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SOME GEOMETRICAL RESULTS ON NEARLY KÄHLER FINSLER MANIFOLDS

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Abstract. This work is intended as an attempt to extend some results of nearly Kählerian Finsler manifolds. We give a condition to generalized $(a, b, \mathbf{J})-$manifolds to be weakly Landsberg metric. Furthermore, we find the conditions under which a nearly Kähler Finsler manifold has relatively isotropic Landsberg curvature and relatively isotropic mean Landsberg curvature.
Keywords:Kähler structure, Nearly Kähler structure, Finsler metric, Landsberg metric

1. Introduction

Nearly Kählerian Finsler manifolds have a wide range of applications in many fields of study. In particular, their applications extend as new approaches are suggested by these manifolds in the fields of physics and mathematics [15]. This fact has motivated us to study nearly Finsler manifolds and their properties.

This paper aims to study some properties of Kähler Finsler manifolds related to the generalized $(a, b, \mathbf{J})-$metric. The generalized $(a, b, \mathbf{J})-$metric was first introduced by Didehkhani and Najafi in [2]. We gain some conditions which determine whether a generalized $(a, b, \mathbf{J})-$manifold is weakly Landsberg metric, also when a

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nearly Kähler Finsler manifold has relatively isotropic Landsberg curvature and relatively isotropic mean Landsberg curvature.

We first recall a quick description of nearly Kählerian Finsler manifolds. For more details and proves the reader is invited to read [1, 10].

For a smooth manifold M with an almost complex structure $\mathbf{J}$, one may consider the following tensor field

$$N_{\mathbf{J}}(X,Y) = [X,Y] + \mathbf{J}[\mathbf{J}X,Y] + \mathbf{J}[X,\mathbf{J}Y] - [\mathbf{J}X,\mathbf{J}Y],$$

where $X, Y \in \chi(M)$. This tensor field is called Nijenhuis tensor. Recall that an almost complex structure is a $(1,1)$-tensor field, $\mathbf{J} = J^i_j dx^j \otimes \frac{\partial}{\partial x^i}$, where $\mathbf{J}^2 = -I_{TM}$. Then $(M, \mathbf{J})$ is said to be a complex manifold if $\mathbf{J}$ is integrable, i.e., $N_{\mathbf{J}} = 0$.

Now let $(M, \mathbf{g})$ be a Riemannian manifold with an almost structure $\mathbf{J}$ on M. We say the triple $(M, \mathbf{g}, \mathbf{J})$ is an almost Hermitian manifold if $\mathbf{J}$ is compatible with the metric $\mathbf{g}$. Means, $\mathbf{g}\big(\mathbf{J}(X), \mathbf{J}(Y)\big) = \mathbf{g}(X,Y)$.

Let $(M, \mathbf{J}, \mathbf{g})$ be an almost Hermitian manifold. Then, following Erich Kähler in [9], one can define the fundamental Kähler form Ω as follows,

$$\Omega(X,Y) = \mathbf{g}(\mathbf{X}, \mathbf{JY}). \tag{1.1}$$

In this case, $(M, \mathbf{J}, \mathbf{g})$ is called an almost Kähler manifold, if $d\Omega = 0$, and is called Kähler manifold, if $d\Omega = 0$ and $N_{\mathbf{J}} = 0$. The conditions for $(M, \mathbf{J}, \mathbf{g})$ to be a Kähler manifold, are equivalent to $\nabla\mathbf{J} = \mathbf{0}$, for the Levi-Civita connection ∇ with respect to $\mathbf{g}$.

