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CONVERGENCE ANALYSIS FOR APPROXIMATING SOLUTION OF VARIATIONAL INCLUSION PROBLEM

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Abstract. This article aims to define a new resolvent operator for variational inclusion problems in the framework of Banach spaces. We design a rapid algorithm using the resolvent operator to approximate the solution of the variational inclusion problem in Banach spaces. Additionally, we show that the algorithm articulated in this article converges faster than the well-known and notable algorithm due to Fang and Huang. To show the superiority and prevalence of the obtained results, we propound a numerical and computational example upholding our claim. Last, a minimization problem is solved with the help of the proposed algorithm, which is the first attempt in the current context of the study.
Keywords: variational inclusion problem, approximation, convergence.

1. Introduction

Variational inclusions, generalized form of variational inequalities have been studied quite extensively and have become an important tool to study a wide range of problems in several branches of pure and applied sciences. Novel and innovative techniques were used to study and explore them in different directions. Studying variational inclusions in all these directions is quite interesting and beneficial,

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but one of the most interesting and important aspects of the theory of variational inclusions is to develop an efficient and implementable algorithm to compute the approximate solution of a variational inclusion problem. Moreover, many authors have constructed different approximation algorithms for different variational inclusions in Hilbert as well as in Banach spaces. In 1976, Rockafellar [13] developed an algorithm for solving variational inclusion problems known as the proximal point algorithm. Many authors have studied the proximal point algorithm like [1, 3, 9, 10, 15]. Verma [14] generalized the relaxed and over-relaxed proximal point algorithm using A-monotonicity for variational inclusion in Hilbert space. Lan [11] introduced the new concept of (A, η, m)-maximal monotone operators, which generalized the existing monotone operators such as A-monotonicity, (H, η)- monotonicity, and other monotone operators as special cases. In 2003, Fang and Huang [4] introduced a new class of maximal η-monotone mapping in Hilbert spaces, which is a generalization of the classical maximal monotone mapping, and studied the properties of the resolvent operator associated with the maximal η-monotone mapping. They also introduced and studied a new class of non-linear variational inclusions involving maximal η-monotone mapping in Hilbert spaces. In 2004, Fang and Huang [5] further extended and generalized their work from Hilbert spaces to Banach spaces by introducing a new class of $\mathbb{H}$-accretive operators. They extend the concept of resolvent operators associated with the classical m-accretive to the new H-accretive operators. By using the resolvent operator technique, they studied the approximate solution of a class of variational inclusions with H-accretive operators in Banach spaces. Let $\mathbb{B}$ be a real Banach space and $G, \mathbb{H} : \mathbb{B} \to \mathbb{B}$ be two single valued operators and $\mathbb{M} : \mathbb{B} \to 2^{\mathbb{B}}$ be a multivalued operator. Fang and Huang [5] considered the problem of finding $b \in \mathbb{B}$, such that

$$0 \in G(b) + M(b) \tag{1.1}$$

and they constructed the following iterative algorithm, for any $b_o \in \mathbb{B}$, the iterative sequence $\{b_n\} \subset \mathbb{B}$ is defined by **Algorithm 1.2**

$$b_{n+1} = \mathbb{R}^{\mathbb{H}}_{M,\lambda}[H(b_n) - \lambda G(b_n)], \quad n = 0, 1, 2, 3, \cdots \tag{1.2}$$

Where $\mathbb{R}^{\mathbb{H}}_{M,\lambda} : \mathbb{B} \to \mathbb{B}$ is defined by

$$\mathbb{R}^{H}_{M,\lambda}(b) = (H + \lambda M)^{-1}(b) \quad \textit{for all} \;\; b \in \mathbb{B}$$

under the condition that M is $\mathbb{H}$-accretive, G is strongly accretive w.r.t. $\mathbb{H}$ and $\mathbb{H}$ is strictly accretive operator. Motivated by the research work in this direction and most particularly taking the results of Fang and Huang [4, 5] into consideration, our motive is to introduce a modified and improved iterative algorithm involving resolvent operator technique in the framework of Banach space for the solution of VIP (1.1). So for it, we took a modified version of the resolvent operator and correspondingly some different constraints on operators, and we constructed a new iterative algorithm sequence for the solution of VIP (1.1) in the framework of Banach space. The convergence of our algorithm is faster than the Algorithm given by Fang and Huang. By Numerical example, we have shown the superiority of our algorithm.

