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WEIGHTED STATISTICAL CONVERGENCE OF ORDER α OF DIFFERENCE SEQUENCES

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Abstract. Study of difference sequences is a recent development in the summability theory. Sometimes a situation may arise that we have a sequence at hand and we are interested in sequences formed by its successive differences and in the structure of these new sequences. Studies on difference sequences were introduced in the 1980s and after that many mathematicians studied these kind of sequences and obtained some generalized difference sequence spaces. In this study, we generalize the concepts of weighted statistical convergence and weighted $\overline{[N_p]}-$summability of real (or complex) numbers sequences to the concepts of Δ^m-weighted statistical convergence of order α and weighted $\left[\overline{N_p}^{\alpha}\right](\Delta^m, r)-$summability of order α by using generalized difference operator Δ^m and examine the relationships between Δ^m-weighted statistical convergence of order α and weighted $\left[\overline{N_p}^{\alpha}\right](\Delta^m, r)-$summability of order α. Our results are more general than the corresponding results in the existing literature.
Keywords: statistical convergence, difference sequences, weighted summability.

1. Introduction, Definitions and Preliminaries

Statistical convergence was introduced by Fast [16] and Steinhaus [34] independently in the same year 1951. Though the notion was firstly handled as a summa-

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bility method by Schoenberg [30]. Salat [28] researched some topological properties of statistical convergence for sequences of real numbers. Fridy [17] defined the concept of statistical Cauchiness and showed that it is equivalent to statistical convergence. He also dealt with some Tauberian theorems. Connor [13] proved that a strongly r-Cesaro summable sequence for $0 < r < \infty$ is statistically convergent and the converse holds for bounded sequences. Recently several generalizations and applications of this concept have been investigated by various authors namely ([3, 4, 5, 6, 7, 8, 11, 14, 20, 21, 22, 26, 31, 32, 33, 35]).

Let $\mathbb{N}$ be the set of all natural numbers, $K \subseteq \mathbb{N}$ and $K(n) = \{k \leq n : k \in K\}$. The natural density of K is defined by $\delta(K) = \lim_n \frac{1}{n} |K(n)|$ if $\lim_n \frac{1}{n} |K(n)|$ exists. The vertical bars indicate the number of the elements in enclosed set.

The number sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$ the set $K(\varepsilon) = \{k \leq n : |x_k - L| \geq \varepsilon\}$ has natural density zero.

Weighted statistical convergence was first defined by Karakaya and Chishti [24] and the concept was modified by Mursaleen et al. [27]. Recently Ghosal [18] revised the definition of weighted statistical convergence as follows.

Let $p = (p_n)$ be a sequence of real numbers such that $\liminf p_n > 0$ and $P_n = p_1 + p_2 + p_3 + ... + p_n$ for all $n \in \mathbb{N}$. A sequence $x = (x_n)$ is said to be weighted statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{P_n} |\{k \leq P_n : p_k |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $S_{\overline{N}} - \lim x = L$. By $S_{\overline{N}}$, we denote the set of all weighted statistically convergent sequences.

Let w be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\|_\infty = \sup |x_k|$, where $k \in \mathbb{N}$. Also by bs, cs, ℓ_1 and ℓ_r ; we denote the spaces of *all bounded*, *convergent*, *absolutely convergent series*, and $r-$*absolutely convergent series*, respectively.

Study of difference sequence spaces is quite new in summability theory. The difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ were first introduced by Kızmaz [25], as the domain of forward difference matrix Δ^F, transforming a sequence $x = (x_k)$ to the difference sequence $\Delta^F x = (x_k - x_{k+1})$, in the classical spaces ℓ_∞, c and c_0. This concept was generalized by Et and Çolak [15]. Afterwards Et et al. ([19],[29]) studied it in order to mainly generalize statistical convergence with respect to Δ^m difference operator as follows

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\}$$

where X is any sequence space, $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$ and so $\Delta^m x_k = \sum_{v=0}^m (-1)^v \binom{m}{v} x_{k+v}$.

