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NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR k-β-CONVEX FUNCTIONS VIA GENERALIZED k-FRACTIONAL CONFORMABLE INTEGRAL OPERATORS

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Abstract. In this paper, we first establish a new Hermite-Hadamard inequality for the class of k-β-convex functions involving the generalized k-fractional conformable integral operators. Then, based on two new identities, we discuss some new k-fractional conformable integral inequalities of midpoint type whose first and second derivatives belong to the class of k-β-convex functions. Several new and known results are derived.
Key words: Hermite-Hadamard inequality, Hölder inequality, power mean inequality, k-β-convex function, generalized k-fractional conformable integral operators.

1. Introduction

If $f : I \to \mathbb{R}$ is a convex function on the interval $I \subset \mathbb{R}$, then for any $a, b \in I$ with $a < b$, we have the following double inequality

$$f\left(\tfrac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \tfrac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the opposite direction if f is concave. This significant result was given in [19], it was first discovered by Hermite in the journal Mathesis in 1881 and it is well known in the literature as the Hermite-Hadamard inequality,

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which provides lower and upper bounds for the integral mean of all convex functions defined on a compact interval, including the middle and endpoints of the domain. For more information on the progress of research on inequality (1.1) we refer the reader to the excellent monograph [14].

Since the discovery of this inequality, many researchers have given considerable attention to study of inequalities in generale in real, fractional and quantum cases see [5, 6, 7, 8, 9, 11, 12, 14, 15, 20, 24, 25, 27, 28, 29, 30, 34, 39, 40].

It is worth noting that fractional integrals and derivatives provide an excellent tool for the description of the memory and hereditary properties of various materials and processes see [18, 31].

In [39], Sarikaya and Yildirim established the analogue fractional of inequality (1.1) as follows

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1 [a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left(J^{\alpha}_{(\frac{a+b}{2})^{+}} f(b) + J^{\alpha}_{(\frac{a+b}{2})^{-}} f(a)\right) \leq \tfrac{f(a)+f(b)}{2}. \tag{1.2}$$

Also, in the same paper they discussed some fractional midpoint inequalities for convex first derivatives and obtained the following results

$$\begin{aligned} &\left|\tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left(J^{\alpha}_{(\frac{a+b}{2})^{+}} f(b) + J^{\alpha}_{(\frac{a+b}{2})^{-}} f(a)\right) - f\left(\tfrac{a+b}{2}\right)\right| \\ \leq\ & \tfrac{b-a}{4(\alpha+1)} \left(\left(\tfrac{(\alpha+1)\left|f'(a)\right|^{q}+(\alpha+3)\left|f'(b)\right|^{q}}{2(\alpha+2)}\right)^{\frac{1}{q}} + \left(\tfrac{(\alpha+3)\left|f'(a)\right|^{q}+(\alpha+1)\left|f'(b)\right|^{q}}{2(\alpha+2)}\right)^{\frac{1}{q}}\right), \end{aligned}$$

and

$$\begin{aligned} &\left|\tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left(J^{\alpha}_{(\frac{a+b}{2})^{+}} f(b) + J^{\alpha}_{(\frac{a+b}{2})^{-}} f(a)\right) - f\left(\tfrac{a+b}{2}\right)\right| \\ \leq\ & \tfrac{b-a}{4} \left(\tfrac{1}{\alpha p+1}\right)^{\frac{1}{p}} \left(\left(\tfrac{\left|f'(a)\right|^{q}+3\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}} + \left(\tfrac{3\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}}\right) \\ \leq\ & \tfrac{b-a}{4} \left(\tfrac{4}{\alpha p+1}\right)^{\frac{1}{p}} \left(|f'(a)| + |f'(b)|\right), \end{aligned}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In [32], Mubeen and Habibullah introduced a new fractional integral operator called k-fractional integrals, regarding some papers involving integral inequalities via this novel operator, we refer readers to [1, 2, 3, 4, 17, 21, 36, 41].

In [17], Farid et al. obtained the following Hermite-Hadamard inequality

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1 [a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for* *k-fractional integrals hold:*

$$f\left(\tfrac{a+b}{2}\right) \le \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{(\frac{a+b}{2})^+,k}f(b)+I^{\alpha}_{(\frac{a+b}{2})^-,k}f(a)\right] \le \frac{f(a)+f(b)}{2}, \tag{1.3}$$

with $\alpha,\ k>0$.

Also, they proved the following identity

Lemma 1.1. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$. *If* $f'\in L_1[a,b]$, *then the following equality for k-fractional integrals holds:*

$$\begin{aligned} &\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{(\frac{a+b}{2})^+,k}f(b)+I^{\alpha}_{(\frac{a+b}{2})^-,k}f(a)\right]-f\left(\tfrac{a+b}{2}\right) \\ = \ &\frac{b-a}{4}\left[\int_0^1 t^{\frac{\alpha}{k}}f'(\tfrac{t}{2}a+\tfrac{2-t}{2}b)dt-\int_0^1 t^{\frac{\alpha}{k}}f'(\tfrac{2-t}{2}a+\tfrac{t}{2}b)dt\right]. \end{aligned} \tag{1.4}$$

By using the above identity, Farid et al. derived the following midpoint inequalities for differentiable mappings

Theorem 1.3. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$. *If* $|f'|^q$ *is convex on* $[a,b]$ *for* $q\ge 1$, *then the following inequality for k-fractional integrals holds:*

$$\begin{aligned} &\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{(\frac{a+b}{2})^+,k}f(b)+I^{\alpha}_{(\frac{a+b}{2})^-,k}f(a)\right]-f\left(\tfrac{a+b}{2}\right)\right| \\ \le \ &\frac{b-a}{4(\frac{\alpha}{k}+1)}\left(\frac{1}{2(\frac{\alpha}{k}+2)}\right)^{\frac{1}{q}}\left[\left((\tfrac{\alpha}{k}+1)|f'(a)|^q+(\tfrac{\alpha}{k}+3)|f'(b)|^q\right)^{\frac{1}{q}}\right. \\ &\left.+\left((\tfrac{\alpha}{k}+3)|f'(a)|^q+(\tfrac{\alpha}{k}+1)|f'(b)|^q\right)^{\frac{1}{q}}\right], \end{aligned} \tag{1.5}$$

with $\alpha,\ k>0$.

Theorem 1.4. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$. *If* $|f'|^q$ *is convex on* $[a,b]$ *for* $q>1$, *then the following inequality for k-fractional integrals holds:*

$$\begin{aligned} &\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha}_{(\frac{a+b}{2})^+,k}f(b)+I^{\alpha}_{(\frac{a+b}{2})^-,k}f(a)\right]-f\left(\tfrac{a+b}{2}\right)\right| \\ \le \ &\frac{b-a}{4}\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left(\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right) \\ \le \ &\frac{b-a}{4}\left(\frac{4k}{\alpha p+k}\right)^{\frac{1}{p}}\left(|f'(a)|+|f'(b)|\right), \end{aligned} \tag{1.6}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

In [21], Huang et al. proved the following Hermite-Hadamard type inequalities for convex functions involving the generalized k-fractional conformable integral operators

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$ be a positive function such that $f \in L_1[a,b]$ and $a < b$. If f is a convex function on $[a,b]$, then*

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{k\Gamma_k(\beta+k)\alpha^{\frac{\beta}{k}}}{2(b-a)^{\frac{\alpha\beta}{k}}}\left[J^{\alpha,\,\beta}_{a^+,k}f(b) + J^{\alpha,\,\beta}_{b^-,k}f(a)\right] \leq \tfrac{f(a)+f(b)}{2}, \tag{1.7}$$

for α, $\beta > 0$.

Theorem 1.6. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) such that $a < b$. and $f' \in L_1[a,b]$. If $|f'|$ is a convex function on $[a,b]$, then*

$$\begin{aligned} &\left|\tfrac{f(a)+f(b)}{2} - \tfrac{k\Gamma_k(\beta+k)\alpha^{\frac{\beta}{k}}}{(b-a)^{\frac{\alpha\beta}{k}}}\left[J^{\alpha,\,\beta}_{a^+,k}f(b) + J^{\alpha,\,\beta}_{b^-,k}f(a)\right]\right| \\ \leq\ & \tfrac{b-a}{2\alpha}\left[2B_{\frac{1}{2^\alpha}}\left(\tfrac{1}{\alpha},\tfrac{\beta}{k}+1\right) - B\left(\tfrac{1}{\alpha},\tfrac{\beta}{k}+1\right)\right]\left(|f'(a)| + |f'(b)|\right), \end{aligned} \tag{1.8}$$

for α, $\beta > 0$.

Recently, in [37], Samraiz et al. established the following inequalities for k-fractional conformable integral operators by using h-convexity

Theorem 1.7. *Let $f : [a,b] \to \mathbb{R}$ be a positive function such that $f \in L_1[a,b]$ and $a < b$. If f is h- convex function on $[a,b]$, then*

$$\begin{aligned} f\left(\tfrac{a+b}{2}\right) \leq\ & \tfrac{h(\frac{1}{2})\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(b-a)^{\frac{\alpha\beta}{k}}}\left[J^{\alpha,\,\beta}_{a^+,k}f(b) + J^{\alpha,\,\beta}_{b^-,k}f(a)\right] \\ \leq\ & \tfrac{\alpha^\beta}{k}h(\tfrac{1}{2})\left[f(a)+f(b)\right]\int_0^1 (1-t^\alpha)^{\frac{\beta}{k}-1}t^{\alpha-1}\left[h(t)+h(1-t)\right]dt. \end{aligned} \tag{1.9}$$

Theorem 1.8. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $a < b$. and $f' \in L_1[a,b]$. If $|f'|$ is an h-convex function on $[a,b]$, then the following inequality for k-conformable fractional integral operators holds*

$$\begin{aligned} &\left|\tfrac{f(a)+f(b)}{2} - \tfrac{\Gamma_k(\beta+k)\alpha^{\frac{\beta}{k}}}{2(b-a)^{\frac{\alpha\beta}{k}}}\left[J^{\alpha,\,\beta}_{a^+,k}f(b) + J^{\alpha,\,\beta}_{b^-,k}f(a)\right]\right| \\ \leq\ & \tfrac{b-a}{2}\left[|f'(a)|\int_0^1\left[(1-t^\alpha)^{\frac{\beta}{k}} + (1-(1-t)^\alpha)^{\frac{\beta}{k}}\right]h(t)dt\right. \\ & \left.+\,|f'(b)|\int_0^1\left[(1-t^\alpha)^{\frac{\beta}{k}} + (1-(1-t)^\alpha)^{\frac{\beta}{k}}\right]h(1-t)dt\right]. \end{aligned} \tag{1.10}$$

In [10], Bayraktar gave some inequalities of midpoint type for (s,m)-convex functions via Riemann-Liouville fractional integral. Among the obtained results we cite

$$\left|\tfrac{2^{\alpha-2}\Gamma(\alpha)}{(mb-a)^{\alpha-1}}\left(J^{\alpha-1}_{(\frac{a+mb}{2})^+}f(mb) + J^{\alpha-1}_{(\frac{a+mb}{2})^-}f(a)\right) - f\left(\tfrac{a+mb}{2}\right)\right|$$

$$\leq \frac{(mb-a)^2}{\alpha 2^{2-\alpha}}\left(\frac{1}{(\alpha+s+1)2^{\alpha+s+1}}+B_{\frac{1}{2}}(\alpha+1,s+1)\right)\left(|f''(a)|+m\,|f''(b)|\right),$$

and

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(mb-a)^{\alpha-1}}\left(J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^+}f(mb)+J^{\alpha-1}_{\left(\frac{a+mb}{2}\right)^-}f(a)\right)-f\left(\frac{a+mb}{2}\right)\right|$$
$$\leq \frac{(mb-a)^2}{\alpha 2^{2-\alpha}}\left(\frac{1}{(\alpha+1)2^{\alpha+1}}\right)^{1-\frac{1}{q}}$$
$$\times\left(\left(\frac{1}{(\alpha+s+1)2^{\alpha+s+1}}\,|f''(a)|^q+mB_{\frac{1}{2}}(\alpha+1,s+1)\,|f''(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(B_{\frac{1}{2}}(\alpha+1,s+1)\,|f''(a)|^q+m\frac{1}{(\alpha+s+1)2^{\alpha+s+1}}\,|f''(b)|^q\right)^{\frac{1}{q}}\right).$$

