under the hypothesis $d(Tx,x_0) \leq r$ for all $x \in D_{x_0,r}$. Again, using the $F_c^M$-contractive property and the hypothesis, we obtain

$$F(d(x,Tx)) \leq F(M_d(x,x_0)) - \tau < F(M_d(x,x_0)) < F(d(x,Tx)).$$

This is a contradiction. Therefore, $d(x,Tx) = 0$ and so $Tx = x$. Consequently, we have $D_{x_0,r} \subset Fix(T)$. □

**Example 3.3.** Let $X = (0,\infty)$ and $(X,d)$ be the usual metric space. Consider the self-mapping $T$ defined by

$$Tx = \left\{ \begin{array}{ccc} x & ; & x \geq 1 \\ 2 & ; & x < 1 \end{array} \right. .$$

It is easy to check that the self-mapping $T$ is an $F_c^M$-contractive self-mapping with $F = \ln x$, $\tau = \ln \frac{3}{2}$ and $x_0 = 6$ and the number $M_d(x,y)$ with the coefficients $\alpha = \frac{2}{3}, \beta = 0, \mu = \frac{1}{3}$. We have

$$r = \min \{d(Tx,x) : x < 1\} = 1.$$

Clearly, we have $Fix(T) = [1,\infty)$ and this set contains the disc $D_{6,1} = [5,7]$.

Now, we obtain another fixed circle theorem with a different technique using the number $N_d^1(x,y)$ defined by

$$\begin{aligned} N_d^1(x,y) \quad = \quad & \alpha \max \{d(x,y), d(x,Tx), d(y,Ty)\} \\ & + \beta \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \theta \frac{d(x,Ty) + d(y,Tx)}{2} \right\} \\ & + \mu \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,y)\, d(y,Ty)}{1 + d(x,Tx)}, \frac{d(x,y)\, d(y,Ty)}{1 + d(Tx,Ty)} \right\}, \end{aligned}$$

where $\alpha, \beta, \mu \in \mathbb{R}^+$ with $\alpha + \beta + \mu = 1$ and $0 \leq \theta < 1$.

**Theorem 3.3.** *Let $(X,d)$ be a metric space, $T$ be a self-mapping on $X$, the number $r$ be defined as in (3.1). Consider the number $N_d^1(x,y)$ with the coefficients $\alpha, \beta, \mu \in \mathbb{R}^+$ with $\alpha + \beta + \mu = 1$ and $0 \leq \theta < 1$. If there exists some $x_0 \in X$ satisfying the following two conditions, then we have $Tx_0 = x_0$ and the set $Fix(T)$ contains the circle $C_{x_0,r}$ :*

(1) *For all $x \in C_{x_0,r}$, there exists $\delta(r) > 0$ such that*

$$r \leq N_d^1(x,x_0) < r + \delta(r) \Rightarrow d(Tx,x_0) \leq r,$$

(2) *For all $x \in X$,*

$$d(Tx,x) > 0 \Rightarrow d(Tx,x) \leq \phi\left(N_d^1(x,x_0)\right),$$

*where $\phi : \mathbb{R}^+ \to \mathbb{R}$ is such that $\phi(t) < t$, for each $t > 0$.*

*Proof.* First, we show that $Tx_0 = x_0$. Suppose that $Tx_0 \neq x_0$. Using the condition (2), we get

$$\begin{aligned} d(Tx_0,x_0) \quad & \leq \quad \phi\left(N_d^1(x_0,x_0)\right) \\ & < \quad N_d^1(x_0,x_0) = (\alpha + \beta + \mu)\, d(x_0,Tx_0) = d(x_0,Tx_0), \end{aligned}$$

a contradiction. Hence we have $Tx_0 = x_0$.