Studying the nearly Kähler manifolds goes back to the 1970s in the studies of Alfred Gray [3]. Gray-Hervella classified almost Hermitian manifolds. One of these classes is known as nearly Kählerian manifolds [4]. A nearly Kähler manifold is an almost Hermitian manifold $(M, \mathbf{J}, \mathbf{g})$ such that

$$(\nabla_X \mathbf{J})X = 0,$$

where X is a vector field on M and ∇ denotes the Levi-Civita connection associated with the metric $\mathbf{g}$. An example of a nearly Kähler manifold that is not Kählerian is S^6. We can also consider G_2-holonomy and super-symmetric models as interesting examples for nearly Kähler structure in six dimension, with regards their relation with torsion. So far, it is known that every nearly Kähler manifold of dimension equal to 6 is isomorphic to a finite quotient of G/K of one of the following forms.

$$S^6 = \frac{G_2}{SU(3)}, \qquad S^3 \times S^3 = \frac{SU_2 \times SU(2)}{\langle 1 \rangle},$$
$$\mathbb{C}P^3 = \frac{Sp(2)}{SU(2).U(1)}, \qquad \mathbb{F}^3 = \frac{SU(3)}{U(1) \times U(1)}.$$

In [11] the author introduces a new condition on an almost complex manifold which is called the Rizza condition. This condition was then developed by Ichijyō on

Finsler manifolds [6] that was leat to introducing Rizza manifolds. To be more precise, let (M, F) be a Finsler manifold. Ichijyō showed that for every $x \in M$ the Minkowski space (T_xM, F_x) is a complex Banach space [6].

Compatibility between $\mathbf{J}$ and F is also proposed by Ichijyō to be the following equation:

$$F\big(x, y\cos\theta + \mathbf{J}_x(y)\sin\theta\big) = F(x, y), \quad \forall\theta \in \mathbb{R}, \quad \forall y \in T_xM. \tag{1.2}$$

The equation 1.2 is called the Rizza condition. Therefore, a Finsler manifold with this condition is called almost Hermitian Finsler manifold or a Rizza manifold [5]. One can consider Rizza manifolds as a natural generalization of almost Hermitian manifolds in the following sense. If F is Riemannian, then it satisfies condition 1.2 if and only if $(M, F, \mathbf{J})$ is an almost Hermitian manifold. The following equivalent conditions to the Rizza condition are suggested by Ichijyō.

- $g_{ij}J_k{}^i y^k y^j = 0$,
- $g_{ir}J^r{}_j + g_{jr}J^r{}_i + 2C_{ijr}J^s{}_r y^s = 0$.

In the papers [5, 6], Ichijyō studied the Kählerian Finsler manifolds. If $|$ is an h-covariant derivative with respect to the Cartan Finsler connection, we say M is a Kählerian Finsler manifold if $J^i_{j|k} = 0$. A Rizza manifold $(M, F, \mathbf{J})$ is called a nearly Kählerian Finsler manifold if the following holds

$$J^i_{j|k} + J^i_{k|j} = 0.$$

Non–Riemannian Rizza manifolds also were studied in [7, 8]. The authors introduced (a, b, J)-manifolds to be this class. To understand this class, let $(M, \alpha, \mathbf{J})$ be a $2n$-dimensional almost Hermitian manifold. The following symmetric quadratic form is defined for a non-vanishing 1-form $b_i(x)$ on M.

$$\beta(x, y) = (b_{ij}(x)y^i y^j)^{\frac{1}{2}}, \tag{1.3}$$

where $b_{ij} = b_i b_j + J_i J_j$ and $J_i = b_r J^r{}_i$ is the local component of the 1-form $b \circ \mathbf{J}$. One can easily see that the Finsler metric $F = \alpha + \beta$ is a typical example of Rizza manifolds [5]. In this case, following [7], $(M, F, \mathbf{J})$ is called an $(a, b, \mathbf{J})$-manifold. An $(a, b, \mathbf{J})$-manifold is called normal if two conditions $\nabla_k b_i = 0$ and $\nabla_k J^i_j = 0$ hold [7].

Consider two 1-forms b_i and J_i on a Riemannian manifold (M, α). Then, we say b_i and J_i are cross-recurrent if there exists a $1-$form λ_k satisfying

$$\nabla_k b_i = \lambda_k J_i, \qquad \nabla_k J_i = -\lambda_k b_i, \tag{1.4}$$

where ∇ is the Levi-Civita connection of α [7].