In [20], Han et al. established the following inequalities related to inequality (1.1) for the generalized fractional integral under the MT-convexity

$$\left|\frac{(1-h)^\mu\Omega(1)-h^\mu\Lambda(1)}{b-a}g(w)-\frac{1}{b-a}\left((1-h)^\mu\left({}_{w^-}I_\rho g(a)\right)+h^\mu\left({}_{w^+}I_\rho g(b)\right)\right)\right|$$
$$\leq \frac{(1-h)^{\mu+1}}{2}\left(|g'(w)|\int_0^1\sqrt{\frac{\tau}{1-\tau}}\,|\Omega(\tau)|\,d\tau+|g'(a)|\int_0^1\sqrt{\frac{1-\tau}{\tau}}\,|\Omega(\tau)|\,d\tau\right)$$
$$+\frac{h^{\mu+1}}{2}\left(|g'(w)|\int_0^1\sqrt{\frac{\tau}{1-\tau}}\,|\Lambda(\tau)|\,d\tau+|g'(b)|\int_0^1\sqrt{\frac{1-\tau}{\tau}}\,|\Lambda(\tau)|\,d\tau\right),$$

and

$$\left|\frac{(1-h)^\mu\Omega(1)-h^\mu\Lambda(1)}{b-a}g(w)-\frac{1}{b-a}\left((1-h)^\mu\left({}_{w^-}I_\rho g(a)\right)+h^\mu\left({}_{w^+}I_\rho g(b)\right)\right)\right|$$
$$\leq \left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left((1-h)^{\mu+1}\left(\int_0^1|\Omega(\tau)|^p\,d\tau\right)^{\frac{1}{p}}\left(|g'(w)|^q+|g'(a)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+h^{\mu+1}\left(\int_0^1|\Lambda(\tau)|^p\,d\tau\right)^{\frac{1}{p}}\left(|g'(w)|^q+|g'(b)|^q\right)^{\frac{1}{q}}\right),$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $w=ha+(1-h)b$ with $h\in(0,1)$ and $\Omega(\tau)=\int_0^\tau\frac{\rho((w-a)u)}{u}du<+\infty, \Lambda(\tau)=\int_0^\tau\frac{\rho((b-w)u)}{u}du<+\infty$.

The main purpose of this paper is to generalize the results obtained in [17], via the k-fractional conformable integral operators. For this we first establish the Hermite-Hadamard inequality for the k-β-convex functions where the obtained result covers several cases already known according to the values of the parameters α,β,k,p and q. Then through two new identities we have established some midpoint type inequalities for functions whose first and second derivatives in absolute value are k-β-convex via k-fractional conformable integral operators several known results can be derived from the obtained results.

2. Preliminaries

In this section, we recall some concepts of convexity that are well known in the literature.

Definition 2.1. [35] A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{2.1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

Definition 2.2. [42] A function f :$I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be β-convex on I, if the inequality

$$f(tx + (1-t)y) \leq t^p(1-t)^q f(x) + t^q(1-t)^p f(y) \tag{2.2}$$

holds for all x, $y \in I$ and $t \in [0, 1]$, where p, $q > -1$. We say that f is β-concave if $(-f)$ is β-convex.

Definition 2.3. [26] A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be k-β-convex on I, if the inequality

$$f(tx + (1-t)y) \leq \tfrac{1}{k} t^{\frac{p}{k}}(1-t)^{\frac{q}{k}} f(x) + \tfrac{1}{k} t^{\frac{q}{k}}(1-t)^{\frac{p}{k}} f(y) \tag{2.3}$$

holds for all x, $y \in I$ and $t \in [0, 1]$, where p, $q > -k$, $k > 0$. We say that f is k-β-concave if $(-f)$ is k-β- convex.

Remark 2.1. In Definition 2.3, if we take $k = 1$ and $p = q = 0$, then obtain P-function (see [13]), if we choose $k = 1$ and $p = -s \in (-1, 0]$ and $q = 0$, then obtain s-Godunova-Levin function of second kind (see [42]), if we take $k = 1$, then obtain β-convex function (see [22]), and if we choose $k = 1$, $p = 1$, $q = 0$, we obtain the classical convex function (see [35]).

Definition 2.4. [16] For $k > 0$, $x \in \mathbb{C} \setminus k\mathbb{Z}^-$, the k-gamma function is defined by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}},$$

where $(x)_{n,k} = \prod_{j=0}^{n-1}(x + jk)$, $k > 0$ is called the Pochhammer k-symbol.
Its integral representation is given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt, \quad \mathrm{Re}(x) > 0. \tag{2.4}$$

One can note that

$$\Gamma_k(x + k) = x\Gamma_k(x).$$

For $k = 1$, (2.4) gives integral representation of gamma function.

Definition 2.5. [16] For $k > 0$, $x \in \mathbb{C} \setminus k\mathbb{Z}^-$, the *k-beta* function with two parameters x and y is defined by

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt, \tag{2.5}$$

and we have

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \ \ \text{Re}(x) > 0, \ \text{Re}(y) > 0.$$

For $k = 1$, (2.5) gives integral representation of the beta function

Definition 2.6. [33] The integral representation of the generalized k-hypergeometric function is given as

$$\begin{aligned} {}_2\mathcal{F}_{1,k}\left((\alpha,k),(\beta,k),(\gamma,k),x\right) = & \sum_{n=0}^{\infty} \tfrac{(\alpha)_{n,\ k}(\beta)_{n,\ k}}{(\gamma)_{n,\ k}} \frac{x^n}{n!}, \quad k > 0 \\ = & \tfrac{1}{kB_k(\beta,\gamma-\beta)} \int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-kxt)^{-\frac{\alpha}{k}}dt, \end{aligned}$$

where $\text{Re}(\gamma) > \text{Re}(\beta) > 0$, $k > 0$ and $|x| < 1$.

Remark 2.2. If we take $k = 1$, we obtain the Euler representation of the Gauss hypergeometric function or ${}_2\mathcal{F}_1$ function which formulated as follows

$${}_2\mathcal{F}_1\left(\alpha,\beta,\gamma,x\right) = \tfrac{1}{B(\beta,\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}dt.$$

Definition 2.7. [32] Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integrals $I_{a^+}^{\alpha} f$ and $I_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$I_{a^+}^{\alpha} f(x) = \tfrac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \tag{2.6}$$

and

$$I_{b^-}^{\alpha} f(x) = \tfrac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b \tag{2.7}$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt, \ \ \alpha > 0$ is the gamma function. Here $I_{a^+}^{0} f(x) = I_{b^-}^{0} f(x) = f(x)$.

In the case where $\alpha = 1$, the fractional integral will be reduced to the classical integral.

Definition 2.8. [32] Let $f \in L_1[a,b]$. Then the left-sided and right-sided k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as

$$I_{a^+,k}^{\alpha} f(x) = \tfrac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x \geq a \tag{2.8}$$

and

$$I_{b^-,k}^{\alpha}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1}f(t)dt, \quad x \le b, \tag{2.9}$$

where $\Gamma_k(\alpha)$ is the k-gamma function. For $k=1$, the k- fractional integrals give Riemann-Liouville fractional integrals.

Definition 2.9. [23] The left and right fractional conformable integral operators $J_{a^+}^{\alpha,\,\beta}$ and $J_{b^-}^{\alpha,\,\beta}$ of order $\beta \in \mathbb{C}$, such that $\mathrm{Re}(\beta) > 0$ and $0 < \alpha \le 1$, for $f \in L_1[a,b]$ are defined by

$$J_{a^+}^{\alpha,\,\beta}f(x) = \frac{1}{\Gamma(\beta)}\int_a^x \left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}(t-a)^{\alpha-1}f(t)dt \tag{2.10}$$

and

$$J_{b^-}^{\alpha,\,\beta}f(x) = \frac{1}{\Gamma(\beta)}\int_x^b \left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}(b-t)^{\alpha-1}f(t)dt \tag{2.11}$$

respectively, where Γ is the Euler gamma function.

Definition 2.10. [36] The generalized left and right k-fractional conformable integral operators $J_{a^+,k}^{\alpha,\,\beta}$ and $J_{b^-,k}^{\alpha,\,\beta}$ of order $\beta \in \mathbb{C}$, such that $\mathrm{Re}(\beta) > 0$, $k > 0$ and $0 < \alpha \le 1$, for $f \in L_1[a,b]$ are defined by

$$J_{a^+,k}^{\alpha,\,\beta}f(x) = \frac{1}{k\Gamma_k(\beta)}\int_a^x \left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}(t-a)^{\alpha-1}f(t)dt, \tag{2.12}$$

and

$$J_{b^-,k}^{\alpha,\,\beta}f(x) = \frac{1}{k\Gamma_k(\beta)}\int_x^b \left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}-1}(b-t)^{\alpha-1}f(t)dt. \tag{2.13}$$

Lemma 2.1. [43] *For any $0 \le a < b$ in $\mathbb{R}$ and $0 < \alpha \le 1$, we have*

$$b^{\alpha} - a^{\alpha} \le (b-a)^{\alpha}.$$

3. Main results

Our first result is to establish the k-fractional conformable Hermite-Hadamard inequality for the k-β-convex functions

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a positive function such that $f \in L_1[a,b]$ and $a < b$. If f is k-β-convex function on $[a,b]$, then the following inequalities for k-fractional conformable integral operators hold*

$$f\left(\tfrac{a+b}{2}\right) \le \quad (1-\tfrac{1}{2^{\alpha}})^{-\frac{\beta}{k}}\left(\tfrac{1}{2}\right)^{\frac{p+q}{k}}\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{k(b-a)^{\frac{\alpha\beta}{k}}}\left(J_{(\frac{a+b}{2})^+,k}^{\alpha,\,\beta}f(b) + J_{(\frac{a+b}{2})^-,k}^{\alpha,\,\beta}f(a)\right)$$

$$\leq \quad (\tfrac{1}{2})^{\frac{p+q}{k}+1}(1-\tfrac{1}{2^{\alpha}})^{-\frac{\beta}{k}}\frac{\alpha\beta}{k}\,[f(a)+f(b)]\,W, \tag{3.1}$$

where

$$\begin{aligned} W = \quad & \Big((\tfrac{1}{2})^{\frac{p+\alpha\beta-k\alpha}{k}} B_k(p+\alpha\beta-k\alpha+k,k) \\ & \times {}_2\mathcal{F}_{1,k}\left((k-k\alpha-q,k),(p+\alpha\beta-k\alpha+k,k),(p+\alpha\beta-k\alpha+2k,k),\tfrac{1}{2k}\right) \\ & + (\tfrac{1}{2})^{\frac{q+\alpha\beta-k\alpha}{k}} B_k(q+\alpha\beta-k\alpha+k,k) \\ & \times {}_2\mathcal{F}_{1,k}\left((k-k\alpha-p,k),(q+\alpha\beta+k-k\alpha,k),(q+\alpha\beta-k\alpha+2k,k),\tfrac{1}{2k}\right)\Big), \end{aligned}$$

with $\mathrm{Re}(\beta)>0$, $k>\frac{1}{2}$, $0<\alpha\leq 1$ *and* $p,\, q>-k$.