In what follows, we always let $\mathbb{B}$ be a real Banach space with dual space $\mathbb{B}^*$, $\langle\cdot,\cdot\rangle$ be the dual pair between B and B^*, and $2^{\mathbb{B}}$ denote the family of all nonempty subsets of $\mathbb{B}$. The generalized duality mapping $J_q : \mathbb{B} \to 2^{\mathbb{B}}$ is defined by

$$J_q(b) = \{f^* \in B^* : \langle b, f^*\rangle = \|b\|^q \ \ \textit{and} \ \ \|f^*\| = \|b\|^{q-1}\} \ \ \textit{for all} \ \ b \in \mathbb{B}$$

Where $q \geq 1$ is a constant.

Definition 1.1. [5] Let $G, \mathbb{H} : \mathrm{B} \to \mathrm{B}$ be two single-valued operators. G is said to be

(i) *accretive*, if
$$\langle G(b) - G(c), J_q(b-c)\rangle \geq 0 \ \ \textit{for all} \ \ b, c \in \mathrm{B};$$

(ii) *strictly accretive*, if G is accretive and
$$\langle G(b) - G(c), J_q(b-c)\rangle = 0 \ \ \textit{iff}\ b = c;$$

(iii) *strongly accretive*, if for some positive constant α
$$\langle G(b) - G(c), J_q(b-c)\rangle \geq \alpha\|b-c\|^q \ \ \textit{for all} \ \ b, c \in \mathrm{B};$$

(iv) *strongly accretive with respect to* $\mathbb{H}$, if for some positive constant β
$$\langle G(b) - G(c), J_q(\mathbb{H}(b) - \mathbb{H}(c))\rangle \geq \beta\|b-c\|^q \ \ \textit{for all} \ \ b, c \in \mathrm{B};$$

(v) *Lipschitz continuous*, if for some positive constant γ
$$\|G(b) - G(c)\| \leq \gamma\|b-c\| \ \ \textit{for all} \ \ b, c \in \mathrm{B}.$$

Definition 1.2. [5] Let $\mathbb{H} : \mathrm{B} \to \mathrm{B}$ be a single valued operator, then a multi-valued mapping $M : \mathrm{B} \to 2^{\mathrm{B}}$ is said to be

(i) *accretive* if
$$\langle s-t, b-c\rangle \geq 0 \ \textit{for all}\ b, c \in \mathrm{B};\ \ s \in M(b),\ \ t \in M(c);$$

(ii) *strongly accretive* if there exists some positive constant η such that
$$\langle s-t, b-c\rangle \geq \eta\|b-c\|^q \ \textit{for all}\ b, c \in \mathrm{B};\ \ s \in M(b),\ \ t \in M(c);$$

(iii) *m-accretive* if M is accretive and $(I + \lambda M)(\mathrm{B}) = \mathrm{B}$ *for all* $\lambda > 0$, where I is the identity mapping on B;

(iv) $\mathbb{H}$-*accretive* if M is accretive and $(\mathbb{H} + \lambda M)(\mathrm{B}) = \mathrm{B}$ holds for every $\lambda > 0$;

(v) *strongly* $\mathbb{H}$-*accretive*, if M is strongly accretive and $(\mathbb{H} + \lambda M)(\mathrm{B}) = \mathrm{B}$ holds for every $\lambda > 0$.

Definition 1.3. Let V be any vector space over field F. Then the algebraic dual space V^* is defined as the set of all linear functionals on V.

Lemma 1.1. *Let $M : \mathrm{B} \to 2^{\mathrm{B}}$ be an m-accretive mapping and $G : \mathrm{B} \to \mathrm{B}$ be an accretive operator. Then a mapping $G + M : \mathrm{B} \to 2^{\mathrm{B}}$ is an m-accretive operator.*

Proof. For $u \in (G + M)(b)$, $v \in (G + M)(c)$, $b, c \in \mathrm{B}$, consider

$$\begin{aligned}&\langle u - v, J_q(b - c)\rangle \\ &= \langle Gb + u - (Gc + v), J_q(b - c)\rangle \\ &= \langle Gb + Gc, J_q(b - c)\rangle + \langle u + v, J_q(b - c)\rangle \geq 0.\end{aligned}$$