If $x \in \Delta^m(X)$ then there exists one and only one $y = (y_k) \in X$ such that

$y_k = \Delta^m x_k$ and

$$x_k = \sum_{v=1}^{k-m} (-1)^m \binom{k-v-1}{m-1},$$
$$y_v = \sum_{v=1}^{k} (-1)^m \binom{k+m-v-1}{m-1} y_{v-m},$$
$$y_{1-m} = y_{2-m} = \ldots = y_0 = 0$$

for sufficiently large k, for instance $k > 2m$. Lately, a great deal of work on generalizations of difference sequence spaces has been achieved by many researchers. Further information can be found in ([1],[2],[9],[10],[12],[23]).

The main goal of this work is to examine the relation between weighted statistical convergence of order α and weighted $\overline{[N_p]}-$summability of order α of difference sequences. Also, we have investigated some properties related to these concepts.

2. Main Results

In this section we give the main results of this article. Now we begin with two new definitions.

Definition 2.1. Let $p = (p_n)$ be a sequence of real numbers such that $\liminf p_n > 0$ and $P_n = p_1 + p_2 + p_3 + ... + p_n$ for all $n \in \mathbb{N}$. A sequence $x = (x_n)$ is said to be Δ^m-weighted statistically convergent of order α $(0 < \alpha \leq 1)$ (or $S^{\alpha}_{\overline{N_p}}(\Delta^m)-$convergent) to L, if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{P_n^{\alpha}} \left|\{k \leq P_n : p_k \left|\Delta^m x_k - L\right| \geq \varepsilon\}\right| = 0.$$

In this case we write $S^{\alpha}_{\overline{N_p}}(\Delta^m) - \lim x = L$ or $x_k \to L\left(S^{\alpha}_{\overline{N_p}}(\Delta^m)\right)$. We denote the set of all Δ^m-weighted statistically convergent sequences of order α by $S^{\alpha}_{\overline{N_p}}(\Delta^m)$.

Definition 2.2. Let $p = (p_n)$ be a sequence of nonnegative real numbers such that $p_1 > 0$ and $P_n = \sum_{k=1}^{n} p_k \to \infty$ as $n \to \infty$, $r > 0$ be a real number. A sequence $x = (x_n)$ is said to be Δ^m-weighted $\overline{[N_p]}-$summable of order α $(0 < \alpha \leq 1)$ (or $\left[\overline{N_p}^{\alpha}\right](\Delta^m, r)-$summable) to L, if

$$\lim_{n\to\infty} \frac{1}{P_n^{\alpha}} \sum_{k=1}^{n} p_k \left|\Delta^m x_k - L\right|^r = 0$$

and we write $x_k \to L\left(\left[\overline{N_p}^{\alpha}\right](\Delta^m, r)\right)$ or $\left[\overline{N_p}^{\alpha}\right](\Delta^m, r) - \lim x = L$. We denote the set of all Δ^m-weighted $\overline{[N_p]}-$summable sequences of order α by $\left[\overline{N_p}^{\alpha}\right](\Delta^m, r)$.

Remark 2.1. In view of Definition 2.1 and Definition 2.2 are clear that

i) If $m = 0$ and $\alpha = 1$, then Δ^m-weighted statistical convergence of order α coincides with weighted statisticall convergence given by Karakaya and Chishti [24] and Mursaleen et al. [27].

ii) If $m = 0$ and $\alpha = 1$, then Δ^m-weighted $\overline{[N_p]}-$summablility of order α coincides with weighted $\overline{[N_p]}-$summablility given by Karakaya and Chishti [24] and Mursaleen et al. [27].

iii) If $m = 0$, then Δ^m-weighted statistical convergence of order α coincides with weighted statisticall convergence of order α given by Ghosal [18].

iv) If $m = 0$, then Δ^m-weighted $\overline{[N_p]}-$summablility of order α coincides with weighted $\overline{[N_p]}-$summablility of order α given by Ghosal [18].

Theorem 2.1. *Let* $S^{\alpha}_{\overline{N_p}}(\Delta^m) - \lim x = L_1$ *and* $S^{\alpha}_{\overline{N_p}}(\Delta^m) - \lim y = L_2$. *Then*

i) $S^{\alpha}_{\overline{N_p}}(\Delta^m) - \lim (x + y) = L_1 + L_2$

ii) $S^{\alpha}_{\overline{N_p}}(\Delta^m) - \lim cx = cL_1, c \in \mathbb{R}$.