Proof. Since f is k-β-convex function, we can write

$$f(\lambda x+(1-\lambda)y) \leq \tfrac{1}{k}\lambda^{\frac{p}{k}}(1-\lambda)^{\frac{q}{k}}f(x)+\tfrac{1}{k}\lambda^{\frac{q}{k}}(1-\lambda)^{\frac{p}{k}}f(y),$$

and

$$f((1-\lambda)x+\lambda y) \leq \tfrac{1}{k}(1-\lambda)^{\frac{p}{k}}\lambda^{\frac{q}{k}}f(x)+\tfrac{1}{k}(1-\lambda)^{\frac{q}{k}}\lambda^{\frac{p}{k}}f(y).$$

Let $\lambda=\frac{1}{2}$, then

$$f\left(\tfrac{x+y}{2}\right) \leq \tfrac{1}{k}(\tfrac{1}{2})^{\frac{p+q}{k}}\,[f(x)+f(y)]\,.$$

Taking $x=\frac{t}{2}a+\frac{2-t}{2}b$ and $y=\frac{2-t}{2}a+\frac{t}{2}b$ for $t\in[0,1]$, clearly $x,y\in[a,b]$ and we have

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{k}(\tfrac{1}{2})^{\frac{p+q}{k}}\left[f(\tfrac{t}{2}a+\tfrac{2-t}{2}b)+f(\tfrac{2-t}{2}a+\tfrac{t}{2}b)\right].$$

Multiplying both sides of the above inequality by $\left[1-(\frac{2-t}{2})^{\alpha}\right]^{\frac{\beta}{k}-1}(\frac{2-t}{2})^{\alpha-1}$, and then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned} & f\left(\tfrac{a+b}{2}\right)\int_0^1 \left[1-(\tfrac{2-t}{2})^{\alpha}\right]^{\frac{\beta}{k}-1}(\tfrac{2-t}{2})^{\alpha-1}dt \\ \leq \quad & \tfrac{1}{k}(\tfrac{1}{2})^{\frac{p+q}{k}}\left\{\int_0^1 \left[1-(\tfrac{2-t}{2})^{\alpha}\right]^{\frac{\beta}{k}-1}(\tfrac{2-t}{2})^{\alpha-1}f(\tfrac{t}{2}a+\tfrac{2-t}{2}b)dt\right. \\ & \left.+\int_0^1 \left[1-(\tfrac{2-t}{2})^{\alpha}\right]^{\frac{\beta}{k}-1}(\tfrac{2-t}{2})^{\alpha-1}f(\tfrac{2-t}{2}a+\tfrac{t}{2}b)dt\right\}. \end{aligned}$$

We can restate the above inequality as follows

$$f\left(\tfrac{a+b}{2}\right)I_1 \leq \tfrac{1}{k}(\tfrac{1}{2})^{\frac{p+q}{k}}\left(I_2+I_3\right), \tag{3.2}$$

where

$$\begin{aligned} I_1 = \quad & \int_0^1 \left[1-(\tfrac{2-t}{2})^{\alpha}\right]^{\frac{\beta}{k}-1}(\tfrac{2-t}{2})^{\alpha-1}dt, \\ I_2 = \quad & \int_0^1 \left[1-(\tfrac{2-t}{2})^{\alpha}\right]^{\frac{\beta}{k}-1}(\tfrac{2-t}{2})^{\alpha-1}f(\tfrac{t}{2}a+\tfrac{2-t}{2}b)dt \end{aligned}$$

and

$$I_3=\int_0^1\left[1-(\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}-1}(\tfrac{2-t}{2})^{\alpha-1}f(\tfrac{2-t}{2}a+\tfrac{t}{2}b)dt.$$

Making the change of variable $u=(\frac{2-t}{2})^\alpha$, I_1 gives

$$\begin{aligned} I_1= & \int_0^1\left[1-(\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}-1}(\tfrac{2-t}{2})^{\alpha-1}dt \\ = & \tfrac{2}{\alpha}\int_{\frac{1}{2^\alpha}}^1(1-u)^{\frac{\beta}{k}-1}du=\tfrac{2k}{\alpha\beta}(1-\tfrac{1}{2^\alpha})^{\frac{\beta}{k}}. \end{aligned} \tag{3.3}$$

Now, let $v=\frac{t}{2}a+\frac{2-t}{2}b$, then we obtain

$$\begin{aligned} I_2= & \int_0^1\left[1-(\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}-1}(\tfrac{2-t}{2})^{\alpha-1}f(\tfrac{t}{2}a+\tfrac{2-t}{2}b)dt \\ = & \frac{2\alpha^{\frac{\beta}{k}-1}}{(b-a)^{\frac{\alpha\beta}{k}}}\int_{\frac{a+b}{2}}^b\left[\tfrac{(b-a)^\alpha-(v-a)^\alpha}{\alpha}\right]^{\frac{\beta}{k}-1}(v-a)^{\alpha-1}f(v)dv \\ = & \frac{2\alpha^{\frac{\beta}{k}-1}k\Gamma_k(\beta)}{(b-a)^{\frac{\alpha\beta}{k}}}J^{\alpha,\,\beta}_{(\frac{a+b}{2})^+,k}f(b). \end{aligned} \tag{3.4}$$

Similarly we get

$$\begin{aligned} I_3= & \int_0^1\left[1-(\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}-1}(\tfrac{2-t}{2})^{\alpha-1}f(\tfrac{2-t}{2}a+\tfrac{t}{2}b)dt \\ = & \frac{2\alpha^{\frac{\beta}{k}-1}k\Gamma_k(\beta)}{(b-a)^{\frac{\alpha\beta}{k}}}J^{\alpha,\,\beta}_{(\frac{a+b}{2})^-,k}f(a). \end{aligned} \tag{3.5}$$

Substituting (3.3)-(3.5) in (3.2), we obtain

$$\tfrac{2k}{\alpha\beta}(1-\tfrac{1}{2^\alpha})^{\frac{\beta}{k}}f\left(\tfrac{a+b}{2}\right)\le\tfrac{1}{k}(\tfrac{1}{2})^{\frac{p+q}{k}}\frac{2\alpha^{\frac{\beta}{k}-1}k\Gamma_k(\beta)}{(b-a)^{\frac{\alpha\beta}{k}}}\left(J^{\alpha,\,\beta}_{(\frac{a+b}{2})^+,k}f(b)+J^{\alpha,\,\beta}_{(\frac{a+b}{2})^-,k}f(a)\right).$$

Now, we will proof of the second inequality in (3.1).

From the k-β-convexity of f , we have

$$f(\tfrac{t}{2}a+\tfrac{2-t}{2}b)\le\tfrac{1}{k}(\tfrac{t}{2})^{\frac{p}{k}}(\tfrac{2-t}{2})^{\frac{q}{k}}f(a)+\tfrac{1}{k}(\tfrac{t}{2})^{\frac{q}{k}}(\tfrac{2-t}{2})^{\frac{p}{k}}f(b),$$

and

$$f(\tfrac{2-t}{2}a+\tfrac{t}{2}b)\le\tfrac{1}{k}(\tfrac{2-t}{2})^{\frac{p}{k}}(\tfrac{t}{2})^{\frac{q}{k}}f(a)+\tfrac{1}{k}(\tfrac{2-t}{2})^{\frac{q}{k}}(\tfrac{t}{2})^{\frac{p}{k}}f(b).$$

By adding the above inequalities, we obtain

$$\begin{aligned} & f(\tfrac{t}{2}a+\tfrac{2-t}{2}b)+f(\tfrac{2-t}{2}a+\tfrac{t}{2}b) \\ \le & \tfrac{1}{k}\left[(\tfrac{t}{2})^{\frac{p}{k}}(\tfrac{2-t}{2})^{\frac{q}{k}}+(\tfrac{t}{2})^{\frac{q}{k}}(\tfrac{2-t}{2})^{\frac{p}{k}}\right][f(a)+f(b)]. \end{aligned}$$

Multiplying both sides of the above inequality by $\left[1-(\frac{2-t}{2})^{\alpha}\right]^{\frac{\beta}{k}-1}(\frac{2-t}{2})^{\alpha-1}$, and then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned}
&\tfrac{1}{k}\int_0^1 \left[1-(\tfrac{2-t}{2})^{\alpha}\right]^{\frac{\beta}{k}-1}(\tfrac{2-t}{2})^{\alpha-1}\left[f(\tfrac{t}{2}a+\tfrac{2-t}{2}b)+f(\tfrac{2-t}{2}a+\tfrac{t}{2}b)\right]dt\\
\leq\ & [f(a)+f(b)]\,\tfrac{1}{k}\int_0^1 \left[1-(\tfrac{2-t}{2})^{\alpha}\right]^{\frac{\beta}{k}-1}(\tfrac{2-t}{2})^{\alpha-1}\left[(\tfrac{t}{2})^{\frac{p}{k}}(\tfrac{2-t}{2})^{\frac{q}{k}}+(\tfrac{t}{2})^{\frac{q}{k}}(\tfrac{2-t}{2})^{\frac{p}{k}}\right]dt\\
=\ & [f(a)+f(b)]\left(\tfrac{1}{k}\int_0^1 \left[1-(\tfrac{2-t}{2})^{\alpha}\right]^{\frac{\beta}{k}-1}(1-\tfrac{t}{2})^{\frac{q}{k}+\alpha-1}(\tfrac{t}{2})^{\frac{p}{k}}dt\right.\\
&\left.+\tfrac{1}{k}\int_0^1 \left[1-(\tfrac{2-t}{2})^{\alpha}\right]^{\frac{\beta}{k}-1}(1-\tfrac{t}{2})^{\frac{p}{k}+\alpha-1}(\tfrac{t}{2})^{\frac{q}{k}}dt\right). \qquad (3.6)
\end{aligned}$$

From Lemma 2.1, we have

$$1-(\tfrac{2-t}{2})^{\alpha}=1^{\alpha}-(\tfrac{2-t}{2})^{\alpha}\leq(\tfrac{t}{2})^{\alpha}. \qquad (3.7)$$

Combining (3.4)-(3.7), we obtain

$$\begin{aligned}
&\tfrac{1}{k}\int_0^1 \left[1-(\tfrac{2-t}{2})^{\alpha}\right]^{\frac{\beta}{k}-1}(\tfrac{2-t}{2})^{\alpha-1}\left[f(\tfrac{t}{2}a+\tfrac{2-t}{2}b)+f(\tfrac{2-t}{2}a+\tfrac{t}{2}b)\right]dt\\
\leq\ & [f(a)+f(b)]\left(\tfrac{1}{k}\int_0^1 \left(\tfrac{t}{2}\right)^{\frac{\alpha\beta}{k}-\alpha}(1-\tfrac{t}{2})^{\frac{q}{k}+\alpha-1}(\tfrac{t}{2})^{\frac{p}{k}}dt\right.\\
&\left.+\tfrac{1}{k}\int_0^1 \left(\tfrac{t}{2}\right)^{\frac{\alpha\beta}{k}-\alpha}(1-\tfrac{t}{2})^{\frac{p}{k}+\alpha-1}(\tfrac{t}{2})^{\frac{q}{k}}dt\right)\\
=\ & [f(a)+f(b)]\left(\tfrac{1}{k}\int_0^1 \left(\tfrac{t}{2}\right)^{\frac{p+\alpha\beta}{k}-\alpha}(1-\tfrac{t}{2})^{\frac{q}{k}+\alpha-1}dt\right.\\
&\left.+\tfrac{1}{k}\int_0^1 \left(\tfrac{t}{2}\right)^{\frac{q+\alpha\beta}{k}-\alpha}(1-\tfrac{t}{2})^{\frac{p}{k}+\alpha-1}dt\right)\\
=\ & [f(a)+f(b)]\left(\left(\tfrac{1}{2}\right)^{\frac{p+\alpha\beta-k\alpha}{k}}\tfrac{1}{k}\int_0^1 t^{\frac{p+\alpha\beta-k\alpha+k}{k}-1}(1-k\tfrac{1}{2k}t)^{-\frac{k-q-k\alpha}{k}}dt\right.\\
&\left.+\left(\tfrac{1}{2}\right)^{\frac{q+\alpha\beta-k\alpha}{k}}\tfrac{1}{k}\int_0^1 t^{\frac{q+\alpha\beta-k\alpha+k}{k}-1}(1-k\tfrac{1}{2k}t)^{-\frac{k-p-k\alpha}{k}}dt\right)\\
=\ & [f(a)+f(b)]\left(\left(\tfrac{1}{2}\right)^{\frac{p+\alpha\beta-k\alpha}{k}}B_k(p+\alpha\beta-k\alpha+k,k)\right.\\
&\times\,{}_2\mathcal{F}_{1,k}\left((k-k\alpha-q,k),(p+\alpha\beta-k\alpha+k,k),(p+\alpha\beta-k\alpha+2k,k),\tfrac{1}{2k}\right)\\
&+\left(\tfrac{1}{2}\right)^{\frac{q+\alpha\beta-k\alpha}{k}}B_k(q+\alpha\beta+k-k\alpha,k)\\
&\left.\times\,{}_2\mathcal{F}_{1,k}\left((k-k\alpha-p,k),(q+\alpha\beta+k-k\alpha,k),(q+\alpha\beta-k\alpha+2k,k),\tfrac{1}{2k}\right)\right).
\end{aligned}$$