Let $x \in C_{x_0,r}$ be any point. Suppose that $Tx \neq x$. Then using the condition (2), we get

$$\begin{aligned}
d(Tx,x) &\leq \phi\left(N_d^1(x,x_0)\right) \\
&< N_d^1(x,x_0) = \alpha \max\{d(x,x_0), d(x,Tx), d(x_0,Tx_0)\} \\
&+ \beta \max\left\{d(x,x_0), d(x,Tx), d(x_0,Tx_0), \theta\frac{(d(x,Tx_0)+d(x_0,Tx))}{2}\right\} \\
&+ \mu \max\left\{d(x,x_0), d(x,Tx), d(x_0,Tx_0), \frac{d(x,x_0)\,d(x_0,Tx_0)}{1+d(x,Tx)},\right. \\
&\quad \left.\frac{d(x,x_0)\,d(x_0,Tx_0)}{1+d(Tx,Tx_0)}\right\}.
\end{aligned}$$

Using the fact that $Tx_0 = x_0$, we obtain

$$\begin{aligned}
d(Tx,x) &< \alpha \max\{r, d(x,Tx), 0\} + \beta \max\left\{r, d(x,Tx), 0, \theta\frac{(r+d(x_0,Tx))}{2}\right\} \\
&\quad +\mu \max\{r, d(x,Tx), 0, 0, 0\}.
\end{aligned}$$

Using the condition (1), we have

$$\theta\frac{r+d(x_0,Tx)}{2} \leq r$$

and so we get

$$d(Tx,x) < (\alpha+\beta+\mu)\,d(Tx,x) = d(Tx,x),$$

which is a contradiction. Consequently, we have $Tx = x$ and $Fix(T)$ contains the circle $C_{x_0,r}$. $\square$

Now we give an example for Theorem 3.3.

**Example 3.4.** Let the set $P = \{x \in \mathbb{C} : |x| = 2\}$ be the metric space with the usual metric and the self mapping $T$ be defined by

$$Tx = \begin{cases} k_x & ; \quad 0 \leq \arg(x) < \frac{\pi}{3} \\ x & ; \quad \frac{\pi}{3} \leq \arg(x) < 2\pi \end{cases},$$

for all $x \in P$, where

$$\arg(k_x) = \arg(x) + \frac{5\pi}{3}, |k_x| = 2.$$

Then the self-mapping $T$ satisfies the conditions of Theorem 3.3 with $\phi(t) = \frac{\sqrt{3}}{2}t$, $\delta(r) = r$, $x_0 = -2$ and $r = 2$.

Let $\arg(k_x) = 2\pi - \beta$ and $\arg(x) = \alpha$ for any $Tx \neq x$, then we get $0 \leq \alpha < \frac{\pi}{3}$ and $2\pi - \beta = \alpha + \frac{5\pi}{3}$. Hence we obtain

$$\alpha + \beta = \frac{\pi}{3}.$$

For any point $x$ with $Tx \neq x$, we get

$$\begin{aligned} |x-k_x| &= \sqrt{(2\cos\alpha - 2\cos\beta)^2 + (2\sin\alpha + 2\sin\beta)^2} \\ &= \sqrt{4\cos^2\alpha - 8\cos\alpha\cos\beta + 4\cos^2\beta + 4\sin^2\alpha + 8\sin\alpha\sin\beta + 4\sin^2\beta} \\ &= \sqrt{8 - 8(\cos\alpha\cos\beta - \sin\alpha\sin\beta)} \\ &= \sqrt{8 - 8\cos(\alpha+\beta)} = \sqrt{8 - 8\frac{1}{2}} = 2. \end{aligned}$$

Now we shall determine the elements of the circle $C_{-2,2} = \{x \in P : d(x,-2) = 2\}$. Since $x = 2e^{i\theta}$ that

$$d(x,-2) = |x+2| = \sqrt{(2\cos\theta + 2)^2 + (2\sin\theta)^2} = 2.$$

Then we get $4\cos^2\theta + 8\cos\theta + 4 + 4\sin^2\theta = 4$, $\cos\theta = -\frac{1}{2}$, $\theta_1 = \frac{2\pi}{3}$, $\theta_2 = \frac{4\pi}{3}$ and $C_{-2,2} = \left\{2e^{i\frac{2\pi}{3}}, 2e^{i\frac{4\pi}{3}}\right\}$. Calculating the number $N_d^1(x,-2)$ for $x \in C_{-2,2}$ we get

$$\begin{aligned} N_d^1(x,-2) &= \alpha\max\{d(x,-2), d(x,Tx)\} \\ &\quad +\beta\max\left\{d(x,-2), d(x,Tx), \theta\frac{(d(x,-2) + d(Tx,-2))}{2}\right\} \\ &\quad +\mu\max\{d(x,-2), d(x,Tx)\} \\ &= (\alpha+\beta+\mu)\, d(x,-2) = 2 \end{aligned}$$

and

$$2 \leq N_d(x,-2) \leq 4 \Rightarrow d(Tx,-2) \leq 2.$$

So, the condition (1) of the Theorem 3.3 is satisfied.

