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ON $f-$LACUNARY STATISTICAL CONVERGENCE OF ORDER β OF DOUBLE SEQUENCES FOR DIFFERENCE SEQUENCES OF FRACTIONAL ORDER

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Abstract. In this study, by using definition of lacunary statistical convergence we introduce the concepts of $f-$ lacunary statistical convergence of order β and strongly $f-$ lacunary summability of order β of double sequences for different sequences of fractional order spaces. Also, we establish some inclusion relations between these concepts.
Keywords: Difference sequences, Lacunary statistical convergence, Modulus function.

1. Introduction

In 1951, Steinhaus [55] and Fast [27] introduced the concept of statistical convergence and later in 1959, Schoenberg [53] reintroduced independently. Bhardwaj and Dhawan [11], Caserta et al. [12], Connor [13], Çakallı [17, 18], Çınar et al. [19], Çolak [20], Et et al. [22, 24], Fridy [29], Işık [35], Salat [51], Di Maio and Kočinac [21], Mursaleen et al. [41, 42, 43], Belen and Mohiuddine [10], Şengül Kandemir [58], Aral [7] and many authors investigated some arguments related to this notion.

Difference sequence spaces were defined by Kızmaz [39] and the concept was generalized by Et et al. [22, 25] as follows:

$$\Delta^m (X) = \{x = (x_k) : (\Delta^m x_k) \in X\},$$

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where X is any sequence space, $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$ and so $\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$.

If $x \in \Delta^m(X)$ then there exists one and only one sequence $y = (y_k) \in X$ such that $y_k = \Delta^m x_k$ and

$$(1.1) x_k = \sum_{v=1}^{k-m} (-1)^m \left(\begin{array}{c} k-v-1 \\ m-1 \end{array} \right) y_v = \sum_{v=1}^{k} (-1)^m \left(\begin{array}{c} k+m-v-1 \\ m-1 \end{array} \right) y_{v-m},$$

$$y_{1-m} = y_{2-m} = \ldots = y_0 = 0$$

for sufficiently large k, for instance $k > 2m$. After then some properties of difference sequence spaces have been studied in [3, 4, 5, 23, 25, 38, 52, 59, 60, 61, 62].

By $\Gamma(r)$, we denote the Gamma function of a real number r and $r > 0$. By the definition, it can be expressed as an improper integral as:

$$\Gamma(r) = \int_0^\infty e^{-t} t^{r-1} dt.$$

From the definition, it is observed that:

(i) For any natural number n, $\Gamma(n+1) = n!$,

(ii) For any real number n and $n \notin \{0, -1, -2, -3, ...\}, \Gamma(n+1) = n\Gamma(n)$,

(iii) For particular cases, we have $\Gamma(1) = \Gamma(2) = 1, \Gamma(3) = 2!, \Gamma(4) = 3!, ...$

For a proper fraction α, we define a fractional difference operator $\Delta^\alpha : w \to w$ defined by

$$\Delta^\alpha(x_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha - i + 1)} x_{k+i}. \tag{1.2}$$

In particular, we have $\Delta^{\frac{1}{2}} x_k = x_k - \frac{1}{2}x_{k+1} - \frac{1}{8}x_{k+2} - \frac{1}{16}x_{k+3} - \frac{5}{128}x_{k+4} - \frac{7}{256}x_{k+5} - \frac{21}{1024}x_{k+6} \cdots$

$\Delta^{-\frac{1}{2}} x_k = x_k + \frac{1}{2}x_{k+1} + \frac{3}{8}x_{k+2} + \frac{5}{16}x_{k+3} + \frac{35}{128}x_{k+4} + \frac{63}{256}x_{k+5} + \frac{231}{1024}x_{k+6} \cdots$

$\Delta^{\frac{1}{3}} x_k = x_k - \frac{1}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{5}{81}x_{k+3} - \frac{10}{243}x_{k+4} - \frac{22}{729}x_{k+5} - \frac{154}{6561}x_{k+6} \cdots$

$\Delta^{\frac{2}{3}} x_k = x_k - \frac{2}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{4}{81}x_{k+3} - \frac{7}{243}x_{k+4} - \frac{14}{729}x_{k+5} - \frac{91}{6561}x_{k+6} \cdots .$

Without loss of generality, we assume throughout that the series defined in (1.2) is convergent. Moreover, if α is a positive integer, then the infinite sum defined in (1.2) reduces to a finite sum i.e.,

$$\sum_{i=0}^{\alpha} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha - i + 1)} x_{k+i}.$$

In fact, this operator generalized the difference operator introduced by Et and Çolak [22].