So, we have

$$\tfrac{2k}{\alpha\beta}(1-\tfrac{1}{2^{\alpha}})^{\frac{\beta}{k}}f\left(\tfrac{a+b}{2}\right)\leq \quad \tfrac{1}{k}(\tfrac{1}{2})^{\frac{p+q}{k}}\frac{2\alpha^{\frac{\beta}{k}-1}k\Gamma_k(\beta)}{(b-a)^{\frac{\alpha\beta}{k}}}\left(J^{\alpha,\,\beta}_{(\frac{a+b}{2})^+,k}f(b)+J^{\alpha,\,\beta}_{(\frac{a+b}{2})^-,k}f(a)\right)$$

$$\leq \left(\tfrac{1}{2}\right)^{\frac{p+q}{k}}\left[f(a)+f(b)\right]W.$$

Rewriting the above inequality, we obtain (3.1). The proof is completed. □

Corollary 3.1. *In* (3.1)*, if we take $k=1$ we obtain the following inequalities for β-convex functions involving the integrals in* (2.10) *and* (2.11)*:*

$$\begin{aligned} f\left(\tfrac{a+b}{2}\right) &\leq \left(1-\tfrac{1}{2^{\alpha}}\right)^{-\beta}\left(\tfrac{1}{2}\right)^{p+q}\frac{\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left(J^{\alpha,\,\beta}_{(\frac{a+b}{2})^{+}}f(b)+J^{\alpha,\,\beta}_{(\frac{a+b}{2})^{-}}f(a)\right)\\ &\leq \left(\tfrac{1}{2}\right)^{p+q+1}\left(1-\tfrac{1}{2^{\alpha}}\right)^{-\beta}\alpha\beta\left[f(a)+f(b)\right]W_1, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} W_1 = & \frac{{}_2\mathcal{F}_1\left(1-\alpha-q,p+\alpha\beta-\alpha+1,p+\alpha\beta-\alpha+2,\frac{1}{2}\right)}{2^{p+\alpha\beta-\alpha}(p+\alpha\beta-\alpha+1)}\\ &+\frac{{}_2\mathcal{F}_1\left(1-\alpha-p,q+\alpha\beta+1-\alpha,q+\alpha\beta-\alpha+2,\frac{1}{2}\right)}{2^{q+\alpha\beta-\alpha}(q+\alpha\beta-\alpha+1)}, \end{aligned}$$

with $\mathrm{Re}(\beta)>0$, $0<\alpha\leq 1$ *and* $p,\ q>-1$.

Corollary 3.2. *In* (3.1)*, if we take $\alpha=1$ we obtain the following inequalities for k-β-convex functions involving the integrals in* (2.8) *and* (2.9)*:*

$$\begin{aligned} f\left(\tfrac{a+b}{2}\right) &\leq \left(\tfrac{1}{2}\right)^{\frac{p+q-\beta}{k}}\frac{\Gamma_k(\beta+k)}{k(b-a)^{\frac{\beta}{k}}}\left(I^{\beta}_{(\frac{a+b}{2})^{+},k}f(b)+I^{\beta}_{(\frac{a+b}{2})^{-},k}f(a)\right)\\ &\leq \left(\tfrac{1}{2}\right)^{\frac{p+q-\beta}{k}+1}\frac{\beta}{k}\left[f(a)+f(b)\right]W_2, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} W_2 = & \frac{B_k(p+\beta,k)\,{}_2\mathcal{F}_{1,k}\left((-q,k),(p+\beta,k),(p+\beta+k,k),\frac{1}{2k}\right)}{2^{\frac{p+\beta-k}{k}}}\\ &+\frac{B_k(q+\beta,k)\,{}_2\mathcal{F}_{1,k}\left((-p,k),(q+\beta,k),(q+\alpha\beta+k,k),\frac{1}{2k}\right)}{2^{\frac{q+\beta-k}{k}}}, \end{aligned}$$

with $\mathrm{Re}(\beta)>0$, $k>\frac{1}{2}$, *and* $p,\ q>-k$.

Our next result is to establish some k-fractional conformable midpoint inequalities for functions whose first derivatives are k-β-convex, for this, we need the following lemma.

Lemma 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L_1[a,b]$, then the following equality for k- fractional conformable integral operators holds:*

$$\begin{aligned} &\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)\left(1-\frac{1}{2^{\alpha}}\right)^{\frac{-\beta}{k}}}{2(b-a)^{\frac{\alpha\beta}{k}}}\left(J^{\alpha,\,\beta}_{(\frac{a+b}{2})^{+},k}f(b)+J^{\alpha,\,\beta}_{(\frac{a+b}{2})^{-},k}f(a)\right)-f\left(\tfrac{a+b}{2}\right)\\ =\ & \tfrac{b-a}{4}\left(1-\tfrac{1}{2^{\alpha}}\right)^{\frac{-\beta}{k}}\left\{\int_0^1\left[1-\left(\tfrac{2-t}{2}\right)^{\alpha}\right]^{\frac{\beta}{k}}\left(f'\left(\tfrac{t}{2}a+\tfrac{2-t}{2}b\right)-f'\left(\tfrac{2-t}{2}a+\tfrac{t}{2}b\right)\right)dt\right\}, \end{aligned} \tag{3.10}$$

with $\mathrm{Re}(\beta)>0$, $k>0$ *and* $0<\alpha\leq 1$.

Proof. Integrating by parts the right side of (3.10) and then making the change of variable $u = \frac{t}{2}a + \frac{2-t}{2}b$, we obtain

$$\begin{aligned} I_1 = & \int_0^1 \left[1 - (\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}} f'(\tfrac{t}{2}a + \tfrac{2-t}{2}b)dt \\ = & \tfrac{2}{a-b}\left[1 - (\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}} f(\tfrac{t}{2}a + \tfrac{2-t}{2}b)\Big|_0^1 \\ & -\tfrac{\alpha\beta}{k(a-b)}\int_0^1 (\tfrac{2-t}{2})^{\alpha-1}\left[1 - (\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}-1} f(\tfrac{t}{2}a + \tfrac{2-t}{2}b)dt \end{aligned}$$

$$\begin{aligned} (3.11) \qquad = & \tfrac{2}{a-b}(1-\tfrac{1}{2^\alpha})^{\frac{\beta}{k}} f\left(\tfrac{a+b}{2}\right) \\ & +\tfrac{2\alpha\beta}{k(b-a)^2}\int_{\frac{a+b}{2}}^b \left(\tfrac{u-a}{b-a}\right)^{\alpha-1}\left[\tfrac{(b-a)^\alpha-(u-a)^\alpha}{(b-a)^\alpha}\right]^{\frac{\beta}{k}-1} f(u)du \end{aligned}$$

$$(3.12) \qquad = \tfrac{-2}{b-a}(1-\tfrac{1}{2^\alpha})^{\frac{\beta}{k}} f(\tfrac{a+b}{2}) + \tfrac{2\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(b-a)^{\frac{\alpha\beta}{k}+1}} J^{\alpha,\,\beta}_{(\frac{a+b}{2})^+,k} f(b).$$

Similarly, we have

$$\begin{aligned} I_2 = & \int_0^1 \left[1 - (\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}} f'(\tfrac{2-t}{2}a + \tfrac{t}{2}b)dt \end{aligned}$$

$$(3.13) \qquad = \tfrac{2}{b-a}(1-\tfrac{1}{2^\alpha})^{\frac{\beta}{k}} f(\tfrac{a+b}{2}) - \tfrac{2\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(b-a)^{\frac{\alpha\beta}{k}+1}} J^{\alpha,\,\beta}_{(\frac{a+b}{2})^-,k} f(a).$$

By Subtracting (3.13) from (3.12), we get

$$\begin{aligned} (3.14) \qquad I_1 - I_2 = & \tfrac{-4}{b-a}(1-\tfrac{1}{2^\alpha})^{\frac{\beta}{k}} f(\tfrac{a+b}{2}) \\ & +\tfrac{2\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{(b-a)^{\frac{\alpha\beta}{k}+1}}\left(J^{\alpha,\,\beta}_{(\frac{a+b}{2})^+,k} f(b) + J^{\alpha,\,\beta}_{(\frac{a+b}{2})^-,k} f(a)\right). \end{aligned}$$

Multiplying both sides of (3.14) by $\frac{b-a}{4}(1-\frac{1}{2^\alpha})^{\frac{-\beta}{k}}$, we get the desired equality in (3.10). □

Remark 3.1. If we take $\alpha = 1$ in (3.10) we obtain (1.3) from Lemma 1.1.

Theorem 3.2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and $f' \in L_1[a,b]$. If $|f'|^\mu$ is k-β-convex function on $[a,b]$ for $\mu \geq 1$, then the following inequality for k-fractional conformable integral operators holds:*

$$\begin{aligned} & \left|\tfrac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)(1-\frac{1}{2^\alpha})^{\frac{-\beta}{k}}}{2(b-a)^{\frac{\alpha\beta}{k}}}\left(J^{\alpha,\,\beta}_{(\frac{a+b}{2})^+,k} f(b) + J^{\alpha,\,\beta}_{(\frac{a+b}{2})^-,k} f(a)\right) - f(\tfrac{a+b}{2})\right| \\ \leq & \tfrac{b-a}{4}(\tfrac{2^\alpha-1}{2^\alpha})^{\frac{-\beta}{k}}\left(\tfrac{2}{\alpha}\right)^{1-\frac{1}{\mu}}\left(B(\tfrac{1}{\alpha},\tfrac{\beta}{k}+1) - B_{\frac{1}{2^\alpha}}(\tfrac{1}{\alpha},\tfrac{\beta}{k}+1)\right)^{1-\frac{1}{\mu}}(|f'(a)| + |f'(b)|) \end{aligned}$$

$$\times\left\{\left(\frac{B_k(\alpha\beta+q+k,k)_2\mathcal{F}_{1,k}\left((-q,k),(\alpha\beta+q+k,k),(\alpha\beta+q+2k,k),\frac{1}{2k}\right)}{2^{\frac{\alpha\beta+q}{k}}}\right)^{\frac{1}{\mu}}\right.$$

$$(3.15)+\left.\left(\frac{B_k(\alpha\beta+p+k,k)_2\mathcal{F}_{1,k}\left((-p,k),(\alpha\beta+p+k,k),(\alpha\beta+p+2k,k),\frac{1}{2k}\right)}{2^{\frac{\alpha\beta+p}{k}}}\right)^{\frac{1}{\mu}}\right\},$$

with $\mathrm{Re}(\beta)>0$, $k>\frac{1}{2}$, $0<\alpha\le 1$ *and* $p,\ q>-k$ *and* $B_x(\alpha,\beta)=\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt$ *is the incomplete beta function.*