□

Using above Lemma 1.1, we have defined a new resolvent operator in the following way:

Definition 1.4. Let $\mathbb{H} : \mathrm{B} \to \mathrm{B}$ be a strictly accretive operator and $G + M : \mathrm{B} \to 2^{\mathrm{B}}$ be strongly $\mathbb{H}-$accretive operator. Then the resolvent operator is defined as:

$$\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) = [\mathbb{H} + \lambda(G + M)]^{-1}(b) \quad \textit{for all} \;\; b \in \mathrm{B}. \tag{1.3}$$

Lemma 1.2. *Let $\mathbb{H} : \mathrm{B} \to \mathrm{B}$ be a strictly accretive operator and $G + M : \mathrm{B} \to 2^{\mathrm{B}}$ be a strongly $\mathbb{H}$-accretive operator. Then the resolvent operator $[\mathbb{H} + \lambda(G + M)]^{-1}$ is a single valued.*

Proof. Suppose for $b \in \mathrm{B}$, and $x, y \in [\mathbb{H}+\lambda(G+M)]^{-1}(b)$. Then from the definition of the resolvent operator, it follows that $-\mathbb{H}x + b \in \lambda(G + M)x$ and $-\mathbb{H}y + b \in \lambda(G + M)y$. Using accretiveness of $G + M$, we get

$$\langle(-\mathbb{H}x + b) - (-\mathbb{H}y + b), x - y\rangle = \langle\mathbb{H}x - \mathbb{H}y, x - y\rangle \geq 0.$$

As $\mathbb{H}$ is strictly accretive operator, so we get $x = y$. Thus $[\mathbb{H} + \lambda(G + M)]^{-1}$ is single valued. □

Lemma 1.3. *Let $\mathbb{H} : \mathrm{B} \to \mathrm{B}$ be a strongly accretive operator with constant $\alpha > 0$ and $G + M : \mathrm{B} \to 2^{\mathrm{B}}$ be a strongly $\mathbb{H}$-accretive operator with positive constant η. Then the resolvent operator $\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) = [\mathbb{H} + \lambda(G + M)]^{-1}(b)$ for all $b \in \mathrm{H}$ and $\lambda > 0$ is $\left(\frac{1}{\alpha+\lambda\eta}\right)$ Lipschitizian continuous i.e.*

$$\left\|\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c)\right\| \leq \left(\frac{1}{\alpha + \lambda\eta}\right)\|b - c\| \textit{ for all } b, c \in \mathrm{B}.$$

Proof. For two given points $b,\; c \in \mathrm{B}$, we have

$$\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) = [\mathbb{H} + \lambda(G + M)]^{-1}(b)$$

and

$$\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c) = [\mathbb{H} + \lambda(G+M)]^{-1}(c).$$

This means that

$$\frac{1}{\lambda}\big(b - \mathbb{H}(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b))\big) \in (G+M)(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b))$$

and

$$\frac{1}{\lambda}\big(c - \mathbb{H}(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c))\big) \in (G+M)(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c)).$$

Since $G+M$ is η strongly accretive, we have

$$\begin{aligned}
&\eta\|\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c)\|^q \\
&\leq \big\langle b - \mathbb{H}(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) - (c - \mathbb{H}(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c)), J_q(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c))\big\rangle \\
&= \frac{1}{\lambda}\big\langle b - c - \big(\mathbb{H}(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b)) - \mathbb{H}(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c))\big), J_q(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c))\big\rangle.
\end{aligned}$$

Now consider

$$\begin{aligned}
&\|b-c\|\|\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c)\|^{q-1} \\
&= \|b-c\|\|J_q(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c))\| \\
&\geq \big\langle b-c, J_q(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c))\big\rangle \\
&\geq \big\langle \mathbb{H}(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b)) - \mathbb{H}(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c), J_q(\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c))\big\rangle \\
&+\lambda\eta\|\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c)\|^q \\
&\geq \alpha\|\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c)\|^q \\
&+\lambda\eta\|\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c)\|^q \\
&= (\alpha+\lambda\eta)\|\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c)\|^q.
\end{aligned}$$

So, we get

$$\|\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(b) - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}(c)\| \leq \left(\frac{1}{\alpha+\lambda\eta}\right)\|b-c\| \textit{ for all } b,c \in \mathrm{B}.$$

□

2. Algorithm for Variational Inclusion Problem

One of the most important and interesting research work in the field of variational inequalities and variational inclusions is to develop algorithms. Many people have made this research work their favorite one like [2, 8, 12] etc. In this part of the article, we design an iterative algorithm for the variational inclusion problem (1.1) using the resolvent operator given by (1.3). In fact problem (1.1) was considered by many authors like [4, 5, 6, 7, 13] etc. for different types of mappings and in different settings.