Recently, using fractional operator Δ^α (fractional order of α, $\alpha \in \mathbb{R}$) Baliarsingh et al. [8, 9, 45] defined the sequence space $\Delta^\alpha(X)$ such as:
$\Delta^\alpha(X) = \{x = (x_k) : (\Delta^\alpha x_k) \in X\}$, where X is any sequence space.

A modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

i) $f(x)=0$ if and only if $x=0$,

ii) $f(x+y)\leq f(x)+f(y)$ for $x,y\geq 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

It follows that f must be continuous in everywhere on $[0,\infty)$. A modulus may be unbounded or bounded.

Aizpuru et al. [1] defined $f-$density of a subset $E\subset\mathbb{N}$ for any unbounded modulus f by

$$d^{f}\left(E\right)=\lim_{n\to\infty}\frac{f\left(\left|\left\{k\leq n:k\in E\right\}\right|\right)}{f\left(n\right)},\textit{ if the limit exists}$$

and defined $f-$statistical convergence for any unbounded modulus f by

$$d^{f}\left(\left\{k\in\mathbb{N}:\left|x_{k}-\ell\right|\geq\varepsilon\right\}\right)=0$$

i.e.

$$\lim_{n\to\infty}\frac{1}{f\left(n\right)}f\left(\left|\left\{k\leq n:\left|x_{k}-\ell\right|\geq\varepsilon\right\}\right|\right)=0,$$

and we write it as $S^{f}-\lim x_{k}=\ell$ or $x_{k}\to\ell\left(S^{f}\right)$. Every $f-$statistically convergent sequence is statistically convergent, but a statistically convergent sequence does not need to be $f-$statistically convergent for every unbounded modulus f.

By a lacunary sequence we mean an increasing integer sequence $\theta=(k_{r})$ of non-negative integers such that $k_{0}=0$ and $h_{r}=(k_{r}-k_{r-1})\to\infty$ as $r\to\infty$. The intervals determined by θ will be denoted by $I_{r}=(k_{r-1},k_{r}]$ and the ratio $\frac{k_{r}}{k_{r-1}}$ will be abbreviated by q_{r}, and $q_{1}=k_{1}$ for convenience.

In [30], Fridy and Orhan introduced the concept of lacunary statistically convergence in the sense that a sequence (x_{k}) of real numbers is called lacunary statistically convergent to a real number ℓ, if

$$\lim_{r\to\infty}\frac{1}{h_{r}}\left|\left\{k\in I_{r}:\left|x_{k}-\ell\right|\geq\varepsilon\right\}\right|=0$$

for every positive real number ε.

Lacunary sequence spaces were studied in [6, 14, 15, 16, 26, 28, 30, 31, 33, 34, 36, 37, 48, 54, 57, 59].

A double sequence $x=(x_{j,k})_{j,k=0}^{\infty}$ has Pringsheim limit ℓ provided that given for every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $|x_{j,k}-\ell|<\varepsilon$ whenever $j,k>N$. In this case, we write $P-\lim x=\ell$ (see Pringsheim [50]).

Let $K\subseteq\mathbb{N}\times\mathbb{N}$ and $K\left(m,n\right)=\left\{(j,k):j\leq m,k\leq n\right\}$. The double natural density of K is defined by

$$\delta_2(K) = P - \lim_{m,n} \frac{1}{mn} |K(m,n)|, \textit{ if the limit exists.}$$

A double sequence $x = (x_{jk})_{j,k\in\mathbb{N}}$ is said to be statistically convergent to a number ℓ if for every $\varepsilon > 0$ the set $\{(j,k) : j \leq m, k \leq n : |x_{jk} - \ell| \geq \varepsilon\}$ has double natural density zero (see Mursaleen and Edely [42]).

In [47], Patterson and Savaş introduced the concept of double lacunary sequence in the sense that double sequence $\theta'' = \{(k_r, l_s)\}$ is called double lacunary sequence, if there exist two increasing sequences of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \textit{ as } r \to \infty$$

and

$$l_0 = 0, \overline{h}_s = l_s - l_{s-1} \to \infty \textit{ as } s \to \infty.$$

where $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \overline{h}_s$ and the following intervals are determined by θ'', $I_r = \{(k) : k_{r-1} < k \leq k_r\}$, $I_s = \{(l) : l_{s-1} < l \leq l_s\}$, $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \textit{ and } l_{s-1} < l \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\overline{q}_s = \frac{l_s}{l_{s-1}}$ and $q_{r,s} = q_r \overline{q}_s$.