Proof. By using Lemma 3.1, and power mean inequality, we have

$$\begin{aligned}
&\left|\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)(1-\frac{1}{2^\alpha})^{\frac{-\beta}{k}}}{2(b-a)^{\frac{\alpha\beta}{k}}}\left(J^{\alpha,\,\beta}_{(\frac{a+b}{2})^+,k}f(b)+J^{\alpha,\,\beta}_{(\frac{a+b}{2})^-,k}f(a)\right)-f(\tfrac{a+b}{2})\right|\\
\le\ & \tfrac{b-a}{4}(1-\tfrac{1}{2^\alpha})^{\frac{-\beta}{k}}\left\{\int_0^1\left[1-(\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}}\left(\left|f'(\tfrac{t}{2}a+\tfrac{2-t}{2}b)\right|+\left|f'(\tfrac{2-t}{2}a+\tfrac{t}{2}b)\right|\right)dt\right\}\\
\le\ & \tfrac{b-a}{4}(1-\tfrac{1}{2^\alpha})^{\frac{-\beta}{k}}\left(\int_0^1\left[1-(\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}}dt\right)^{1-\frac{1}{\mu}}\\
&\times\left\{\left(\int_0^1\left[1-(\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}}\left|f'(\tfrac{t}{2}a+\tfrac{2-t}{2}b)\right|^\mu dt\right)^{\frac{1}{\mu}}\right.\\
&\left.+\left(\int_0^1\left[1-(\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}}\left|f'(\tfrac{2-t}{2}a+\tfrac{t}{2}b)\right|^\mu dt\right)^{\frac{1}{\mu}}\right\}\\
=\ & \tfrac{b-a}{4}(1-\tfrac{1}{2^\alpha})^{\frac{-\beta}{k}}\left(\tfrac{2}{\alpha}\right)^{1-\frac{1}{\mu}}\left(B(\tfrac{1}{\alpha},\tfrac{\beta}{k}+1)-B_{2^{\frac{1}{\alpha}}}(\tfrac{1}{\alpha},\tfrac{\beta}{k}+1)\right)^{1-\frac{1}{\mu}}\\
&\times\left\{\left(\int_0^1\left[1-(\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}}\left|f'(\tfrac{t}{2}a+\tfrac{2-t}{2}b)\right|^\mu dt\right)^{\frac{1}{\mu}}\right.\\
(3.16)\ &\left.+\left(\int_0^1\left[1-(\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}}\left|f'(\tfrac{2-t}{2}a+\tfrac{t}{2}b)\right|^\mu dt\right)^{\frac{1}{\mu}}\right\}.
\end{aligned}$$

Using (3.7), the k-β-convexity of $|f'|^\mu$, and the fact that $\int_0^1\left[1-(\frac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}}dt=\left(\frac{2}{\alpha}\right)\left(B(\frac{1}{\alpha},\frac{\beta}{k}+1)-B_{\frac{1}{2^\alpha}}(\frac{1}{\alpha},\frac{\beta}{k}+1)\right)$, in (3.16) we obtain

$$\begin{aligned}
&\left|\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)(1-\frac{1}{2^\alpha})^{\frac{-\beta}{k}}}{2(b-a)^{\frac{\alpha\beta}{k}}}\left(J^{\alpha,\,\beta}_{(\frac{a+b}{2})^+,k}f(b)+J^{\alpha,\,\beta}_{(\frac{a+b}{2})^-,k}f(a)\right)-f(\tfrac{a+b}{2})\right|\\
\le\ & \tfrac{b-a}{4}(1-\tfrac{1}{2^\alpha})^{\frac{-\beta}{k}}\left(\tfrac{2}{\alpha}\right)^{1-\frac{1}{\mu}}\left(B(\tfrac{1}{\alpha},\tfrac{\beta}{k}+1)-B_{\frac{1}{2^\alpha}}(\tfrac{1}{\alpha},\tfrac{\beta}{k}+1)\right)^{1-\frac{1}{\mu}}\\
&\times\left\{\left(\int_0^1(\tfrac{t}{2})^{\frac{\alpha\beta}{k}}\left(\tfrac{1}{k}\left(\tfrac{t}{2}\right)^{\frac{p}{k}}(1-\tfrac{t}{2})^{\frac{q}{k}}|f'(a)|^\mu+\tfrac{1}{k}\left(\tfrac{t}{2}\right)^{\frac{q}{k}}(1-\tfrac{t}{2})^{\frac{p}{k}}|f'(b)|^\mu\right)dt\right)^{\frac{1}{\mu}}\right.
\end{aligned}$$

$$+\left(\int_0^1 \left(\tfrac{t}{2}\right)^{\frac{\alpha\beta}{k}} \left(\tfrac{1}{k}\left(\tfrac{t}{2}\right)^{\frac{q}{k}}\left(1-\tfrac{t}{2}\right)^{\frac{p}{k}}|f'(a)|^{\mu}+\tfrac{1}{k}\left(\tfrac{t}{2}\right)^{\frac{p}{k}}\left(1-\tfrac{t}{2}\right)^{\frac{q}{k}}|f'(b)|^{\mu}\right)dt\right)^{\frac{1}{\mu}}\Bigg\}$$

$$= \quad \tfrac{b-a}{4}\left(1-\tfrac{1}{2^{\alpha}}\right)^{\frac{-\beta}{k}}\left(\tfrac{2}{\alpha}\right)^{1-\frac{1}{\mu}}\left(B\left(\tfrac{1}{\alpha},\tfrac{\beta}{k}+1\right)-B_{\frac{1}{2^{\alpha}}}\left(\tfrac{1}{\alpha},\tfrac{\beta}{k}+1\right)\right)^{1-\frac{1}{\mu}}$$

$$\times\Bigg\{\left(\frac{|f'(a)|^{\mu}}{k2^{\frac{\alpha\beta+p}{k}}}\int_0^1 t^{\frac{\alpha\beta+p+k}{k}-1}\left(1-k\tfrac{1}{2k}t\right)^{\frac{q}{k}}dt\right.$$

$$\left.+\frac{|f'(b)|^{\mu}}{k2^{\frac{\alpha\beta+q}{k}}}\int_0^1 t^{\frac{\alpha\beta+q+k}{k}-1}\left(1-k\tfrac{1}{2k}t\right)^{\frac{p}{k}}dt\right)^{\frac{1}{\mu}}$$

$$+\left(\frac{|f'(a)|^{\mu}}{k2^{\frac{\alpha\beta+q}{k}}}\int_0^1 t^{\frac{\alpha\beta+q+k}{k}-1}\left(1-\tfrac{t}{2}\right)^{\frac{p}{k}}dt+\frac{|f'(b)|^{\mu}}{k2^{\frac{\alpha\beta+p}{k}}}\int_0^1 t^{\frac{\alpha\beta+p+k}{k}-1}\left(1-\tfrac{t}{2}\right)^{\frac{q}{k}}dt\right)^{\frac{1}{\mu}}\Bigg\}$$

$$= \quad \tfrac{b-a}{4}\left(1-\tfrac{1}{2^{\alpha}}\right)^{\frac{-\beta}{k}}\left(\tfrac{2}{\alpha}\right)^{1-\frac{1}{\mu}}\left(B\left(\tfrac{1}{\alpha},\tfrac{\beta}{k}+1\right)-B_{\frac{1}{2^{\alpha}}}\left(\tfrac{1}{\alpha},\tfrac{\beta}{k}+1\right)\right)^{1-\frac{1}{\mu}}$$

$$\times\Bigg\{\left(\frac{B_k(\alpha\beta+q+k,k)\,{}_2\mathcal{F}_{1,k}\left((-q,k),(\alpha\beta+q+k,k),(\alpha\beta+q+2k,k),\frac{1}{2k}\right)}{2^{\frac{\alpha\beta+q}{k}}}|f'(a)|^{\mu}\right.$$

$$\left.+\frac{B_k(\alpha\beta+p+k,k)\,{}_2\mathcal{F}_{1,k}\left((-p,k),(\alpha\beta+p+k,k),(\alpha\beta+p+2k,k),\frac{1}{2k}\right)}{2^{\frac{\alpha\beta+p}{k}}}|f'(b)|^{\mu}\right)^{\frac{1}{\mu}}$$

$$\left(\frac{B_k(\alpha\beta+p+k,k)\,{}_2\mathcal{F}_{1,k}\left((-p,k),(\alpha\beta+p+k,k),(\alpha\beta+p+2k,k),\frac{1}{2k}\right)}{2^{\frac{\alpha\beta+p}{k}}}|f'(a)|^{\mu}\right.$$

$$\left.+\frac{B_k(\alpha\beta+q+k,k)\,{}_2\mathcal{F}_{1,k}\left((-q,k),(\alpha\beta+q+k,k),(\alpha\beta+q+2k,k),\frac{1}{2k}\right)}{2^{\frac{\alpha\beta+q}{k}}}|f'(b)|^{\mu}\right)^{\frac{1}{\mu}}\Bigg\}. \tag{3.17}$$

Using the following algebraic inequality $(a_1+a_2)^s \le a_1^s + a_2^s$, for $0 \le s < 1$ and $a_1, a_2 \ge 0$, and since $\mu > 1$, i.e. $0 < \frac{1}{\mu} < 1$, (3.17) gives

$$\left|\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)\left(1-\frac{1}{2^{\alpha}}\right)^{\frac{-\beta}{k}}}{2(b-a)^{\frac{\alpha\beta}{k}}}\left(J^{\alpha,\,\beta}_{\left(\frac{a+b}{2}\right)^+,k}f(b)+J^{\alpha,\,\beta}_{\left(\frac{a+b}{2}\right)^-,k}f(a)\right)-f\left(\tfrac{a+b}{2}\right)\right|$$

$$\le \quad \tfrac{b-a}{4}\left(\tfrac{2^{\alpha}-1}{2^{\alpha}}\right)^{\frac{-\beta}{k}}\left(\tfrac{2}{\alpha}\right)^{1-\frac{1}{\mu}}\left(B\left(\tfrac{1}{\alpha},\tfrac{\beta}{k}+1\right)-B_{\frac{1}{2^{\alpha}}}\left(\tfrac{1}{\alpha},\tfrac{\beta}{k}+1\right)\right)^{1-\frac{1}{\mu}}\left(|f'(a)|+|f'(b)|\right)$$

$$\times\Bigg\{\left(\frac{B_k(\alpha\beta+q+k,k)\,{}_2\mathcal{F}_{1,k}\left((-q,k),(\alpha\beta+q+k,k),(\alpha\beta+q+2k,k),\frac{1}{2k}\right)}{2^{\frac{\alpha\beta+q}{k}}}\right)^{\frac{1}{\mu}}$$

$$+\left(\frac{B_k(\alpha\beta+p+k,k)\,{}_2\mathcal{F}_{1,k}\left((-p,k),(\alpha\beta+p+k,k),(\alpha\beta+p+2k,k),\frac{1}{2k}\right)}{2^{\frac{\alpha\beta+p}{k}}}\right)^{\frac{1}{\mu}}\Bigg\}.$$

The proof is completed. $\square$

Corollary 3.3. *In* (3.15), *if we take $k=1$ we obtain the following inequalities for β-convex functions involving the integrals in* (2.10) *and* (2.11)*:*

$$\begin{aligned}
(3.18)\qquad & \left|\frac{(1-\frac{1}{2^\alpha})^{-\beta}\alpha^\beta\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left(J^{\alpha,\,\beta}_{(\frac{a+b}{2})^+}f(b)+J^{\alpha,\,\beta}_{(\frac{a+b}{2})^-}f(a)\right)-f\left(\tfrac{a+b}{2}\right)\right| \\
\le\qquad & \tfrac{b-a}{4}\left(\tfrac{2^\alpha}{2^\alpha-1}\right)^\beta\left(\tfrac{2}{\alpha}\right)^{1-\frac{1}{\mu}}\left(|f'(a)|+|f'(b)|\right) \\
& \times\left(B(\tfrac{1}{\alpha},\beta+1)-B_{\frac{1}{2^\alpha}}(\tfrac{1}{\alpha},\beta+1)\right)^{1-\frac{1}{\mu}} \\
& \times\left\{\left(\frac{{}_2\mathcal{F}_1\left(-q,\alpha\beta+q+1,\alpha\beta+q+2,\frac{1}{2}\right)}{2^{\alpha\beta+q}(\alpha\beta+q+1)}\right)^{\frac{1}{\mu}}+\left(\frac{{}_2\mathcal{F}_1\left(p,\alpha\beta+p+1,\alpha\beta+p+2,\frac{1}{2}\right)}{2^{\alpha\beta+p}(\alpha\beta+p+1)}\right)^{\frac{1}{\mu}}\right\},
\end{aligned}$$

with $\operatorname{Re}(\beta)>0$, $0<\alpha\le 1$ *and* $p,\,q>-1$.