In addition, $\left[\overline{N_p}^{\alpha}\right](\Delta^m, r)$ *is also a linear space.*

Theorem 2.2. *Let* $\alpha \in (0, 1]$ *and* $1 \leq r < \infty$ *be a positive real number. The sequence space* $\left[\overline{N_p}^{\alpha}\right](\Delta^m, r)$ *is a Banach space normed by*

$$\|x\|_P = \sum_{k=1}^{m} |x_k| + \sup_n \left(\frac{1}{P_n^{\alpha}} \sum_{k=1}^{n} p_k |\Delta^m x_k|^r \right)^{\frac{1}{r}}.$$

Proof. It is a routine verification that $\left[\overline{N_p}^{\alpha}\right](\Delta^m, r)$ is a normed space with the above norm. Let (x^s) be a Cauchy sequence such that $x^s = (x_k^s)_k \in \left[\overline{N_p}^{\alpha}\right](\Delta^m, r)$ for each $s \in \mathbb{N}$. Then we have

$$\left\|x^s - x^t\right\|_P = \sum_{k=1}^{m} \left|x_k^s - x_k^t\right| + \sup_n \left(\frac{1}{P_n^{\alpha}} \sum_{k=1}^{n} p_k \left|\Delta^m \left(x_k^s - x_k^t\right)\right|^r \right)^{\frac{1}{r}} \to 0$$

as $s, t \to \infty$ and so that

$$\sum_{k=1}^{m} \left|x_k^s - x_k^t\right| \to 0$$

and

$$\left(\frac{1}{P_n^{\alpha}} \sum_{k=1}^{n} p_k \left|\Delta^m \left(x_k^s - x_k^t\right)\right|^r \right)^{\frac{1}{r}} \to 0,$$

for all $n \in \mathbb{N}$ as $s, t \to \infty$. Hence we get $|x_k^s - x_k^t| \to 0$ as $s, t \to \infty$, and for all $k \in \mathbb{N}$. This implies that the sequence (x_k^s) is a Cauchy in $\mathbb{R}$ for each $k \in \mathbb{N}$.

Also, it is convergent, because $\mathbb{R}$ is complete. Assume that $\lim_s (x_k^s) = x_k$ for each $k \in \mathbb{N}$. Since (x_k^s) is a Cauchy sequence, then for every $\varepsilon > 0$ there exists a number $n_0 = n_0(\varepsilon)$ such that

$$\left\| x^s - x^t \right\|_P < \varepsilon$$

for all $s, t \geq n_0$. Hence for all $n \in \mathbb{N}$ and all $s, t \geq n_0$ we get

$$\sum_{k=1}^{m} \left| x_k^s - x_k^t \right| < \varepsilon$$

and

$$\frac{1}{P_n^\alpha} \sum_{k=1}^{n} p_k \left| \sum_{v=0}^{m} (-1)^v \binom{m}{v} \left(x_k^s - x_k^t \right) \right|^r < \varepsilon^r < \varepsilon_1.$$

Taking limit as $t \to \infty$ in the last inequalities, we get

$$\lim_t \sum_{k=1}^{m} \left| x_k^s - x_k^t \right| = \sum_{k=1}^{m} \left| x_k^s - x_k \right| < \varepsilon$$

and

$$\begin{aligned} & \lim_t \frac{1}{P_n^\alpha} \sum_{k=1}^{n} p_k \left| \sum_{v=0}^{m} (-1)^v \binom{m}{v} \left(x_k^s - x_k^t \right) \right|^r \\ = \; & \frac{1}{P_n^\alpha} \sum_{k=1}^{n} p_k \left| \sum_{v=0}^{m} (-1)^v \binom{m}{v} \left(x_k^s - x_k \right) \right|^r \\ < \; & \varepsilon_1 \end{aligned}$$

for $s \geq n_0$ and all $n \in \mathbb{N}$. This implies that

$$\left\| x^s - x \right\|_P < \varepsilon + \varepsilon_1,$$

and hence $x^s \to x \left(\left[\overline{N_p}^\alpha \right] (\Delta^m, r) \right)$ as $s \to \infty$. Since

$$\begin{aligned} & \frac{1}{P_n^\alpha} \sum_{k=1}^{n} p_k \left| \Delta^m x_k - L \right|^r \\ \leq \; & 2^r \left\{ \frac{1}{P_n^\alpha} \sum_{k=1}^{n} p_k \left| \Delta^m x_k^n - L \right|^r + \frac{1}{P_n^\alpha} \sum_{k=1}^{n} p_k \left| \Delta^m \left(x_k^{n_0} - x_k \right) \right|^r \right\} \end{aligned}$$

we get $x \in \left[\overline{N_p}^\alpha \right] (\Delta^m, r)$. $\square$

The proof of each of the following results is straightforward, so we choose to state these results without proof.