Now we show that if the argument of $x$ approaches to 0, then $d(x,-2)$ is increasing. We have

$$\begin{aligned} d(x,-2) &= |x+2| = \sqrt{(2\cos\alpha + 2)^2 + (2\sin\alpha)^2} \\ &= \sqrt{4\cos^2\alpha + 8\cos\alpha + 4 + 4\sin^2\alpha} \\ &= \sqrt{8\cos\alpha + 8}. \end{aligned}$$

Hence for $\alpha \to 0$ we get $d(x,-2) > 2\sqrt{3}$. Consequently, we obtain

$$\begin{aligned} 2 &= d(Tx,x) \leq \frac{\sqrt{3}}{2} N_d(x,-2) \\ &= \alpha\max\{d(x,-2), d(x,Tx)\} + \beta\max\left\{d(x,-2), d(x,Tx), \theta\frac{(d(x,-2) + d(Tx,-2))}{2}\right\} \\ &\quad +\mu\max\{d(x,-2), d(x,Tx)\} \end{aligned}$$

and so the condition (2) of the Theorem 3.3 is satisfied.

**Remark 3.1.** The converse statement of Theorem 3.3 is not always true. Let the self-mapping $T$ be as in Example in 3.2. Since $d(Tz,z) > 0$ for $|z-1| > \xi$, we get

$$d(Tz,z) = d(1,z)$$

$$
\begin{aligned}
&\leq \phi\left(N_d(z,1)\right) \\
&= \phi\left(\begin{array}{c}
\alpha \max\left\{d(z,1), d(z,Tz), d(1,T1)\right\} \\
+\beta \max\left\{d(z,1), d(z,Tz), d(1,T1), \theta\frac{(d(z,T1)+d(1,Tz))}{2}\right\} \\
+\mu \max\left\{d(z,1), d(z,Tz), d(1,T1), \frac{d(z,1)d(1,T1)}{1+d(z,Tz)}, \frac{d(z,1)d(1,T1)}{1+d(Tz,T1)}\right\}
\end{array}\right) \\
&= \phi\left(d(z,1)\right) \\
&< d(z,1),
\end{aligned}
$$

which is a contradiction. Hence the self-mapping $T$ does not satisfy the condition (2) of Theorem 3.3 but $C_{1,\xi}$ is a fixed circle of $T$. Moreover, the self-mapping $T$ fixes the disc $D_{1,\xi}$.

Now we give another fixed-circle theorem using a classical technique.

**Theorem 3.4.** *Let $(X,d)$ be a metric space and $C_{x_0,r}$ be any circle on $X$. Let us define the mapping*

$$\varphi : X \to [0,\infty), \varphi(x) = d(x,x_0),$$

*for all $x \in X$. If there exists a self-mapping $T : X \to X$ satisfying*

$$
\begin{array}{ll}
(1) & d(x,Tx) \leq \max\{\varphi(x), \varphi(Tx)\} - r, \\
(2) & d(Tx,x_0) - hd(x,Tx) \leq r,
\end{array}
$$

*for all $x \in C_{x_0,r}$ and $h \in [0,1)$, then $C_{x_0,r}$ is a fixed circle of $T$.*

*Proof.* Let $x \in C_{x_0,r}$ be an arbitrary point. If $\max\{\varphi(x), \varphi(Tx)\} = \varphi(x)$ then using the condition (1) we have

$$d(x,Tx) \leq \max\{\varphi(x), \varphi(Tx)\} - r = \varphi(x) - r = r - r = 0$$

and so $d(x,Tx) = 0$. Hence we get $Tx = x$.