We consider the problem while taking M as an $\mathbb{H}$-accretive and G as strongly accretive so that $G+M : \mathrm{B} \to 2^{\mathrm{B}}$ is also an $\mathbb{H}$-accretive operator. Fang and Huang had also taken the same mappings. First, we state the fixed point formulation for our problem.

Lemma 2.1. *Let $\mathbb{H} : \mathrm{B} \to \mathrm{B}$ be a strictly accretive operator and $G+M : \mathrm{B} \to 2^{\mathrm{B}}$ be a strongly $\mathbb{H}$-accretive operator. Then $b \in \mathrm{B}$ is a solution of problem (1.1) iff*

$$b = \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}[\mathbb{H}(b)]. \tag{2.1}$$

Based on the above-fixed point formulation, we propose our algorithm as:
Algorithm 2.2 For any $b_0 \in \mathrm{B}$, the iterative sequence $\{b_n\} \subset \mathrm{B}$ is defined by

$$b_{n+1} = R^{\mathbb{H}}_{G+M,\lambda}[\mathbb{H}(c_n)], c_n = R^{\mathbb{H}}_{G+M,\lambda}[\mathbb{H}(b_n)], \quad n = 0, 1, 2, \cdots. \tag{2.2}$$

3. Main Result

In this part of the article, we have shown that the sequence generated by Algorithm (2.2) converges to the unique solution of problem (1.1) strongly. A numerical example is given to show that the convergence rate of our algorithm is faster than that of Fang and Huang [5].

Theorem 3.1. *Let $\mathbb{H} : \mathrm{B} \to \mathrm{B}$ be a strongly accretive and Lipschitz continuous operator with positive constants α and β, respectively, $G+M : \mathrm{B} \to 2^{\mathrm{B}}$ be a strongly $\mathbb{H}$-accretive operator with positive constant η and there exists some positive constant λ such that $\left(\frac{\beta}{\alpha+\lambda\eta}\right) < 1$.*
Then the sequence constructed by algorithm (2.2) converges to the unique solution of problem (1.1) strongly.

Proof. Let b^* be the solution of the problem (1.1), then it follows that

$$\begin{aligned} \|b_{n+1} - b^*\| &= \|\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}[\mathbb{H}(c_n)] - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}[\mathbb{H}(b^*)]\| \\ &\leq \frac{1}{\alpha+\lambda\eta}\|\mathbb{H}(c_n) - \mathbb{H}(b^*)\| \\ &\leq \frac{\beta}{\alpha+\lambda\eta}\|c_n - b^*\| \\ &= \frac{\beta}{\alpha+\lambda\eta}\|\mathbb{R}^{\mathbb{H}}_{G+M,\lambda}[\mathbb{H}(b_n)] - \mathbb{R}^{\mathbb{H}}_{G+M,\lambda}[\mathbb{H}(b^*)]\| \\ &\leq \left(\frac{\beta}{\alpha+\lambda\eta}\right)^2 \|b_n - b^*\|. \end{aligned}$$

Continuing in this way, we obtain

$$\|b_{n+1} - b^*\| \leq \left(\frac{\beta}{\alpha+\lambda\eta}\right)^{2(n+1)} \|b_0 - b^*\|. \tag{3.1}$$