Corollary 3.4. *In* (3.18), *if we take $\alpha=1$ we obtain the following inequalities for k-β-convex functions involving the integrals in* (2.8) *and* (2.9)*:*

$$\begin{aligned}
& \left|\frac{2^{\frac{\beta}{k}-1}\Gamma_k(\beta+k)}{(b-a)^{\frac{\beta}{k}}}\left(I^{\beta}_{(\frac{a+b}{2})^+,k}f(b)+I^{\beta}_{(\frac{a+b}{2})^-,k}f(a)\right)-f\left(\tfrac{a+b}{2}\right)\right| \\
\le\quad & \frac{b-a}{2^{2-\frac{\beta}{k\mu}}}\left(\frac{k}{\beta+k}\right)^{1-\frac{1}{\mu}}\left(2^{\frac{\beta}{k}+1}-1\right)^{1-\frac{1}{\mu}}\left(|f'(a)|+|f'(b)|\right) \\
& \times\left\{\left(\frac{B_k(\beta+q+k,k)\,{}_2\mathcal{F}_{1,k}\left((-q,k),(\beta+q+k,k),(\beta+q+2k,k),\frac{1}{2k}\right)}{2^{\frac{\beta+q}{k}}}\right)^{\frac{1}{\mu}}\right. \\
(3.19)\qquad & \left.+\left(\frac{B_k(\beta+p+k,k)\,{}_2\mathcal{F}_{1,k}\left((-p,k),(\beta+p+k,k),(\beta+p+2k,k),\frac{1}{2k}\right)}{2^{\frac{\beta+p}{k}}}\right)^{\frac{1}{\mu}}\right\},
\end{aligned}$$

with $\operatorname{Re}(\beta)>0$, $k>\frac{1}{2}$ *and* $p,\,q>-k$.

Remark 3.2. If we choose $\alpha=p=1$ and $q=0$ in (3.15) we obtain the inequality in (1.4) (for $k=1$).

Theorem 3.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $f'\in L_1[a,b]$. If $|f'|^\mu$ is k-β-convex function on $[a,b]$ for $\mu>1$, then the following inequality for k-fractional conformable integral operators holds:*

$$\begin{aligned}
& \left|\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)(1-\frac{1}{2^\alpha})^{\frac{-\beta}{k}}}{2(b-a)^{\frac{\alpha\beta}{k}}}\left(J^{\alpha,\,\beta}_{(\frac{a+b}{2})^+,k}f(b)+J^{\alpha,\,\beta}_{(\frac{a+b}{2})^-,k}f(a)\right)-f\left(\tfrac{a+b}{2}\right)\right| \\
\le\quad & \tfrac{b-a}{4}\left(1-\tfrac{1}{2^\alpha}\right)^{\frac{-\beta}{k}}\left(\tfrac{2}{\alpha}\right)^{\frac{1}{\lambda}}\left(B(\tfrac{1}{\alpha},\tfrac{\lambda\beta}{k}+1)-B_{\frac{1}{2^\alpha}}(\tfrac{1}{\alpha},\tfrac{\lambda\beta}{k}+1)\right)^{\frac{1}{\lambda}}\left(|f'(a)|+|f'(b)|\right) \\
& \times\left(\left(\frac{B_k(p+k,k)\,{}_2\mathcal{F}_{1,k}\left((-q,k),(p+k,k),(p+2k,k),\frac{1}{2k}\right)}{2^{\frac{p}{k}}}\right)^{\frac{1}{\mu}}\right.
\end{aligned}$$

$$+\left(\frac{B_k(q+k,k)\,{}_2\mathcal{F}_{1,k}\left((-p,k),(q+k,k),(q+2k,k),\frac{1}{2k}\right)}{2^{\frac{q}{k}}}\right)^{\frac{1}{\mu}}\Bigg), \tag{3.20}$$

with $\mathrm{Re}(\beta)>0, k>\frac{1}{2}, 0<\alpha\le 1,$ p, $q>-k$, *and* $\frac{1}{\mu}+\frac{1}{\lambda}=1$.

Proof. From Lemma 3.1, properties of modulus, and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)(1-\frac{1}{2^\alpha})^{\frac{-\beta}{k}}}{2(b-a)^{\frac{\alpha\beta}{k}}}\left(J^{\alpha,\,\beta}_{(\frac{a+b}{2})^+,k}f(b)+J^{\alpha,\,\beta}_{(\frac{a+b}{2})^-,k}f(a)\right)-f\left(\tfrac{a+b}{2}\right)\right|\\
\le\ & \tfrac{b-a}{4}(1-\tfrac{1}{2^\alpha})^{\frac{-\beta}{k}}\\
&\times\left\{\int_0^1\left[1-(\tfrac{2-t}{2})^\alpha\right]^{\frac{\beta}{k}}\left(\left|f'(\tfrac{t}{2}a+\tfrac{2-t}{2}b)\right|+\left|f'(\tfrac{2-t}{2}a+\tfrac{t}{2}b)\right|\right)dt\right\}\\
\le\ & \tfrac{b-a}{4}(1-\tfrac{1}{2^\alpha})^{\frac{-\beta}{k}}\left(\int_0^1\left[1-(\tfrac{2-t}{2})^\alpha\right]^{\frac{\lambda\beta}{k}}dt\right)^{\frac{1}{\lambda}}\\
&\times\left\{\left(\int_0^1\left|f'(\tfrac{t}{2}a+\tfrac{2-t}{2}b)\right|^\mu dt\right)^{\frac{1}{\mu}}+\left(\int_0^1\left|f'(\tfrac{2-t}{2}a+\tfrac{t}{2}b)\right|^\mu dt\right)^{\frac{1}{\mu}}\right\}\\
=\ & \tfrac{b-a}{4}(1-\tfrac{1}{2^\alpha})^{\frac{-\beta}{k}}\left(\tfrac{2}{\alpha}\right)^{\frac{1}{\lambda}}\left[B(\tfrac{1}{\alpha},\tfrac{\lambda\beta}{k}+1)-B_{\frac{1}{2^\alpha}}(\tfrac{1}{\alpha},\tfrac{\lambda\beta}{k}+1)\right]^{\frac{1}{\lambda}}\\
&\times\left\{\left(\int_0^1\left|f'(\tfrac{t}{2}a+\tfrac{2-t}{2}b)\right|^\mu dt\right)^{\frac{1}{\mu}}+\left(\int_0^1\left|f'(\tfrac{2-t}{2}a+\tfrac{t}{2}b)\right|^\mu dt\right)^{\frac{1}{\mu}}\right\}.
\end{aligned}$$

The k-β-convexity of $|f'|^\mu$ gives

$$\begin{aligned}
&\left|\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)(1-\frac{1}{2^\alpha})^{\frac{-\beta}{k}}}{2(b-a)^{\frac{\alpha\beta}{k}}}\left(J^{\alpha,\,\beta}_{(\frac{a+b}{2})^+,k}f(b)+J^{\alpha,\,\beta}_{(\frac{a+b}{2})^-,k}f(a)\right)-f\left(\tfrac{a+b}{2}\right)\right|\\
\le\ & \tfrac{b-a}{4}(1-\tfrac{1}{2^\alpha})^{\frac{-\beta}{k}}\left(\tfrac{2}{\alpha}\right)^{\frac{1}{\lambda}}\left(B(\tfrac{1}{\alpha},\tfrac{\lambda\beta}{k}+1)-B_{\frac{1}{2^\alpha}}(\tfrac{1}{\alpha},\tfrac{\lambda\beta}{k}+1)\right)^{\frac{1}{\lambda}}\\
&\times\left\{\left(\frac{|f'(a)|^\mu}{k}\int_0^1(\tfrac{t}{2})^{\frac{p}{k}}(1-\tfrac{1}{2}t)^{\frac{q}{k}}dt+\frac{|f'(b)|^\mu}{k}\int_0^1(\tfrac{t}{2})^{\frac{q}{k}}(1-\tfrac{1}{2}t)^{\frac{p}{k}}dt\right)^{\frac{1}{\mu}}\right.\\
&\left.+\left(\frac{|f'(a)|^\mu}{k}\int_0^1(\tfrac{t}{2})^{\frac{q}{k}}(1-\tfrac{1}{2}t)^{\frac{p}{k}}dt+\frac{|f'(b)|^\mu}{k}\int_0^1(\tfrac{t}{2})^{\frac{p}{k}}(1-\tfrac{1}{2}t)^{\frac{q}{k}}dt\right)^{\frac{1}{\mu}}\right\}\\
\le\ & \tfrac{b-a}{4}(1-\tfrac{1}{2^\alpha})^{\frac{-\beta}{k}}\left(\tfrac{2}{\alpha}\right)^{\frac{1}{\lambda}}\left(B(\tfrac{1}{\alpha},\tfrac{\lambda\beta}{k}+1)-B_{\frac{1}{2^\alpha}}(\tfrac{1}{\alpha},\tfrac{\lambda\beta}{k}+1)\right)^{\frac{1}{\lambda}}(|f'(a)|+|f'(b)|)\\
&\times\left(\left(\frac{1}{k}\int_0^1(\tfrac{t}{2})^{\frac{p}{k}}(1-\tfrac{1}{2}t)^{\frac{q}{k}}dt\right)^{\frac{1}{\mu}}+\left(\frac{1}{k}\int_0^1(\tfrac{t}{2})^{\frac{q}{k}}(1-\tfrac{1}{2}t)^{\frac{p}{k}}dt\right)^{\frac{1}{\mu}}\right)\\
=\ & \tfrac{b-a}{4}(1-\tfrac{1}{2^\alpha})^{\frac{-\beta}{k}}\left(\tfrac{2}{\alpha}\right)^{\frac{1}{\lambda}}\left[B(\tfrac{1}{\alpha},\tfrac{\lambda\beta}{k}+1)-B_{\frac{1}{2^\alpha}}(\tfrac{1}{\alpha},\tfrac{\lambda\beta}{k}+1)\right]^{\frac{1}{\lambda}}(|f'(a)|+|f'(b)|)\\
&\times\left(\left(\frac{1}{2^{\frac{p}{k}}k}\int_0^1 t^{\frac{p+k}{k}-1}(1-k\tfrac{1}{2k}t)^{-\frac{(-q)}{k}}dt\right)^{\frac{1}{\mu}}\right.
\end{aligned}$$

$$+\left(\frac{1}{2^{\frac{q}{k}}k}\int_0^1 t^{\frac{q+k}{k}-1}(1-k\tfrac{1}{2k}t)^{-\frac{(-p)}{k}}dt\right)^{\frac{1}{\mu}}\Bigg)$$

$$= \quad \tfrac{b-a}{4}(1-\tfrac{1}{2^\alpha})^{\frac{-\beta}{k}}\left(\tfrac{2}{\alpha}\right)^{\frac{1}{\lambda}}\left[B(\tfrac{1}{\alpha},\tfrac{\lambda\beta}{k}+1)-B_{\frac{1}{2^\alpha}}(\tfrac{1}{\alpha},\tfrac{\lambda\beta}{k}+1)\right]^{\frac{1}{\lambda}}(|f'(a)|+|f'(b)|)$$

$$\times\left(\left(\frac{B_k(p+k,k)_2\mathcal{F}_{1,k}\left((-q,k),(p+k,k),(p+2k,k),\frac{1}{2k}\right)}{2^{\frac{p}{k}}}\right)^{\frac{1}{\mu}}\right.$$

$$\left.+\left(\frac{B_k(q+k,k)_2\mathcal{F}_{1,k}\left((-p,k),(q+k,k),(q+2k,k),\frac{1}{2k}\right)}{2^{\frac{q}{k}}}\right)^{\frac{1}{\mu}}\right).$$

Here we used again the fact that $(a_1+a_2)^{\frac{1}{\mu}} \le a_1^{\frac{1}{\mu}}+a_2^{\frac{1}{\mu}}$, for $0<\frac{1}{\mu}<1$ and $a_1, a_2 \ge 0$. □

In ordre to establish the k-fractional conformal midpoint type inequalities for twice differentiable and k-β-convex functions, we need the following lemma.