If $\max\{\varphi(x), \varphi(Tx)\} = \varphi(Tx)$ then we obtain

$$d(x,Tx) \leq \max\{\varphi(x), \varphi(Tx)\} - r = \varphi(Tx) - r,$$

and using the condition (2) we find

$$d(x,Tx) \leq \varphi(Tx) - r \leq hd(x,Tx) + r - r = hd(x,Tx).$$

This implies $d(x,Tx) = 0$ since $h \in [0,1)$. Hence, we get $Tx = x$.

Consequently, $C_{x_0,r}$ is a fixed circle of $T$. □

We give some illustrative examples.

**Example 3.5.** Let $X = \mathbb{R}$ be the metric space with the usual metric. Let us consider the circle $C_{0,5}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tx = \left\{\begin{array}{ccc}
1 & ; & x \in \left\{-\frac{10}{\sqrt{5}}, 1\right\} \\
\frac{10x+25\sqrt{5}}{\sqrt{5}x+10} & ; & x \in \mathbb{R} \backslash \left\{-\frac{10}{\sqrt{5}}, 1\right\}
\end{array}\right.,$$

for all $x \in \mathbb{R}$. Then the self-mapping $T$ satisfies the conditions (1) and (2) in Theorem 3.4. Hence $C_{0,5}$ is a fixed circle of $T$. Notice that $C_{3,2}$ is another fixed circle of $T$ and so the number of the fixed circles need not be unique for a given self-mapping.

**Example 3.6.** Let $X = \mathbb{R}$ be the usual metric space. Let us consider the circle $C_{0,4}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tx = \begin{cases} \frac{10x+40\sqrt{5}}{\sqrt{5}x+4} & ; \quad x \in (-5,5) \\ 15 & ; \quad otherwise \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping $T$ satisfies the condition (1) but does not satisfy the condition (2) in Theorem 3.4. Clearly, $C_{0,4}$ is not a fixed circle of $T$. This example shows the importance of the condition (2) of Theorem 3.4.

In the following example, we give an example of a self-mapping which satisfies the condition (2) and does not satisfy the condition (1) of Theorem 3.4.

**Example 3.7.** Let $X = \mathbb{C}$ be the metric space with the usual metric. Let us consider the circle $C_{0,10}$ and define the self-mapping $T : \mathbb{C} \to \mathbb{C}$ by

$$Tx = \begin{cases} \mathtt{Re}\,(z) + i\frac{\mathtt{Im}(z)}{2} & ; \quad \mathtt{Im}\,(z) \geq 0 \\ \mathtt{Re}\,(z) - i\frac{\mathtt{Im}(z)}{2} & ; \quad \mathtt{Im}\,(z) < 0 \end{cases},$$

for all $z \in \mathbb{C}$. Then it is easy to check that the self-mapping $T$ satisfies the condition (2) but does not satisfy the condition (1) of Theorem 3.4. Clearly, $C_{0,10}$ is not a fixed circle of $T$.

Now we use Theorem 2.2 to obtain a uniqueness theorem for fixed circles of self-mappings.

**Theorem 3.5.** *$(X, d)$ be a metric space and $T : X \to X$ be a self-mapping with the fixed-circle $C_{x_0,r}$. If the contractive condition (2.3) is satisfied for all $x \in C_{x_0,r}$, $y \in X \backslash C_{x_0,r}$ by $T$ then $C_{x_0,r}$ is the unique fixed circle of $T$.*

*Proof.* Assume that there exist two fixed circles $C_{x_0,r}$ and $C_{x_0,\rho}$ of the self-mapping $T$. Let $x \in C_{x_0,r}$ and $y \in C_{x_0,\rho}$ be arbitrary points with $x \neq y$. If the contractive condition (2.3) is satisfied by $T$ then we obtain