An (a, b, J)-manifold is called nearly normal if b_i and J_i are cross-recurrent and $\nabla_k J^i_j + \nabla_j J^i_k = 0$. As an example, the class of a normal $(a, b, \mathbf{J})$-manifold is a

Kählerian Finsler manifold. Also, as it is shown in [8], a nearly normal $(a, b, \mathbf{J})$-manifold is a nearly Kählerian Finsler manifold.

As a substitute for $\beta = b_i(x)dx^i$, one can consider symmetric quadratic form $\beta = b_{ij}dx^i \otimes dx^j$. Then $\beta(\mathbf{J}(y)) = \beta(y)$, and therefore $\beta(\mathbf{J}^2(y)) = \beta(\mathbf{J}(y))$. The last result is $\beta(y) = 0$. Assume that $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric. In the paper [2] Didehkhani and Najafi introduce generalized $(a, b, \mathbf{J})-$metrics. They consider an $(a, b, \mathbf{J})-$metric $F = \alpha + \beta$. Now if $\psi : (-b_0, b_0) \to \mathbb{R}$ be a positive smooth function, then $F = \alpha\psi(\frac{\beta}{\alpha})$ is said to be the generalized $(a, b, \mathbf{J})-$metric. They also proved that this metric defines a Rizza manifold.

In what follows we first recall some concepts of Landsberg curvature and Finsler. In section 3., we investigate a condition under which the nearly Kähler Finsler manifold $(M, F, \mathbf{J})$ is a weakly Landsberg metric. Then, we obtain the condition under which F has relatively isotropic Landsberg curvature and relatively isotropic mean Landsberg curvature.

2. Preliminary

In this section, we briefly recall some preliminaries we will be using throughout this thesis. For the omitted details, we refer the reader to [14, 15].

Let M be an n-dimensional C^∞ manifold, with the tangent bundle $TM = \bigcup_{x\in M} T_xM$ and the slit tangent bundle $TM_0 := TM - \{0\}$. Let (M, F) be a Finsler manifold. Then the fundamental tensor, $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$, is the following quadratic form,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. In this case, one can define an operator $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ as follows,

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]_{t=0}, \quad u, v, w \in T_xM.$$

It is easily seen that $\mathbf{C}_y$ measures the non-Euclidean feature of F_x. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well-known that $\mathbf{C} = 0$ if and only if F is Riemannian.

Let $x \in M$. We define a family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$, where for any $y \in T_xM_0$, the maps $\mathbf{I}_y : T_xM \to \mathbb{R}$ are defined as follows.

$$\mathbf{I}_y(u) := \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at a point $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. Then, $\mathbf{I}_y(y) = 0$ and $\mathbf{I}_{\lambda y} = \lambda^{-1}\mathbf{I}_y$, for $\lambda > 0$. Therefore, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

Let (M, F) be an n-dimensional Finsler manifold. Then F induces a global vector field $\mathbf{G}$ on TM_0 as follows. Let (x^i, y^i) be a standard coordinate for TM_0. Then $\mathbf{G}$ is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}.$$

The coefficients $G^i = G^i(x, y)$ are called spray coefficients and given by

$$G^i = \frac{1}{4}g^{il}\Big[\frac{\partial^2 F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big].$$

The vector field $\mathbf{G}$ is called the spray associated with F.

The Berwald curvature, $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$, is defined by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j\partial y^k\partial y^l}.$$

The Finsler metric F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

The Landsberg curvature, $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$, is also defined by

$$\mathbf{L}_y(u, v, w) := -\frac{1}{2}\mathbf{g}_y\big(\mathbf{B}_y(u, v, w), y\big), \quad y \in T_xM. \tag{2.1}$$

The Landsberg curvature in local coordinates is of the form $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k$, where

$$L_{ijk} := -\frac{1}{2}y_lB^l{}_{ijk}.$$

The quantity $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM}$ is called the Landsberg curvature. If $\mathbf{L} = 0$, then F is called a Landsberg metric. According to the definition, every Berwald metric is a Landsberg metric (see [12] and [13]).