Theorem 2.3. *(i) If a sequence $x=(x_k)$ is Δ^m-weighted statistical convergence of order α, then $S^{\alpha}_{\overline{N_p}}(\Delta^m)-$limit is unique.*

(ii) If a sequence $x=(x_k)$ is Δ^m-weighted $\overline{[N_p]}-$summable of order α, then $\left[\overline{N_p}^{\alpha}\right](\Delta^m,r)-$limit is unique.

Theorem 2.4. *Let $p=(p_n)$ be a sequence of real numbers such that $\liminf p_n > 0$ and x be a $\left[\overline{N_p}^{\alpha}\right](\Delta^m,r)-$summable sequence to L. If $0<r<1$ and $0 \leq |\Delta^m x_k - L| < 1$, then x is $S^{\alpha}_{\overline{N_p}}(\Delta^m)-$statistically convergent to L.*

Proof. Since $x=(x_k)$ is $\left[\overline{N_p}^{\alpha}\right](\Delta^m,r)-$summable to L we have

$$\frac{1}{P_n^{\alpha}}\sum_{k=1}^{n} p_k\left|\Delta^m x_k - L\right|^r = 0.$$

If $0<r<1$ and $0 \leq |\Delta^m x_k - L| < 1$, then we can write

$$p_k\left|\Delta^m x_k - L\right|^r \geq p_k\left|\Delta^m x_k - L\right|.$$

So we have

$$\begin{aligned}\sum_{k=1}^{n} p_k\left|\Delta^m x_k - L\right|^r &\geq \sum_{k=1}^{n} p_k\left|\Delta^m x_k - L\right| \\ &\geq \sum_{k=1}^{[P_n]} p_k\left|\Delta^m x_k - L\right| \\ &\geq \left|\left\{k \leq P_n : p_k\left|\Delta^m x_k - L\right| \geq \varepsilon\right\}\right|\varepsilon\end{aligned}$$

and so that

$$\frac{1}{P_n^{\alpha}}\left|\left\{k \leq P_n : p_k\left|\Delta^m x_k - L\right| \geq \varepsilon\right\}\right|\varepsilon \leq \frac{1}{P_n^{\alpha}}\sum_{k=1}^{n} p_k\left|\Delta^m x_k - L\right|^r \to 0.$$

This means that $x=(x_k)$ is $S^{\alpha}_{\overline{N_p}}(\Delta^m)-$statistically convergent to L. □

Theorem 2.5. *Let $p=(p_n)$ be a sequence of real numbers such that $\liminf p_n > 0$ and x be a $\left[\overline{N_p}^{\alpha}\right](\Delta^m,r)-$summable sequence to L. If $1 \leq r < \infty$ and $1 \leq |\Delta^m x_k - L| < \infty$, then x is $S^{\alpha}_{\overline{N_p}}(\Delta^m)-$statistically convergent to L.*

Proof. Proof is similar to that of Theorem 2.4. □

Theorem 2.6. *Let x be a $S_{\overline{N_p}}(\Delta^m)-$statistically convergent sequence and $p_k|\Delta^m x_k - L| \leq M$. If the following assertions hold, then x is $\left[\overline{N_p}\right](\Delta^m,r)-$summable sequence to L.*

i) $0<r<1$ and $1 \leq M < \infty$,

ii) $1 \leq r < \infty$ and $0 \leq M < 1$.