Lemma 3.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° (I° is the interior of I) such that $f'' \in L_1[a,b]$ where $a, b \in I^\circ$, and $a < \frac{m}{n}b$, $n, m \in \mathbb{N}^*$. Then the following equality holds:*

$$\frac{2^{\frac{\alpha}{k}-2}\Gamma_k(\alpha)}{(mb-na)^{\frac{\alpha}{k}-1}}\left(I^{\alpha-k}_{(\frac{na+mb}{2})^+,k}f(mb)+I^{\alpha-k}_{(\frac{na+mb}{2})^-,k}f(na)\right)-f\left(\tfrac{na+mb}{2}\right)$$

$$= \frac{k(mb-na)^2}{\alpha 2^{2-\frac{\alpha}{k}}}\left(\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}}f''(nat+m(1-t)b)dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1(1-t)^{\frac{\alpha}{k}}f''(nat+m(1-t)b)dt\right). \tag{3.21}$$

Proof. Integrating by parts twice, we have

$$\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}}f''(nat+m(1-t)b)dt$$

$$= -\frac{1}{2^{\frac{\alpha}{k}}(mb-na)}f'\left(\tfrac{na+mb}{2}\right)-\frac{\alpha}{2^{\frac{\alpha}{k}-1}k(mb-na)^2}f\left(\tfrac{na+mb}{2}\right)$$

$$+\frac{\alpha(\alpha-k)}{k^2(mb-na)^2}\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}-2}f(nat+m(1-t)b)dt. \tag{3.22}$$

Similarly, we obtain

$$\int_{\frac{1}{2}}^1(1-t)^{\frac{\alpha}{k}}f''(nat+m(1-t)b)dt$$

$$= \frac{1}{2^{\frac{\alpha}{k}}(mb-na)}f'\left(\tfrac{na+mb}{2}\right)-\frac{\alpha}{2^{\frac{\alpha}{k}-1}k(mb-na)^2}f\left(\tfrac{na+mb}{2}\right)$$

$$+\frac{\alpha(\alpha-k)}{k^2(mb-na)^2}\int_{\frac{1}{2}}^{1}(1-t)^{\frac{\alpha}{k}-2}f(nat+m(1-t)b)dt. \tag{3.23}$$

Now, making the change of variable $u = nat + m(1-t)b$ in both integrals in (3.22) and (3.23), and then summing the resulting equalities, we obtain

$$\begin{aligned}
&\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}} f''(nat+m(1-t)b)dt + \int_{\frac{1}{2}}^{1}(1-t)^{\frac{\alpha}{k}} f''(nat+m(1-t)b)dt \\
= \quad & -\frac{2\alpha}{2^{\frac{\alpha}{k}-1}k(mb-na)^2} f\left(\frac{na+mb}{2}\right) \\
& +\frac{\alpha(\alpha-k)\Gamma_k(\alpha-k)}{k(mb-na)^{\frac{\alpha}{k}+1}}\left(I^{\alpha-k}_{(\frac{na+mb}{2})^+,k}f(mb)+I^{\alpha-k}_{(\frac{na+mb}{2})^-,k}f(na)\right). \qquad (3.24)
\end{aligned}$$

Multiplying both sides of equality (3.24) by $\frac{k(mb-na)^2}{\alpha 2^{2-\frac{\alpha}{k}}}$ and using the property $(\alpha-k)\Gamma_k(\alpha-k) = \Gamma_k(\alpha)$, we complete the proof of Lemma 3.2. □

Remark 3.3. If we take $n = k = 1$ in (3.21) we obtain Lemma 2.1 in [10].

Theorem 3.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° (I° is the interior of I) such that $f'' \in L_1[a,b]$ where $a, b \in I^\circ$, and $a < \frac{mb}{n}$, n, $m \in \mathbb{N}^*$. If $|f''|^\mu$ is k-β-convex function on $[a,b]$ for $\mu \geq 1$. Then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-2}\Gamma_k(\alpha)}{(mb-na)^{\frac{\alpha}{k}-1}}\left(I^{\alpha-k}_{(\frac{na+mb}{2})^+,k}f(mb)+I^{\alpha-k}_{(\frac{na+mb}{2})^-,k}f(na)\right)-f\left(\frac{na+mb}{2}\right)\right| \qquad (3.25)\\
\leq \quad & \frac{k(mb-na)^2}{\alpha 2^{3-\frac{\alpha}{\mu k}-\frac{1}{\mu}}}\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{\mu}} \\
& \times\left\{\left(\left(\frac{1}{k}B_{\frac{1}{2}}(\frac{\alpha+p+k}{k},\frac{q+k}{k})\right)^{\frac{1}{\mu}}+\left(\frac{B\left(\frac{p+k}{k},\frac{\alpha+q+k}{k}\right)-B_{\frac{1}{2}}\left(\frac{p+k}{k},\frac{\alpha+q+k}{k}\right)}{k}\right)^{\frac{1}{\mu}}\right)|f''(na)|\right. \\
& \left.+\left(\left(\frac{1}{k}B_{\frac{1}{2}}(\frac{\alpha+q+k}{k},\frac{p+k}{k})\right)^{\frac{1}{\mu}}+\left(\frac{B\left(\frac{q+k}{k},\frac{\alpha+p+k}{k}\right)-B_{\frac{1}{2}}\left(\frac{q+k}{k},\frac{\alpha+p+k}{k}\right)}{k}\right)^{\frac{1}{\mu}}\right)|f''(mb)|\right\}
\end{aligned}$$

for $k > 0$, $\alpha > k$ and p, $q > -k$.

Proof. By using Lemma 3.2, and power mean inequality, we obtain

$$\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-2}\Gamma_k(\alpha)}{(mb-na)^{\frac{\alpha}{k}-1}}\left(I^{\alpha-k}_{(\frac{na+mb}{2})^+,k}f(mb)+I^{\alpha-k}_{(\frac{na+mb}{2})^-,k}f(na)\right)-f\left(\frac{na+mb}{2}\right)\right| \\
\leq \quad & \frac{k(mb-na)^2}{\alpha 2^{2-\frac{\alpha}{k}}}\left(\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}}\left|f''(nat+m(1-t)b)\right|dt+\int_{\frac{1}{2}}^{1}(1-t)^{\frac{\alpha}{k}}\left|f''(nat+m(1-t)b)\right|dt\right) \\
\leq \quad & \frac{k(mb-na)^2}{\alpha 2^{2-\frac{\alpha}{k}}}\left\{\left(\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{\mu}}\left(\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}}\left|f''(nat+m(1-t)b)\right|^\mu dt\right)^{\frac{1}{\mu}}\right.
\end{aligned}$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-t)^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{\mu}}\left(\int_{\frac{1}{2}}^{1}(1-t)^{\frac{\alpha}{k}}\left|f''(nat+m(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\Bigg\}$$

$$= \frac{k(mb-na)^2}{\alpha 2^{3-\frac{\alpha}{\mu k}-\frac{1}{\mu}}}\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{\mu}}\left\{\left(\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}}\left|f''(nat+m(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}(1-t)^{\frac{\alpha}{k}}\left|f''(nat+m(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right\}. \tag{3.26}$$

The k-β-convexity of $|f''|^{\mu}$ gives

$$\left(\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}}\left|f''(nat+m(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}}$$

$$\leq \left(\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}}\left[\frac{1}{k}t^{\frac{p}{k}}(1-t)^{\frac{q}{k}}\left|f''(na)\right|^{\mu}+\frac{1}{k}t^{\frac{q}{k}}(1-t)^{\frac{p}{k}}\left|f''(mb)\right|^{\mu}\right]dt\right)^{\frac{1}{\mu}}$$

$$= \left(\frac{1}{k}\left|f''(na)\right|^{\mu}\int_0^{\frac{1}{2}}t^{\frac{\alpha+p}{k}}(1-t)^{\frac{q}{k}}dt+\frac{1}{k}\left|f''(mb)\right|^{\mu}\int_0^{\frac{1}{2}}t^{\frac{\alpha+q}{k}}(1-t)^{\frac{p}{k}}dt\right)^{\frac{1}{\mu}}.$$

$$\leq \left(\frac{1}{k}B_{\frac{1}{2}}\left(\frac{\alpha+p+k}{k},\frac{q+k}{k}\right)\left|f''(na)\right|^{\mu}+\frac{1}{k}B_{\frac{1}{2}}\left(\frac{\alpha+q+k}{k},\frac{p+k}{k}\right)\left|f''(mb)\right|^{\mu}\right)^{\frac{1}{\mu}}.$$

Using the following algebraic inequality:

$$(a_1+a_2)^s \leq a_1^s + a_2^s, \text{ for } 0 \leq s < 1 \text{ and } a_1, a_2 \geq 0,$$

we get

$$\left(\int_0^{\frac{1}{2}}t^{\frac{\alpha}{k}}\left|f''(nat+m(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}} \tag{3.27}$$

$$\leq \left(\frac{1}{k}B_{\frac{1}{2}}\left(\frac{\alpha+p+k}{k},\frac{q+k}{k}\right)\right)^{\frac{1}{\mu}}\left|f''(na)\right|+\left(\frac{1}{k}B_{\frac{1}{2}}\left(\frac{\alpha+q+k}{k},\frac{p+k}{k}\right)\right)^{\frac{1}{\mu}}\left|f''(mb)\right|.$$

Similarly, we obtain

$$\left(\int_{\frac{1}{2}}^{1}(1-t)^{\frac{\alpha}{k}}\left|f''(nat+m(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}}$$

$$\leq \left(\frac{1}{k}\left(B\left(\frac{p+k}{k},\frac{\alpha+q+k}{k}\right)-B_{\frac{1}{2}}\left(\frac{p+k}{k},\frac{\alpha+q+k}{k}\right)\right)\left|f''(na)\right|^{\mu}\right.$$

$$\left.+\frac{1}{k}\left(B\left(\frac{q+k}{k},\frac{\alpha+p+k}{k}\right)-B_{\frac{1}{2}}\left(\frac{q+k}{k},\frac{\alpha+p+k}{k}\right)\right)\left|f''(mb)\right|^{\mu}\right)^{\frac{1}{\mu}}$$

$$\leq \left(\frac{1}{k}\left(B\left(\frac{p+k}{k},\frac{\alpha+q+k}{k}\right)-B_{\frac{1}{2}}\left(\frac{p+k}{k},\frac{\alpha+q+k}{k}\right)\right)\right)^{\frac{1}{\mu}}|f''(na)|$$

$$+\left(\frac{1}{k}\left(B\left(\frac{q+k}{k},\frac{\alpha+p+k}{k}\right)-B_{\frac{1}{2}}\left(\frac{q+k}{k},\frac{\alpha+p+k}{k}\right)\right)\right)^{\frac{1}{\mu}}|f''(mb)|. \tag{3.28}$$

Using (3.27) and (3.28) in (3.26), we get the desired inequality in (3.25). □

Remark 3.4. Theorem 3.4 will be reduced to Theorem 2.1 from [10], if we choose $k = n = \mu = 1, p = s$ and $q = 0$. Moreover, if we take $\alpha = 2$ and $m = s = 1$, we obtain Prposition from [38].