The relative rate of change of $\mathbf{C}$ along Finslerian geodesics is $\mathbf{L}/\mathbf{C}$, by the definition. In addition, F is said to be a relatively isotropic Landsberg metric if

$$\mathbf{L} + cF\mathbf{C} = 0,$$

where $c = c(x)$ is a scalar function on M.

Let $x \in M$ and $y \in T_xM$. Define $\mathfrak{J}_y : T_xM \to \mathbb{R}$ by $\mathfrak{J}_y(u) := J_i(y)u^i$, where

$$\mathfrak{J}_i := g^{jk}L_{ijk}. \tag{2.2}$$

The quantity $\mathfrak{J}$ is called the mean Landsberg curvature. A Finsler metric F is called a weakly Landsberg metric if $\mathbf{J} = 0$. It is clear that every Landsberg metric is a weakly Landsberg metric.

Let $x \in M$ and $F_x := F_{|T_xM}$. Put $G^i_j = \frac{\partial G^i}{\partial y^j}$. We denote the Cartan connection of the Finsler metric F by $CF = (F^i_{jk}, G^i_j, C^i_{jk})$. Here F^i_{jk} and C^i_{jk} are as follows,

$$(2.3)\ F^i_{jk} = \frac{1}{2}g^{ir}(\delta_k g_{jr} + \delta_j g_{rk} - \delta_r g_{kj}), \qquad C^i_{jk} = \frac{1}{2}g^{ir}(\frac{\partial g_{jr}}{\partial y^k} + \frac{\partial g_{rk}}{\partial y^j} - \frac{\partial g_{kj}}{\partial y^r}).$$

where $\delta_k = \frac{\partial}{\partial x^k} - G^i_k \frac{\partial}{\partial y^i}$. Indeed, $C^i_{jk} = g^{ir}C_{rjk}$, where $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ is the Cartan tensor of F.
For any Finsler tensor $S^i_j(x, y)$, the $h-$covariant and $v-$covariant derivatives with respect to $C\Gamma$, are defined as follows, respectively

$$(2.4) \qquad S^i_{j|k} = \frac{\delta S^i_j}{\delta x^k} + S^m_j \Gamma^i_{mk} - S^i_m \Gamma^m_{jk}, \quad S^i_{j|k} = \frac{\partial S^i_j}{\partial y^k} + S^m_j C^i_{mk} - S^i_m C^m_{jk}.$$

Put $G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}$. One may see that F^i_{jk} and G^i_{jk} are positively homogeneous functions of degree 0 with respect to y. Also, we have $G^i_j = F^i_{jk}y^k$. Furthermore, an important identity, $F^i_{jk} = G^i_{jk} - L^i_{jk}$, holds, where $L^i_{jk} = g^{is}L_{sjk}$. For a Finsler metric F, we can define the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$. Then if $S^i_j(x, y)$ be any Finsler tensor

$$(2.5) \qquad S^i_{j;k} = \frac{\partial S^i_j}{\partial x^k} + S^m_j G^i_{mk} - S^i_m G^m_{jk}$$

is then the $h-$covariant with respect to $B\Gamma$.

3. Main Results

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric and $\beta = \beta_i(x)dx^i$ be a non-vanishing $1-$form on a differentiable manifold M with $\|\beta\|_\alpha < 1$. Then $F = \alpha + \beta$ is called a Randers metric. In [6] Ichijiō generalized Randers metric by replacing the $1-$form β with a symmetric quadratic form $\beta = b_{ij}dx^i \otimes dx^j$. Also, he introduced $(a, b, \mathbf{J})-$ manifolds as a special class of generalized Randers manifold [7]. He showed that a normal $(a, b, \mathbf{J})-$metric gives a non-trivial example of a Kähler Finsler manifold. In order to extend the class of Rizza manifolds introduced by Ichijiō, one can define a generalized $(a, b, \mathbf{J})-$metric as follows.