Proof. Suppose that $x = (x_k)$ is a $S_{\overline{N}_p}(\Delta^m)$ −statistically convergent sequence to L. Then for every $\varepsilon > 0$ we have $\delta_{\bar{N}}(K(\varepsilon)) = 0$, where $K(\varepsilon) = \{k \in \mathbb{N} : p_k |\Delta^m x_k - L| \geq \varepsilon\}$. Write $K_{P_n}(\varepsilon) = \{k \leq P_n : p_k |\Delta^m x_k - L| \geq \varepsilon\}$. Since $p_k |\Delta^m x_k - L| \leq M \ (k = 1, 2, ...)$ we have

$$\begin{aligned} & \frac{1}{P_n} \sum_{k=1}^{n} p_k |\Delta^m x_k - L|^r \\ = & \frac{1}{P_n} \sum_{k=1, k \notin K_{P_n}(\varepsilon)}^{n} p_k |\Delta^m x_k - L|^r + \frac{1}{P_n} \sum_{k=1, k \in K_{P_n}(\varepsilon)}^{n} p_k |\Delta^m x_k - L|^r \\ \leq & \varepsilon + M \frac{|K_{P_n}(\varepsilon)|}{P_n} \to 0. \end{aligned}$$

Hence $x_k \to L\left(\left[N_p\right](\Delta^m, r)\right)$. □

Theorem 2.7. *Let* $\lim_{n\to\infty} \frac{p_{n+1}}{P_n^\alpha} = 0$ *and* $S_{\overline{N}_p}^\alpha(\Delta^m) - \lim x = L$, *then* $S^\alpha(\Delta^m) - \lim x = L$.

Proof. Let $S_{\overline{N}_p}^\alpha(\Delta^m) - \lim x = L$, $\liminf p_n > c > 0$ and n be a sufficiently large number, then there exists a positive integer m such that $P_m < n \leq P_{m+1}$. Then for $\varepsilon > 0$,

$$\begin{aligned} & \frac{1}{n^\alpha} |\{k \leq n : |\Delta^m x_k - L| \geq \varepsilon\}| \\ \leq & \frac{1}{P_m^\alpha} |\{k \leq P_{m+1} : p_k |\Delta^m x_k - L| \geq c\varepsilon\}| \\ = & \frac{1}{P_m^\alpha} |\{k \leq P_m : p_k |\Delta^m x_k - L| \geq c\varepsilon\}| + \frac{p_{m+1}}{P_m^\alpha} \end{aligned}$$

Consequently $S^\alpha(\Delta^m) - \lim x = L$. □

The following example shows that in general the converse of Theorem 2.7 is not true.

Example 2.1. Define a sequence $x = (x_n)$ by

$$\Delta^m x_n = \begin{cases} 1, n = k^2 \\ \frac{1}{\sqrt{n}}, otherwise \end{cases}, k \in \mathbb{N}.$$

It is clear that x is Δ^m−statistically convergent sequence of order α to 0, but not weighted Δ^m−statistically convergent sequence of order α to 0, for $p_n = n$ for all $n \in \mathbb{N}$ and $\frac{1}{2} < \alpha \leq 1$.

Theorem 2.8. *Let* α *and* β *are fixed real numbers such that* $0 < \alpha \leq \beta \leq 1$. *Then the inclusion* $S_{\overline{N}_p}^\alpha(\Delta^m) \subseteq S_{\overline{N}_p}^\beta(\Delta^m)$ *is strict for some* α *and* β *such that* $\alpha < \beta$.

Proof. The inclusion part of the proof follows from the following inequality:

$$\frac{1}{P_n^{\beta}}\left|\{k\le P_n : p_k\left|\Delta^m x_k - L\right|\ge\varepsilon\}\right| \le \frac{1}{P_n^{\alpha}}\left|\{k\le P_n : p_k\left|\Delta^m x_k - L\right|\ge\varepsilon\}\right|.$$

To prove that the inclusions is strict, choose $p_n = n$ for all $n \in \mathbb{N}$ and define a sequence $x = (x_n)$ by

$$\Delta^m x_n = \begin{cases} 1, & n = k^2 \\ \frac{1}{\sqrt{n}}, & n \neq k^2 \end{cases}, k \in \mathbb{N}.$$

Hence $x \in S_{\overline{N_p}}^{\beta}(\Delta^m)$ for $\frac{1}{2} < \beta \le 1$, but $x \notin S_{\overline{N_p}}^{\alpha}(\Delta^m)$ for $0 < \alpha \le \frac{1}{2}$. □

Corollary 2.1. *If we take $\beta = 1$ then $S_{\overline{N}_p}^{\alpha}(\Delta^m) \subseteq S_{\overline{N_p}}(\Delta^m)$ strictly holds.*