The double number sequence x is $S_{\theta''}-$convergent to ℓ provided that for every $\varepsilon > 0$,

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : |x_{k,l} - \ell| \geq \varepsilon\}|) = 0.$$

In this case write $S_{\theta''} - \lim x_{k,l} = \ell$ or $x_{k,l} \to \ell\,(S_{\theta''})$ (see [47]).

The notion of a modulus was given by Nakano [44]. Maddox [40] used a modulus function to construct some sequence spaces. Afterwards different sequence spaces defined by modulus have been studied by Altın and Et [2], Et et al. [23], Işık [35], Gaur and Mursaleen [32], Nuray and Savaş [46], Pehlivan and Fisher [49], Şengül [56] and everybody else.

2. Main Results

In this section we will introduce the concepts of $f-$lacunary statistical convergence of order β and strong $f-$lacunary summability of order β of double sequences for difference sequences of fractional order, where f is an unbounded modulus and give some results related to these concepts.

Definition 2.1. Let f be an unbounded modulus, $\theta'' = \{(k_r, l_s)\}$ be a double lacunary sequence and β be a real number such that $0 < \beta \leq 1$ and α be a proper fraction. We say that the double sequence $x = (x_{k,l})$ is $\Delta_f^\alpha-$lacunary statistically convergent of order β, if there is a real number ℓ such that

$$\lim_{r,s\to\infty} \frac{1}{[f(h_{r,s})]^\beta} f(|\{(k,l) \in I_{r,s} : |\Delta^\alpha x_{k,l} - \ell| \geq \varepsilon\}|) = 0.$$

This space will be denoted by $\Delta^{\alpha}(S_{\theta''}^{f,\beta})$. In this case, we write $\Delta^{\alpha}(S_{\theta''}^{f,\beta})-\lim x_{k,l}=\ell$ or $x_{k,l}\to\ell\left(\Delta^{\alpha}(S_{\theta''}^{f,\beta})\right)$. In the special case $\theta''=\{(2^r,2^s)\}$, we shall write $\Delta^{\alpha}(S''^{f,\beta})$ instead of $\Delta^{\alpha}(S_{\theta''}^{f,\beta})$.

Definition 2.2. Let f be a modulus function, $\theta''=\{(k_r,l_s)\}$ be a double lacunary sequence, $p=(p_k)$ be a sequence of strictly positive real numbers and β be a positive real number and α be a proper fraction. We say that the double sequence $x=(x_{k,l})$ is strongly $\Delta^{\alpha}\left(w^{\beta}\left[\theta'',f,p\right]\right)-$summable to ℓ (a real number), if there is a real number ℓ such that

$$\lim_{r,s\to\infty}\frac{1}{[h_{r,s}]^{\beta}}\sum_{(k,l)\in I_{r,s}}\left[f\left(\left|\Delta^{\alpha}x_{k,l}-\ell\right|\right)\right]^{p_k}=0.$$

In this case we write $\Delta^{\alpha}\left(w^{\beta}\left[\theta'',f,p\right]\right)-\lim x_{k,l}=\ell$. The set of all strongly $\Delta^{\alpha}\left(w^{\beta}\left[\theta'',f,p\right]\right)-$ summable sequences will be denoted by $\Delta^{\alpha}\left(w^{\beta}\left[\theta'',f,p\right]\right)$. If we take $p_k=1$ for all $k\in\mathbb{N}$, we write $\Delta^{\alpha}\left(w^{\beta}\left[\theta'',f\right]\right)$ instead of $\Delta^{\alpha}\left(w^{\beta}\left[\theta'',f,p\right]\right)$.

Definition 2.3. Let f be an unbounded modulus, $\theta''=\{(k_r,l_s)\}$ be a double lacunary sequence, $p=(p_k)$ be a sequence of strictly positive real numbers and β be a positive real number and α be a proper fraction. We say that the double sequence $x=(x_{k,l})$ is strongly $\Delta^{\alpha}\left(w_{\theta''}^{f,\beta}(p)\right)-$summable to ℓ (a real number), if there is a real number ℓ such that

$$\lim_{r,s\to\infty}\frac{1}{\left[f\left(h_{r,s}\right)\right]^{\beta}}\sum_{(k,l)\in I_{r,s}}\left[f\left(\left|\Delta^{\alpha}x_{k,l}-\ell\right|\right)\right]^{p_k}=0.$$

In the present case, we write $\Delta^{\alpha}\left(w_{\theta''}^{f,\beta}(p)\right)-\lim x_{k,l}=\ell$. The set of all strongly $\Delta^{\alpha}\left(w_{\theta''}^{f,\beta}(p)\right)-$ summable sequences will be denoted by $\Delta^{\alpha}\left(w_{\theta''}^{f,\beta}(p)\right)$. In case of $p_k=p$ for all $k\in\mathbb{N}$ we write $\Delta^{\alpha}\left(w_{\theta''}^{f,\beta}[p]\right)$ instead of $\Delta^{\alpha}\left(w_{\theta''}^{f,\beta}(p)\right)$.