Definition 3.1. ([2]) Consider an $(a, b, \mathbf{J})-$metric $F = \alpha + \beta$. Let $\psi : (-b_0, b_0) \to \mathbb{R}$ be a positive smooth function. Then, a Finsler metric in the form $F = \alpha\psi(\frac{\beta}{\alpha})$ is called a generalized $(a, b, \mathbf{J})-$metric.

In [6], Ichijyō proved that a Kählerian Finsler manifold is a Landsberg manifold. In the following, Didehkhani and Najafi generalized this fact to nearly Kählerian Finsler manifold. For this, they proved that the Berwald curvature of a nearly Kähler Finsler manifold and its almost complex structure has a delicate relation.

Proposition 3.1. ([2]) Let $(M, F, \mathbf{J})$ be a nearly Kähler manifold. Then the following holds

$$y^k J_k^r B_{rjm}^i = 0. \tag{3.1}$$

Now, we get the condition under which a nearly Kähler Finsler manifold $(M, F, \mathbf{J})$ is a weakly Landsberg metric. Consequently, we need the following lemma. Let us recall two important identities

$$g_{ij;k} = -2L_{ijk}, \qquad F_{jk}^i = G_{jk}^i - L_{jk}^i, \tag{3.2}$$

where ";" denote the $h-$covariant derivative with respect to the Berwald connection $B\Gamma = (G_{jk}^i, G_j^i, 0)$.

Lemma 3.1. *Let $(M, F, \mathbf{J})$ be a nearly Kähler Finsler manifold. Then the following hold*

(i) $J_m^r \Im_s + g^{rj} g_{is} y^k J_k^r L_{rj,m}^i = 0,$

(ii) $J_j^r \Im_m - 2g^{rs} g_{is} J_r^i L_{mj}^r + J_m^r \Im_j + g^{rs} g_{is} y^k J_k^r B_{rjm}^i = 0.$

where, $L_{rj,m}^i$ denote the vertical derivation of Landsberg curvature L_{rj}^i with respect to y^m.

Proof. Part (i): Let $(M, F, \mathbf{J})$ be a nearly Kähler Finsler manifold. Using relation (2.5), we rewrite $J_{j|k}^i + J_{k|j}^i = 0$ as follows

$$\partial_k J_j^i + J_j^r F_{rk}^i + \partial_j J_k^i + J_k^r F_{rj}^i - 2J_r^i F_{jk}^r = 0. \tag{3.3}$$

Multiplying relation (3.3) with y^k implies that

$$y^k \partial_k J_j^i + J_j^r G_r^i + y^k \partial_j J_k^i + y^k J_k^r F_{rj}^i - 2J_r^i G_j^r = 0, \tag{3.4}$$

where $y^k F_{kj}^r = G_j^r$. Taking a vertical derivation of relation (3.4) with respect to y^m yields

$$\partial_m J_j^i + \partial_j J_m^i + J_j^r G_{rm}^i - 2J_r^i G_{jm}^r + J_m^r G_{rj}^i - J_m^r L_{rj}^i + y^k J_k^r \frac{\partial F_{rj}^i}{\partial y^m} = 0. \tag{3.5}$$

where $F_{rj}^i := G_{rj}^i - L_{rj}^i$. By (3.2), we have

$$J_{j;m}^i + J_{m;j}^i = J_m^r L_{rj}^i - y^k J_k^r B_{rjm}^i + y^k J_k^r \frac{\partial L_{rj}^i}{\partial y^m}. \tag{3.6}$$

By contracting relation (3.6) with $g_{is} g^{rj}$, one can get

$$g^{rj} g_{is} (J_{j;m}^i + J_{m;j}^i) = J_m^r \Im_s - g^{rj} g_{is} y^k J_k^r B_{rjm}^i + g^{rj} g_{is} y^k J_k^r \frac{\partial L_{rj}^i}{\partial y^m}. \tag{3.7}$$