Theorem 2.9. *Let α and β are fixed real numbers such that $0 < \alpha \le \beta \le 1$. Then the inclusion $\left[\overline{N_p}^{\alpha}\right](\Delta^m, r) \subseteq \left[\overline{N_p}^{\beta}\right](\Delta^m, r)$ is strict for some α and β such that $\alpha < \beta$.*

Proof. The inclusion part of the proof follows from the following inequality:

$$\frac{1}{P_n^{\beta}}\sum_{k=1}^{n} p_k\left|\Delta^m x_k - L\right|^r \le \frac{1}{P_n^{\alpha}}\sum_{k=1}^{n} p_k\left|\Delta^m x_k - L\right|^r.$$

To show that the inclusion is strict, choose $p_n = n$ for all $n \in \mathbb{N}$ and define a sequence $x = (x_n)$ by

$$\Delta^m x_k = \begin{cases} 1, if\ k\ is\ square \\ 0,\ otherwise \end{cases}.$$

Then $x \in \left[\overline{N_p}^{\beta}\right](\Delta^m, r)$ for $\frac{1}{2} < \beta \le 1$ but $x \notin \left[\overline{N_p}^{\alpha}\right](\Delta^m, r)$ for $0 < \alpha \le \frac{1}{2}$. □

Corollary 2.2. *If we take $\beta = 1$ then $\left[\overline{N_p}^{\alpha}\right](\Delta^m, r) \subseteq \left[\overline{N_p}\right](\Delta^m, r)$ strictly holds.*

In the following results we take the sequences (p_n) and (q_n) of real numbers such that $\liminf p_n > 0$, $\liminf q_n > 0$ and $P_n = p_1 + p_2 + p_3 + ... + p_n$, $Q_n = q_1 + q_2 + q_3 + ... + q_n$ with $p_n \le q_n$ for all $n \in \mathbb{N}$.

Theorem 2.10. *If*

$$\liminf \frac{P_n}{Q_n} > 0 \tag{2.1}$$

then $S_{\overline{N_q}}(\Delta^m) \subseteq S_{\overline{N}_p}(\Delta^m)$.

Proof. Let $S_{\overline{N_q}}(\Delta^m) - \lim x = L$, then

$$\lim_{n\to\infty} \frac{1}{Q_n} \left|\{k \le Q_n : q_k \left|\Delta^m x_k - L\right| \ge \varepsilon\}\right| = 0.$$

For given $\varepsilon > 0$ we have the following inclusion

$$\{k \le Q_n : q_k \left|\Delta^m x_k - L\right| \ge \varepsilon\} \supset \{k \le P_n : p_k \left|\Delta^m x_k - L\right| \ge \varepsilon\}$$

is satisfied and so we have the following inequality since $P_n \le Q_n$ for all $n \in \mathbb{N}$

$$\begin{aligned}\frac{1}{Q_n} \left|\{k \le Q_n : q_k \left|\Delta^m x_k - L\right| \ge \varepsilon\}\right| &\ge \tfrac{1}{Q_n} \left|\{k \le P_n : p_k \left|\Delta^m x_k - L\right| \ge \varepsilon\}\right| \\ &= \tfrac{P_n}{Q_n} \tfrac{1}{P_n} \left|\{k \le P_n : p_k \left|\Delta^m x_k - L\right| \ge \varepsilon\}\right|.\end{aligned}$$

Hence $S_{\overline{N_p}}(\Delta^m) - \lim x = L$. □

Theorem 2.11. *If the condition* (2.1) *is satisfied, then* $\left[\overline{N_q}\right](\Delta^m, r) \subset \left[\overline{N_p}\right](\Delta^m, r)$.

Proof. Proof is similar to that of Theorem 2.10. □

3. Conclusion

In this paper, we have studied the concepts of Δ^m-weighted statistical convergence of order α and Δ^m-weighted $[\overline{N_p}]-$summability of order α for $0 < \alpha \le 1$ and related theorems.The relationships between these concepts were examined for different α values. It has been shown that $\left[\overline{N_p}^{\alpha}\right](\Delta^m, r)$ is a Banach space. For a further study, we suggest to investigate the present work for the fuzzy case. However, due to changes in the setting, the definitions and methods of proofs will not always be analogous to those of the present work (see [1, 2, 12] for the definitions and related concepts in the fuzzy setting).

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