Definition 2.4. Let f be an unbounded modulus, $\theta''=\{(k_r,l_s)\}$ be a double lacunary sequence, $p=(p_k)$ be a sequence of strictly positive real numbers and β be a positive real number and α be a proper fraction. We say that the double sequence $x=(x_{k,l})$ is strongly $\Delta^{\alpha}\left(w_{\theta'',f}^{\beta}(p)\right)-$summable to ℓ (a real number), if there is a real number ℓ such that

$$\frac{1}{\left[f\left(h_{r,s}\right)\right]^{\beta}}\sum_{(k,l)\in I_{r,s}}\left|\Delta^{\alpha}x_{k,l}-\ell\right|^{p_k}=0.$$

In the present case, we write $\Delta^{\alpha}\left(w_{\theta'',f}^{\beta}(p)\right)-\lim x_{k,l}=\ell$. The set of all strongly $\Delta^{\alpha}\left(w_{\theta'',f}^{\beta}(p)\right)-$ summable sequences will be denoted by $\Delta^{\alpha}\left(w_{\theta'',f}^{\beta}(p)\right)$. In case of $p_k=p$ for all $k\in\mathbb{N}$ we write $\Delta^{\alpha}\left(w_{\theta'',f}^{\beta}[p]\right)$ instead of $\Delta^{\alpha}\left(w_{\theta'',f}^{\beta}(p)\right)$.

The proof of each of the following results is fairly straightforward, so we choose to state these results without proof, where we shall assume that the sequence $p=(p_k)$ is bounded and $0<h=\inf_k p_k \le p_k \le \sup_k p_k = H<\infty$.

Theorem 2.1. *The space* $\Delta^\alpha\left(w_{\theta''}^{f,\beta}(p)\right)$ *is paranormed by*

$$g(x)=\sup_{r,s}\left\{\frac{1}{[f(h_{r,s})]^\alpha}\sum_{(k,l)\in I_{r,s}}[f(|\Delta^\alpha x_{k,l}|)]^{p_k}\right\}^{\frac{1}{M}}$$

where $M=\max(1,H)$.

Proposition 2.1. [49] *Let* f *be a modulus and* $0<\delta<1$. *Then for each* $\|u\|\ge\delta$, *we have* $f(\|u\|)\le 2f(1)\delta^{-1}\|u\|$.

Theorem 2.2. *Let* f *be an unbounded modulus,* β *be a real number such that* $0<\beta\le 1$, α *be a proper fraction and* $p>1$. *If* $\lim_{u\to\infty}\inf\frac{f(u)}{u}>0$, *then* $\Delta^\alpha\left(w_{\theta''}^{f,\beta}[p]\right)=\Delta^\alpha\left(w_{\theta'',f}^{\beta}[p]\right)$.

Proof. Let $p>1$ be a positive real number and $x\in\Delta^\alpha\left(w_{\theta''}^{f,\beta}[p]\right)$. If $\lim_{u\to\infty}\inf\frac{f(u)}{u}>0$ then there exists a number $c>0$ such that $f(u)>cu$ for $u>0$. Clearly

$$\begin{aligned}\frac{1}{[f(h_{r,s})]^\beta}\sum_{(k,l)\in I_{r,s}}[f(|\Delta^\alpha x_{k,l}-\ell|)]^p &\ge \frac{1}{[f(h_{r,s})]^\beta}\sum_{(k,l)\in I_{r,s}}[c\,|\Delta^\alpha x_{k,l}-\ell|]^p\\ &= \frac{c^p}{[f(h_{r,s})]^\beta}\sum_{(k,l)\in I_{r,s}}|\Delta^\alpha x_{k,l}-\ell|^p\end{aligned}$$

and therefore $\Delta^\alpha\left(w_{\theta''}^{f,\beta}[p]\right)\subset\Delta^\alpha\left(w_{\theta'',f}^{\beta}[p]\right)$.