Taking $\lim n \to \infty$, we get

$$\lim_{n\to\infty} \|b_{n+1} - b^*\| = 0.$$

This implies that b_n converges to b^* strongly. □

4. Numerical Example

Example 4.1. Let $\mathbb{B} = \mathbb{R}$, the set of reals, $\mathbb{H}, G : \mathbb{R} \to \mathbb{R}$ and $\mathbb{M} : \mathbb{R} \to 2^{\mathbb{R}}$ be defined as $\mathbb{H}(b) = 2b \ \ \forall s \in \mathbb{R}$, $G(b) = b/3, \forall s \in \mathbb{R}$ and $\mathbb{M}(b) = 3b, \forall s \in \mathbb{R}$. Then $\mathbb{H}$ is strongly accretive with constant $\alpha = 1.9$ and Lipschitz continuous with constant $\beta = 2.1$. $\mathbb{B}$ is lipschitz continuos with constant $\gamma = 0.4$ and strongly accretive w.r.to $\mathbb{H}$ with constant $\delta = 0.5$ and $\mathbb{G} + \mathbb{M}$ is strongly $\mathbb{H}$-accretive operator with constant $\eta = 3.1$. Under these conditions, the example satisfies the conditions of both theorems, the theorem of Fang and Huang, as well as conditions of our theorem 3.1. Utilizing MATLAB 2012, Figure 4.1 depicts the convergence of $\{b_n\}$ by taking initial value $b_1 = 1$, where one can easily see that our algorithm converges faster than the algorithm due to Fang and Huang [5].

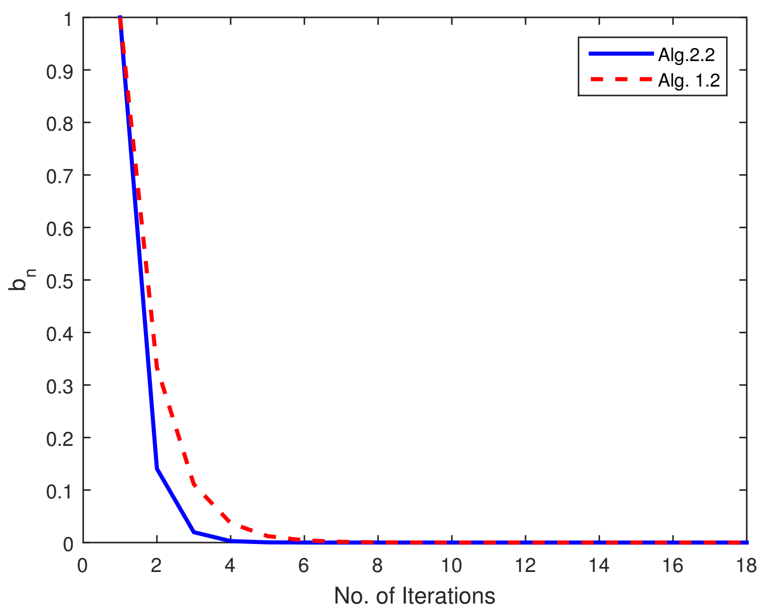

Fig. 4.1: The convergence of $\{b_n\}$ with initial value $b_1 = 1$.

No. of Iterations	Algorithm 2.2	Algorithm 1.2
1	1.0000	1.0000
2	0.1406	0.3333
3	0.0198	0.1111
4	0.0028	0.0370
5	0.0004	0.0123
6	0.0001	0.0041
7	0.0000	0.0014
8	0.0000	0.0005
9	0.0000	0.0002
10	0.0000	0.0001
11	0.0000	0.0000

Table 4.1: Comparison Table: Algorithm 2.2 vs Algorithm 1.2 with initial value $b_1 = 1$.

5. Application

In this section, we discuss the application of our algorithm for solving minimization problem. We consider the minimization problem on R, a Banach space. Let $f : \mathrm{R} \to \mathrm{R}$ be a function defined by $f(b) = \frac{b^2}{4}$ and $g : \mathrm{R} \to \mathrm{R} \cup \{+\infty\}$ be a proper convex and lower semi-continuous function defined as $g(b) = b^2$. we want to find the minimum of the function $\{f(b) + g(b)\}$, i.e. to find $b^* \in \mathrm{R}$ such that

$$\{f(b^*) + g(b^*)\} = \min_{b \in \mathrm{R}} \{f(b) + g(b)\}. \tag{5.1}$$

Then by Fermat's rule problem (5.1) is equivalent to find $b \in \mathrm{R}$ such that