$$\begin{aligned} d(x,y) &= d(Tx,Ty) \leq N_d^*(x,y) \\ &= \alpha \max\{d(x,Tx), d(y,Ty)\} + \beta \max\left\{ \begin{array}{c} d(x,y), d(x,Tx), d(y,Ty), \\ \theta\frac{d(x,Ty)+d(y,Tx)}{2} \end{array} \right\} \\ &\quad +\mu \max\left\{ d(x,Tx), d(y,Ty), \frac{d(x,y)}{1+d(x,y)}, \frac{d(y,Tx)}{1+d(y,Tx)}, \frac{d(Tx,Ty)}{1+d(Tx,Ty)} \right\} \\ &= \beta d(x,y) + \mu\frac{d(x,y)}{1+d(x,y)} \\ &\leq (\alpha+\beta+\mu)\, d(x,y) \\ &= d(x,y), \end{aligned}$$

which is a contradiction. Hence, we get $x = y$. Consequently, $C_{x_0,r}$ is the unique fixed circle of $T$. $\square$

Finally, we note that the identity map $I_X : X \to X$, defined by $I_X(x) = x$ for all $x \in X$, satisfies the conditions of Theorem 3.4 (resp. Theorem 3.1, Theorem 3.2 and Theorem 3.3). Now, we determine a condition which excludes the identity map in Theorem 3.4 (resp. Theorem 3.1, Theorem 3.2 and Theorem 3.3).

**Theorem 3.6.** *Let $(X, d)$ be a metric space, $T : X \to X$ be a self-mapping and the mapping $\psi_r : \mathbb{R}_+ \to \mathbb{R}$ $(r > 0)$ be defined as follows:*

$$\psi_r(k) = \begin{cases} k - 2r & ; \quad k > 0 \\ 0 & ; \quad k = 0 \end{cases},$$

*for all $k \in \mathbb{R}_+$. The self-mapping $T : X \to X$ satisfies the condition*

$$d(x, Tx) < \psi_r(d(x, Tx)) + 2r \tag{3.2}$$

*for all $x \in X$ if and only if $T = I_X$.*

*Proof.* Let $x \in X$ be any point and assume that $Tx \neq x$. Using the inequality (3.2), we get

$$d(x, Tx) < \psi_r(d(x, Tx)) + 2r = d(x, Tx) - 2r + 2r = d(x, Tx),$$

a contradiction. Then, we have $Tx = x$ and hence $T = I_X$.

Conversely, it is clear that the identity map $I_X$ satisfies the inequality (3.2). $\square$

**Corollary 3.1.** *If a self-mapping $T : X \to X$ satisfies the conditions of Theorem 3.4 (resp. Theorem 3.1, Theorem 3.2 and Theorem 3.3) and does not satisfy the inequality (3.2) then $T \neq I_X$.*

## 4. An overview of activation functions

Several fixed point results such as Banach fixed point theorem and Brouwer's fixed point theorem have been extensively used in the theoretical studies of neural networks. It is well known that the type of activation functions plays an important role in the multistability analysis of neural networks. Especially, continuity and discontinuity of activation functions are crucial (see, for example, [11, 17, 18, 19] and the references therein). Geometric viewpoint is also an efficient tool for many studies. For example, using the Brouwer's Fixed Point Theorem and a geometric approach to locate where the fixed points are, the existence of a fixed point for every recurrent neural network was proved [13].

In this section, we consider some continuous and discontinuous activation functions used in the artificial neural networks and associate them with the results we have obtained. First, we give the following proposition to determine discontinuity (or continuity) of a self-mapping $T$ on its fixed points without any hypothesis on the metric space and the self-mapping.

**Proposition 4.1.** *Let $(X, d)$ be a metric space and $T$ be a self-mapping on $X$. Then $T$ is continuous at $z \in Fix(T)$ if and only if $\lim_{x \to z} N_d(x, z) = 0$.*

Gaussian-wavelet-type functions are one of the classes of activation functions used in the study of neural networks to increase the storage capacity of the neural network (see for example [18] and [19]). Gaussian-wavelet-type functions are defined by

$$f_i(x) = \begin{cases} u_i & , \quad -\infty < x < p_i \\ l_{i,1}x + c_{i,1} & , \quad p_i \leq x \leq r_i \\ l_{i,2}x + c_{i,2} & , \quad r_i < x < q_i \\ l_{i,3}x + c_{i,3} & , \quad q_i \leq x \leq s_i \\ v_i & , \quad s_i < x < +\infty \end{cases}, \tag{4.1}$$

where $p_i$, $r_i$, $q_i$, $s_i$, $u_i$, $v_i$, $l_{i,1}$, $l_{i,2}$, $l_{i,3}$, $c_{i,1}$, $c_{i,2}$ and $c_{i,3}$ are constants with $-\infty < p_i < r_i < q_i < s_i < +\infty$, $l_{i,1} > 0$, $l_{i,2} < 0$, $l_{i,3} > 0$, $i = 1, 2, \cdots, n$.