We multiply relation (3.4) by y^j and obtain

$$y^j y^k \partial_k J^i_j + y^j J^r_j G^i_r + y^k y^j \partial_j J^i_k - 2y^j J^i_r G^r_j + y^k y^j J^r_k G^i_{rj} - y^k y^j J^r_k L^i_{rj} = 0, \tag{3.8}$$

where we have used $y^k F^r_{kj} = G^r_j$, $F^i_{jk} = G^i_{jk} - L^i_{jk}$. Differentiating (3.8) with respect to y^j and y^m, leads us to

$$\partial_m J^i_j + \partial_j J^i_m + J^r_j G^i_{rm} - 2J^i_r G^r_{mj} + J^r_m G^i_{rj} + y^k J^r_k B^i_{rjm} = 0, \tag{3.9}$$

where we have used $y^j B^i_{jkl} = 0$ and $y^j L^i_{rj} = 0$. One can rewrite relation (3.9) as follows

$$g^{rj} g_{is} (J^i_{j;m} + J^i_{m;j}) = -g^{rj} g_{is} y^k J^r_k B^i_{rjm}. \tag{3.10}$$

Comparing relation (3.7) and relation (3.10) imply that

$$J^r_m \mathfrak{I}_s + g^{rj} g_{is} y^k J^r_k \frac{\partial L^i_{rj}}{\partial y^m} = 0.$$

Part (ii): Differentiating relation (3.4) with respect to y^s we get

$$y^k \partial_k J^i_s + J^r_s G^i_r + y^j \partial_s J^i_j - 2J^i_r G^r_s + y^k J^r_k G^i_{rs} = 0, \tag{3.11}$$

by differentiating relation (3.11) with respect to y^t we have

$$\partial_t J^i_s + J^r_s G^i_{rt} + \partial_s J^i_t - 2J^i_r G^r_{st} + J^r_t G^i_{rs} + y^k J^r_k B^i_{rst} = 0. \tag{3.12}$$

Using $F^i_{jk} = G^i_{jk} - L^i_{jk}$ we rewrite (3.12), as follows

$$\begin{aligned} \partial_m J^i_j &+ J^r_j F^i_{rm} - J^i_r F^r_{jm} + \partial_j J^i_m + J^r_m F^i_{rj} - J^i_r F^r_{jm} + \\ J^r_j L^i_{rm} &- J^i_r L^r_{jm} + J^r_m L^i_{rj} - J^i_r L^r_{jm} + y^k J^r_k B^i_{rjm} = 0. \end{aligned}$$

According to the relation $J^i_{j|k} + J^i_{k|j} = 0$, the above equation is reduced as follows

$$J^r_j L^i_{rm} - 2J^i_r L^r_{jm} + J^r_m L^i_{rj} + y^k J^r_k B^i_{rjm} = 0. \tag{3.13}$$

Finally by contracting (3.13), with $g_{is} g^{rj}$, one can get

$$J^r_j \mathfrak{I}_m - 2g^{rs} g_{is} J^i_r L^r_{mj} + J^r_m \mathfrak{I}_j + g^{rs} g_{is} y^k J^r_k B^i_{rjm} = 0. \tag{3.14}$$

We get the proof. □

As a direct consequence of the above lemma, the following proposition holds.

Proposition 3.2. *Let $(M, F, \mathbf{J})$ be a nearly Kähler Finsler manifold. Then F is a weakly Landsberg metric if the following holds*

$$J^r_j \mathfrak{I}_m = J^r_m \mathfrak{I}_j. \tag{3.15}$$

Proof. Let $(M, F, \mathbf{J})$ be a nearly Kähler Finsler manifold so $J^i_{j|k} + J^i_{k|j} = 0$. Using relation (3.15) we get

$$J^p_{i|m} = J^p_{i;m} + J^p_i \mathfrak{J}_m - J^p_m \mathfrak{J}_i = J^p_{i;m}. \tag{3.16}$$

Consider $g^{rs} g_{is} y^k J^r_k B^i_{rjm} = 0$ so the relation (3.1)ii, is reduced to

$$J^r_j \mathfrak{J}_m + J^r_m \mathfrak{J}_j = 2g^{rs} g_{is} J^i_r L^r_{mj}. \tag{3.17}$$