$$0 \in \frac{b}{2} + 2b, \tag{5.2}$$

which is same as problem (1.3) by taking $G(b) = \frac{b}{2}$ and $M(b) = 2b$. This problem can be easily solved by algorithm (2.2). Let us take $\mathbb{H}(b) = 3b$, then it is strongly accretive with constant $\alpha = 2.9$ and Lipschitz continuous with constant $\beta = 3.1$. G is Lipschitz continuous with constant $\gamma = 0.6$ and is strongly accretive with constant $\delta = 1.4$ and G+M is strongly $\mathbb{H}-$ accretive with constant $\eta = 7.4$. So, it satisfies all the conditions given in theorem (3.1). Figure 5.1 shows the convergence of $\{b_n\}$ by taking different initial values. Moreover, Figure 5.2 shows that the minimum acquired is 0.

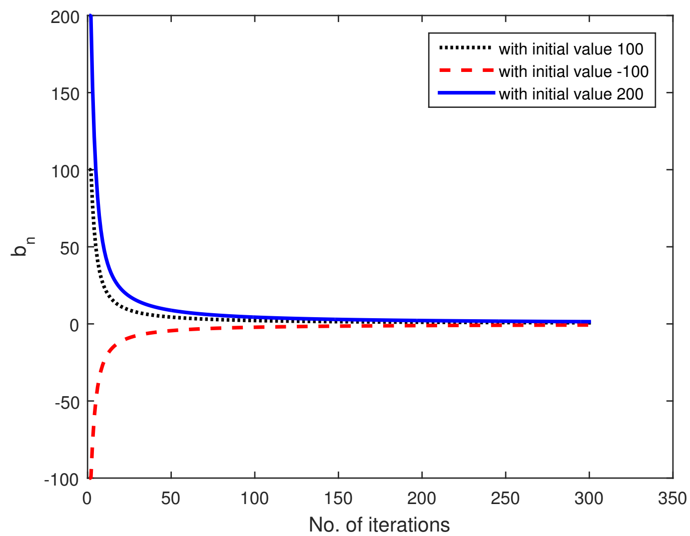

FIG. 5.1: The minimum of problem (5.1) is 0 while starting with different initial values $100, -100$ and 200.

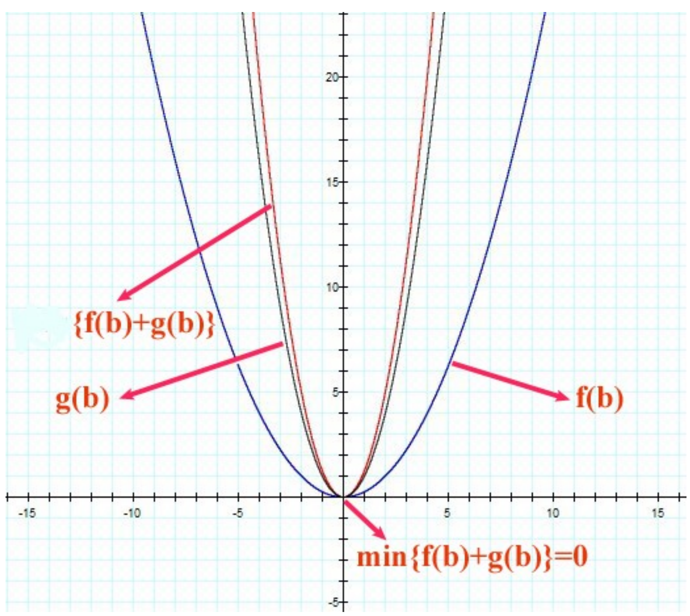

FIG. 5.2: $\min_{b\in\mathbb{R}}\{f(b)+g(b)\}$.

6. Conclusions

We introduced a modified and improved iterative algorithm involving the resolvent operator technique in the framework of Banach spaces for the existence of the solution of VIP (1.1). We constructed a novel iterative algorithm sequence for the solution of VIP (1.1) in the setting of Banach spaces by taking a modified version of the resolvent operator and, correspondingly, some different constraints on operators. The convergence of our algorithm is more rapid than the well-known algorithm due to Fang and Huang. A numerical example with computer simulation is adopted to uphold our claims.

Last, an open inquiry is whether we could foster an algorithm having a quicker pace of assembly than the algorithm introduced in this article and, furthermore, loosen up conditions on mappings included.

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