Corollary 3.5. *In* (3.25)*, if we choose* $k = 1$*,* $p = -s \in (-1, 0], q = 0$ *and for* $\alpha > -1$*, we obtain the following inequality for s-Godunova-Levin function involving the integrals in* (2.6) *and* (2.7)*:*

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(mb-na)^{\alpha-1}}\left(I^{\alpha-1}_{(\frac{na+mb}{2})^+}f(mb)+I^{\alpha-1}_{(\frac{na+mb}{2})^-}f(na)\right)-f\left(\frac{na+mb}{2}\right)\right|$$

$$\leq \frac{(mb-na)^2}{\alpha 2^{3-\frac{\alpha}{\mu k}-\frac{1}{\mu}}}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{\mu}}\times\left\{\left(\left(\left(\frac{1}{(\alpha+1-s)2^{\alpha+1-s}}\right)^{\frac{1}{\mu}}\right.\right.\right.$$

$$\left.\left.+\left(B(1-s,\alpha+1)-B_{\frac{1}{2}}(1-s,\alpha+1)\right)^{\frac{1}{\mu}}\right)\left|f''(na)\right|\right.$$

$$\left.+\left(\left(B_{\frac{1}{2}}(\alpha+1,1-s)\right)^{\frac{1}{\mu}}+\left(\frac{2^{\alpha+1-s}-1}{(\alpha+1-s)2^{\alpha+1-s}}\right)^{\frac{1}{\mu}}\right)\left|f''(mb)\right|\right\}.$$

Theorem 3.5. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a twice differentiable mapping on* I° *(*I° *is the interior of* I*) such that* $f'' \in L_1[a, b]$ *where* $a, b \in I^\circ$*, and* $a < \frac{mb}{n}$*,* $n, m \in \mathbb{N}^*$*. If* $|f''|^\mu$ *is k-β-convex function on* $[a, b]$ *for* $\mu > 1$*. Then the following inequality holds:*

$$\left|\frac{2^{\frac{\alpha}{k}-2}\Gamma_k(\alpha)}{(mb-na)^{\frac{\alpha}{k}-1}}\left(I^{\alpha-k}_{(\frac{na+mb}{2})^+,k}f(mb)+I^{\alpha-k}_{(\frac{na+mb}{2})^-,k}f(na)\right)-f\left(\frac{na+mb}{2}\right)\right|$$

$$\leq \frac{k(mb-na)^2}{\alpha 2^{2+\frac{1}{\lambda}}}\left(\frac{k}{\lambda\alpha+k}\right)^{\frac{1}{\lambda}}$$

$$\times\left(\left(\left(\frac{1}{k}B_{\frac{1}{2}}\left(\frac{p+k}{k},\frac{q+k}{k}\right)\right)^{\frac{1}{\mu}}+\left(\frac{B(\frac{p+k}{k},\frac{q+k}{k})-B_{\frac{1}{2}}(\frac{p+k}{k},\frac{q+k}{k})}{k}\right)^{\frac{1}{\mu}}\right)|f''(na)|\right.$$

$$\left.+\left(\left(\frac{1}{k}B_{\frac{1}{2}}\left(\frac{q+k}{k},\frac{p+k}{k}\right)\right)^{\frac{1}{\mu}}+\left(\frac{B(\frac{q+k}{k},\frac{p+k}{k})-B_{\frac{1}{2}}(\frac{q+k}{k},\frac{p+k}{k})}{k}\right)^{\frac{1}{\mu}}\right)|f''(mb)|\right), \tag{3.29}$$

with $k > 0$*,* $\alpha > k$*,* $p, q > -k$ *and* $\frac{1}{\mu} + \frac{1}{\lambda} = 1$*.*

Proof. From Lemma 3.2, and Hölder's inequality, we have

$$\left|\frac{2^{\frac{\alpha}{k}-2}\Gamma_k(\alpha)}{(mb-na)^{\frac{\alpha}{k}-1}}\left(I^{\alpha-k}_{(\frac{na+mb}{2})^+,k}f(mb)+I^{\alpha-k}_{(\frac{na+mb}{2})^-,k}f(na)\right)-f\left(\frac{na+mb}{2}\right)\right|$$

$$
\begin{aligned}
&\leq \frac{k(mb-na)^2}{\alpha 2^{2-\frac{\alpha}{k}}}\left(\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}}\left|f''(nat+m(1-t)b)\right|dt\right. \\
&\quad \left.+\int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}}\left|f''(nat+m(1-t)b)\right|dt\right) \\
&\leq \frac{k(mb-na)^2}{\alpha 2^{2-\frac{\alpha}{k}}}\left\{\left(\int_0^{\frac{1}{2}} t^{\frac{\lambda\alpha}{k}}dt\right)^{\frac{1}{\lambda}}\left(\int_0^{\frac{1}{2}}\left|f''(nat+m(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right. \\
&\quad \left.+\left(\int_{\frac{1}{2}}^1 (1-t)^{\frac{\lambda\alpha}{k}}dt\right)^{\frac{1}{\lambda}}\left(\int_{\frac{1}{2}}^1\left|f''(nat+m(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right\} \\
&= \frac{k(mb-na)^2}{\alpha 2^{2+\frac{1}{\lambda}}}\left(\frac{k}{\lambda\alpha+k}\right)^{\frac{1}{\lambda}}\left(\left(\int_0^{\frac{1}{2}}\left|f''(nat+m(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right. \\
&\quad \left.+\left(\int_{\frac{1}{2}}^1\left|f''(nat+m(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}}\right). \qquad (3.30)
\end{aligned}
$$

The k-β-convexity of $|f''|^{\mu}$ gives

$$
\begin{aligned}
&\left(\int_0^{\frac{1}{2}}\left|f''(nat+m(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}} \\
&\leq \left(\frac{1}{k}\int_0^{\frac{1}{2}} t^{\frac{p}{k}}(1-t)^{\frac{q}{k}}\left|f''(na)\right|^{\mu}dt+\frac{1}{k}\int_0^{\frac{1}{2}} t^{\frac{q}{k}}(1-t)^{\frac{p}{k}}\left|f''(mb)\right|^{\mu}dt\right)^{\frac{1}{\mu}} \\
&= \left(\frac{1}{k}B_{\frac{1}{2}}\left(\frac{p+k}{k},\frac{q+k}{k}\right)\left|f''(na)\right|^{\mu}+\frac{1}{k}B_{\frac{1}{2}}\left(\frac{q+k}{k},\frac{p+k}{k}\right)\left|f''(mb)\right|^{\mu}\right)^{\frac{1}{\mu}} \\
&\leq \left(\frac{1}{k}B_{\frac{1}{2}}\left(\frac{p+k}{k},\frac{q+k}{k}\right)\right)^{\frac{1}{\mu}}\left|f''(na)\right|+\left(\frac{1}{k}B_{\frac{1}{2}}\left(\frac{q+k}{k},\frac{p+k}{k}\right)\right)^{\frac{1}{\mu}}\left|f''(mb)\right|. \qquad (3.31)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&\left(\int_{\frac{1}{2}}^1\left|f''(nat+m(1-t)b)\right|^{\mu}dt\right)^{\frac{1}{\mu}} \\
&\leq \left(\frac{1}{k}\int_{\frac{1}{2}}^1 t^{\frac{p}{k}}(1-t)^{\frac{q}{k}}\left|f''(na)\right|^{\mu}dt+\frac{1}{k}\int_{\frac{1}{2}}^1 t^{\frac{q}{k}}(1-t)^{\frac{p}{k}}\left|f''(mb)\right|^{\mu}dt\right)^{\frac{1}{\mu}} \\
&\leq \left(\frac{B\left(\frac{p+k}{k},\frac{q+k}{k}\right)-B_{\frac{1}{2}}\left(\frac{p+k}{k},\frac{q+k}{k}\right)}{k}\right)^{\frac{1}{\mu}}\left|f''(na)\right| \\
&\quad +\left(\frac{B\left(\frac{q+k}{k},\frac{p+k}{k}\right)-B_{\frac{1}{2}}\left(\frac{q+k}{k},\frac{p+k}{k}\right)}{k}\right)^{\frac{1}{\mu}}\left|f''(mb)\right|. \qquad (3.32)
\end{aligned}
$$

Substituting (3.31) and (3.32) in (3.30), we obtain the desired inequality in (3.29). □

Corollary 3.6. *In (3.29), if we put $k = 1$ and $p = q = 0$, we get the following inequality for P-function involving the integrals in (2.6) and (2.7):*

$$\left|\frac{2^{\alpha-2}\Gamma(\alpha)}{(mb-na)^{\alpha-1}}\left(I^{\alpha-1}_{\left(\frac{na+mb}{2}\right)^+}f(mb)+I^{\alpha-1}_{\left(\frac{na+mb}{2}\right)^-}f(na)\right)-f\left(\frac{na+mb}{2}\right)\right|$$

$$(3.33)\quad \leq \frac{(mb-na)^2}{4\alpha}\left(\frac{1}{\lambda\alpha+1}\right)^{\frac{1}{\lambda}}\left(|f''(na)|+|f''(mb)|\right),$$

with $\alpha > 1$ and $\frac{1}{\mu}+\frac{1}{\lambda}=1$.

4. Applications to special means

For arbitrary real numbers a, b we have:

The Arithmetic mean: $A(a,b) = \frac{a+b}{2}$.

The p-Logarithmic mean: $L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a, b > 0, a \neq b$ and $p \in \mathbb{R}\setminus\{0,-1\}$.

Proposition 4.1. *Let $a, b \in \mathbb{R}$ with $0 < a < b$ and $n > 2$, then we have*

$$\left|L_n^n(a,b)-A^n(a,b)\right| \leq \frac{n(b-a)\sqrt{2}}{12}\left(a^{n-1}+b^{n-1}\right).$$

Proof. The assertion follows from Theorem 3.3 with $\alpha = \beta = k = 1$ and $\mu = 2$, applied to the function $f(x) = x^n$. □

Proposition 4.2. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$\left|L_3^3(na,mb)-A^3(na,mb)\right| \leq \frac{3\sqrt{5}(mb-na)^2}{20}(na+mb).$$

Proof. The assertion follows from Corollary 3.6 with $\alpha = \mu = 2$, applied to the function $f(x) = t^3$. □

5. Conclusion

The main results of the paper can be summarized as follows:

1. Hermite-Hadamard inequality for the class of k-β-convex functions involving the generalized k-fractional conformable integral operators is established.
2. Two new fractional identities regarding midpoint type inequalities are established.
3. Some k-fractional conformable midpoint type inequalities for functions whose first derivatives are k-β-convex are discussed.
4. Some k-fractional conformable midpoint type inequalities for functions whose second derivatives are k-β-convex are given.
5. Various special cases have been studied in details.

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