Multiplying relation (3.17) with J^k_i implies that

$$J^k_i J^r_j \mathfrak{J}_m + J^k_i J^r_m \mathfrak{J}_j = -2g^{rs} g_{is} L^k_{mj}. \tag{3.18}$$

Contracting relation (3.18) with y^j and using the relation $y^j L^r_{mj} = 0$ and $y^j \mathfrak{J}_j = 0$, we have

$$y^j J^k_i J^r_j \mathfrak{J}_m = 0. \tag{3.19}$$

By differentiating (3.19) with respect to y^l we have

$$J^k_i J^r_l \mathfrak{J}_m = 0. \tag{3.20}$$

According to relation (3.20) and (3.1), the proof is complete. □

Proposition 3.3. *Let $(M, F, \mathbf{J})$ be a nearly Kähler manifold. Then F has relatively isotropic Landsberg curvature if and only if it is Riemannian or Landsbergian metric.*

Proof. Let F has relatively isotropic Landsberg curvature $\mathbf{L} = cF\mathbf{C}$, where $c = c(x)$ is a scalar function on M. We rewrite the relation (3.1) ii, using $\mathbf{L}_{ijk} = cF\mathbf{C}_{ijk}$, $y^k J^r_k B^i_{rjm} = 0$ and $\mathfrak{J}_i = \mathbf{g}^{it}\mathbf{L}_{tjk}$. Therefore, we have

$$cF(J^r_j C^i_{rm} - 2J^i_r C^r_{jm} + J^r_m C^i_{rj}) = 0. \tag{3.21}$$

Multiplying relation (3.21) with y_i implies that

$$-2cF J^i_r y_i C^r_{jm} = 0. \tag{3.22}$$

By relation (3.22), it follows that $\mathbf{C} = 0$ or $c = 0$. If $\mathbf{C} = 0$, then F is Riemannian. Nevertheless, $c = 0$ and F is reduced to a Landsberg metric. □

Proposition 3.4. *Let $(M, F, \mathbf{J})$ be a nearly Kähler manifold. Then F has relatively isotropic mean Landsberg curvature if and only if it is Landsbergian metric or satisfies the following*

$$J^r_j I_r = J^r_r I_j. \tag{3.23}$$

Proof. Let F has relatively isotropic mean Landsberg curvature $\mathfrak{J} = cF\mathbf{I}$, where $c = c(x)$ is a scalar function on M. We rewrite the relation (3.1) *ii*, using $\mathfrak{J}_j = cF\mathbf{I}_i$ and $y^k J^r_k B^i_{rjm} = 0$. Multiplying relation (3.1) *ii* with g_{tr}, implies that

$$g_{tr} J^r_j \mathfrak{J}_m - 2g^{rs} g_{is} J^i_r L_{tmj} + g_{tr} J^r_m \mathfrak{J}_j = 0. \tag{3.24}$$

By contracting relation (3.24) whit g^{tm}, one can get

$$\delta^m_r J^r_j \mathfrak{J}_m - 2\delta^r_i J^i_r \mathfrak{J}_j + \delta^m_r J^r_m \mathfrak{J}_j = 0, \tag{3.25}$$

where we have used $g^{it} g_{tj} = \delta^i_j$. Replacing $\mathfrak{J}_j = cF\mathbf{I}_i$ in relation (3.25), we have

$$cF(\delta^m_r J^r_j I_m - 2\delta^r_i J^i_r I_j + \delta^m_r J^r_m I_j) = 0. \tag{3.26}$$

We rewrite relation (3.26), as follows

$$cF(J^r_j I_r - J^r_r I_j) = 0. \tag{3.27}$$

By relation (3.27), it follows that $c = 0$ or

$$J^r_j I_r = J^r_r I_j. \tag{3.28}$$

If $c = 0$, then the function F is reduced to a Landsberg metric. □

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