In [17], it was shown that the storage capacity of the neural networks can be considerably expanded by use of discontinuous activation functions. Brouwer's fixed point theorem is used to establish the existence of multiple equilibrium points for neural networks considered in [17]. The following class of discontinuous nonmonotonic piecewise linear activation functions is introduced in [17] :

$$f_i(x) = \begin{cases} u_i & , \quad -\infty < x < p_i \\ l_{i,1}x + c_{i,1} & , \quad p_i \leq x \leq r_i \\ l_{i,2}x + c_{i,2} & , \quad r_i < x \leq q_i \\ v_i & , \quad q_i < x < +\infty \end{cases}, \tag{4.2}$$

where $p_i$, $r_i$, $q_i$, $u_i$, $v_i$, $l_{i,1}$, $l_{i,2}$, $c_{i,1}$ and $c_{i,2}$ are constants with $-\infty < p_i < r_i < q_i < +\infty$, $l_{i,1} > 0$, $l_{i,2} < 0$, $u_i = f_i(p_i) = f_i(q_i)$, $f_i(r_i) = l_{i,2}r_i + c_{i,2}$ and $v_i > f_i(r_i)$, $i = 1, 2, \cdots, n$.

Let $X = \mathbb{R}$ and the function $d : X^2 \to \mathbb{R}$ be defined by $d(x, y) = |x - y| + ||x| - |y||$ for all $x, y \in \mathbb{R}$. Then the function $d : X^2 \to \mathbb{R}$ is a metric on $\mathbb{R}$. Now, we consider the circle $C_{2,4}$. We get

$$C_{2,4} = \{4\} \cup [-2, 0].$$

By choosing $p_i = -2$, $r_i = 0$, $q_i = 2$, $s_i = 4$, $u_i = -2$, $v_i = 4$, $l_{i,1} = 1$, $l_{i,2} = -1$, $l_{i,3} = \frac{5}{2}$, $c_{i,1} = 0$, $c_{i,2} = 1$ and $c_{i,3} = -6$, we get the following discontinuous activation function $f_1(x)$ belonging to the class defined in (4.1) :

$$f_1(x) = \begin{cases} -2 & , \quad -\infty < x < -2 \\ x & , \quad -2 \leq x \leq 0 \\ -x + 1 & , \quad 0 < x < 2 \\ \frac{5}{2}x - 6 & , \quad 2 \leq x \leq 4 \\ 4 & , \quad 4 < x < +\infty \end{cases}.$$

Obviously, we get $C_{2,4} = Fix\,(f_1\,(x))$. The continuous activation function $f_1\,(x)$ does not satisfy the conditions of Theorem 3.1, Theorem 3.2 and Theorem 3.3 but satisfies the conditions of Theorem 3.4. We determine the continuity of $f_1\,(x)$ at its fixed points by use of the number $N_d\,(x, y)$. For any $t \in [-2, 0]$, we have $\lim_{x\to t} N_d\,(x, t) = 0$ and hence $f_1\,(x)$ is continuous at $x = t$. Also, we have $\lim_{x\to 4} N_d\,(x, 4) = 0$ and hence $f_1\,(x)$ is continuous at $x = 4$.

By choosing $p_i = -5$, $r_i = -3$, $q_i = -2$, $u_i = -2$, $v_i = 10$, $l_{i,1} = 1$, $l_{i,2} = -2$, $c_{i,1} = +3$, $c_{i,2} = -6$ and $v_i = 10$, we get the following discontinuous activation function $f_2\,(x)$ belonging to the class (4.2) :

$$f_2\,(x) = \begin{cases} -2 & , \quad -\infty < x < -5 \\ x+3 & , \quad -5 \le x \le -3 \\ -2x-6 & , \quad -3 < x \le -2 \\ 10 & , \quad -2 < x < +\infty \end{cases} .$$

Now we consider the usual metric on $\mathbb{R}$ and the circle $C_{4,6} = \{-2, 10\}$. Obviously we get $C_{4,6} = Fix\,(f_2\,(x))$. The discontinuous activation function $f_2\,(x)$ does not satisfy the conditions of Theorem 3.1, Theorem 3.2 and Theorem 3.3 but satisfies the conditions of Theorem 3.4. Since $\lim_{x\to -2} N_d\,(x, -2)$ does not exist, $f_2\,(x)$ is discontinuous at $x = -2$. We have $\lim_{x\to 10} N_d\,(x, 10) = 0$ and hence $f_2\,(x)$ is continuous at $x = 10$.

These examples show the effectiveness of our theoretical results for contribution to the study of neural networks in the context of designing a new neural network with a more generalized activation function. Similar results can be investigated based on new contractions, and their possible applications can be discussed (see, for example [1]).

## REFERENCES


1. H. Ahmad, M. Younis and M. E. Köksal: *Double controlled partial metric type spaces and convergence results.* J. Math. **2021** (2021), Art. ID 7008737, 11 pp.
2. İ Altun and D. Türkoğlu: *Some fixed point theorems for weakly compatible mappings satisfying an implicit relation.* Taiwanese J. Math. **13** (2009), 1291–1304.
3. T. V. An, N. V. Dung and V. T. L. Hang: *General fixed point theorems on metric spaces and 2-metric spaces.* Filomat **28** (2014), 2037–2045.
4. H. Aydi, N. Taş, N. Özgür and N. Mlaiki: *Fixed-discs in rectangular metric spaces.* Symmetry **11** (2019), 294.
5. R. K. Bisht and N. Özgür: *Discontinuous convex contractions and their applications in neural networks.* Comput. Appl. Math. **40** (2021), Paper No. 11, 11 pp.
6. S. H. Chang: *Existence-uniqueness and fixed-point iterative method for general nonlinear fourth-order boundary value problems.* J. Appl. Math. Comput. **67** (2021), 221–231.
7. U. Çelik: *Geometry of fixed points and discontinuity at fixed points.* Ph. D. Thesis, Balıkesir University, Balıkesir, 2021.
8. U. Çelik and N. Özgür: *A new solution to the discontinuity problem on metric spaces.* Turkish J. Math **44** (2020), 1115–1126.

9. U. Çelik and N. Özgür: *On the fixed-circle problem.* Facta Univ. Ser. Math. Inform. **35** (2020), 1273–1290.
10. K. Ezzinbi and M. A. Taoudi: *Sadovskii-Krasnosel'skii type fixed point theorems in Banach spaces with application to evolution equations.* J. Appl. Math. Comput. **49** (2015), 243–260.
11. Y. J. Huang, S. J. Chen, X. H. Yang and J. Xiao: *Coexistence and local Mittag-Leffler stability of fractional-order recurrent neural networks with discontinuous activation functions.* Chinese Physics B **28** (2019), 040701.
12. M. Imdad, S. Kumar and M. S. Khan: *Remarks on some fixed point theorems satisfying implicit relations.* Dedicated to the memory of Prof. Dr. Naza Tanović-Miller. Rad. Mat. **11** (2002), 1–9.
13. L. K. Li: *Fixed point analysis for discrete-time recurrent neural networks.* In: [Proceedings 1992] IJCNN International Joint Conference on Neural Networks, IEEE, Vol. 4, 1992, 134–139.
14. N. Mlaiki, U. Çelik, N. Taş, N. Özgür and A. Mukheimer: *Wardowski type contractions and the fixed-circle problem on $S$-metric spaces.* J. Math. **2018** (2018), Art. ID 9127486, 9 pp.
15. N. Mlaiki, N. Taş and N. Özgür: *On the fixed-circle problem and Khan type contractions.* Axioms **7** (2018), 80.
16. N. Mlaiki, N. Özgür and N. Taş: *New fixed-circle results related to $F_c$-contractive and $F_c$-expanding mappings on metric space.* (2021). Available from: https://arxiv.org/pdf/2101.10770.pdf.
17. X. Nie and W. X. Zheng: *On Stability of Multiple Equilibria for Delayed Neural Networks with Discontinuous activation Functions.* In:Proceeding of the 34th Chinese Control Conference July 28-30, 2015.
18. X. Nie and W. X. Zheng: *Complete stability of neural networks with nonmonotonic piecewise linear activation functions.* IEEE Transactions on Circuits and Systems II: Express Briefs **62** (2015), 1002–1006.
19. X. Nie, J. Cao and S. Fei: *Multistability and instability of competitive neural networks with non-monotonic piecewise linear activation functions.* Nonlinear Anal., Real World Appl. **45** (2019), 799–821.
20. N. Özgür: *Fixed-disc results via simulation functions.* Turkish J. Math. **43** (2019), 2794–2805.
21. N. Özgür and N. Taş: *Some fixed-circle theorems and discontinuity at fixed circle.* In: AIP Conference Proceedings, AIP Publishing LLC **1926** (2018), 020048.
22. N. Y. Özgür, N. Taş and U. Çelik: *New fixed-circle results on $S$-metric spaces.* Bull. Math. Anal. Appl. **9** (2017), 10–23.
23. N. Taş and N. Özgür: *A new contribution to discontinuity at fixed point.* Fixed Point Theory **20** (2019), 715–728.
24. A. Tomar, J. Meena and S. K. Padaliya: *Fixed point to fixed circle and activation function in partial metric space.* J. Appl. Anal. **28** (2022), 57–66.
25. N. Y. Özgür and N. Taş: *Fixed-circle problem on $S$-metric spaces with a geometric viewpoint.* Facta Univ. Ser. Math. Inform. **34** (2019), 459–472.
26. N. Y. Özgür and N. Taş: *Some fixed-circle theorems on metric spaces.* Bull. Malays. Math. Sci. Soc. **42** (2019), 1433–1449.

27. R. P. Pant, N. Y. Özgür and N. Taş: *On Discontinuity Problem at Fixed Point.* Bull. Malays. Math. Sci. Soc. **43** (2020), 499–517.

28. R. P. Pant, N. Y. Özgür and N. Taş: *Discontinuity at fixed points with applications.* Bull. Belg. Math. Soc. Simon Stevin **26** (2019), 571–589.

29. H. K. Pathak, R. Rodríguez-López and R. K. Verma: *A common fixed point theorem using implicit relation and property $(E.A)$ in metric spaces.* Filomat **21** (2007), 211–234.

30. V. Popa and M. Mocanu: *Altering distance and common fixed points under implicit relations.* Hacet. J. Math. Stat. **38** (2009), 329–337.

31. V. Popa: *Some fixed point theorems for compatible mappings satisfying an implicit relation.* Demonstratio Math. **32** (1999), 157–163.

32. H. N. Saleh, S. Sessa, W. M. Alfakih, M. Imdad and N. Mlaiki: *Fixed circle and fixed disc results for new types of $\theta_c-$contractive mappings in metric spaces.* Symmetry **12** (2020), 1825.

33. S. Sedghi and N. V. Dung: *Fixed point theorems on $S$-metric spaces.* Mat. Vesnik **66** (2014), 113–124.

34. N. Taş, N. Özgür and N. Mlaiki: *New types of $F_c-$contractions and the fixed-circle problem.* Mathematics **6** (2018), 188.

35. D. Wardowski: *Fixed points of a new type of contractive mappings in complete metric spaces.* Fixed Point Theory Appl. **2012** (2012) 94, 6 pp.

36. M. Younis, D. Singh and A. A. N. Abdou: *A fixed point approach for tuning circuit problem in dislocated b-metric spaces.* Math. Methods Appl. Sci. **45** (2022), 2234–2253.

37. M. Younis and D. Singh: *On the existence of the solution of Hammerstein integral equations and fractional differential equations.* J. Appl. Math. Comput. **68** (2022), 1087–1105.

38. M. Younis, A. Sretenović and S. Radenović: *Some critical remarks on "Some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations"* . Nonlinear Anal. Model. Control **27** (2022), 163–178.