Now let $x\in\Delta^\alpha\left(w_{\theta'',f}^{\beta}[p]\right)$. Then we have

$$\frac{1}{[f(h_{r,s})]^\beta}\sum_{(k,l)\in I_{r,s}}|\Delta^\alpha x_{k,l}-\ell|^p\to 0 \text{ as } r,s\to\infty.$$

Let $0<\delta<1$. We can write

$$\frac{1}{[f(h_{r,s})]^\beta}\sum_{(k,l)\in I_{r,s}}|\Delta^\alpha x_{k,l}-\ell|^p \ge \frac{1}{[f(h_{r,s})]^\beta}\sum_{\substack{(k,l)\in I_{r,s}\\ |\Delta^\alpha x_{k,l}-\ell|\ge\delta}}|\Delta^\alpha x_{k,l}-\ell|^p$$

$$\geq \frac{1}{\left[f\left(h_{r,s}\right)\right]^{\beta}} \sum_{\substack{(k,l)\in I_{r,s} \\ |\Delta^{\alpha}x_{k,l}-\ell|\geq\delta}} \left[\frac{f\left(\left|\Delta^{\alpha}x_{k,l}-\ell\right|\right)}{2f\left(1\right)\delta^{-1}}\right]^{p}$$
$$\geq \frac{1}{\left[f\left(h_{r,s}\right)\right]^{\beta}} \frac{\delta^{p}}{2^{p}f\left(1\right)^{p}} \sum_{(k,l)\in I_{r,s}} \left[f\left(\left|\Delta^{\alpha}x_{k,l}-\ell\right|\right)\right]^{p}$$

by Proposition 2.1. Therefore $x\in\Delta^{\alpha}\left(w_{\theta''}^{f,\beta}\left[p\right]\right)$. □

If $\lim_{u\to\infty}\inf\frac{f(u)}{u}=0$, the equality $\Delta^{\alpha}\left(w_{\theta''}^{f,\beta}\left[p\right]\right)=\Delta^{\alpha}\left(w_{\theta'',f}^{\beta}\left[p\right]\right)$ can not be hold as shown the following example:

Example 2.1. Let $f\left(x\right)=2\sqrt{x}$ and define a double sequence $x=\left(x_{k,l}\right)$ by

$$\Delta^{\alpha}x_{k,l}=\begin{cases}\sqrt[3]{h_{r,s}}, & \text{if } k=k_{r} \text{ and } l=l_{s} \\ 0, & \text{otherwise}\end{cases} \quad r,s=1,2,....$$

For $\ell=0$, $\beta=\frac{3}{4}$ and $p=\frac{6}{5}$, we have

$$\frac{1}{\left[f\left(h_{r,s}\right)\right]^{\beta}} \sum_{(k,l)\in I_{r,s}} \left[f\left(\left|\Delta^{\alpha}x_{k,l}\right|\right)\right]^{p} = \frac{\left(2\left[h_{r,s}\right]^{\frac{1}{6}}\right)^{\frac{6}{5}}}{\left(2\sqrt{h_{r,s}}\right)^{\frac{3}{4}}} = \frac{\left(2\left(h_{r}\overline{h_{s}}\right)^{\frac{1}{6}}\right)^{\frac{6}{5}}}{\left(2\sqrt{h_{r}\overline{h_{s}}}\right)^{\frac{3}{4}}} \to 0 \text{ as } r,s\to\infty$$

hence $x\in\Delta^{\alpha}\left(w_{\theta''}^{f,\alpha}\left[p\right]\right)$, but

$$\frac{1}{\left[f\left(h_{r,s}\right)\right]^{\beta}} \sum_{(k,l)\in I_{r,s}} \left|\Delta^{\alpha}x_{k,l}\right|^{p} = \frac{\left(\sqrt[3]{h_{r,s}}\right)^{\frac{6}{5}}}{\left(2\sqrt{h_{r,s}}\right)^{\frac{3}{4}}} \to \infty \text{ as } r,s\to\infty$$

and so $x\notin\Delta^{\alpha}\left(w_{\theta'',f}^{\beta}\left[p\right]\right)$.

Maddox [40] showed that the existence of an unbounded modulus f for which there is a positive constant c such that $f\left(xy\right)\geq cf\left(x\right)f\left(y\right)$, for all $x\geq0$, $y\geq0$.

Theorem 2.3. *Let f be an unbounded modulus and β be a positive real number and α be a proper fraction. If $\lim_{u\to\infty}\frac{\left[f(u)\right]^{\beta}}{u^{\beta}}>0$, then $\Delta^{\alpha}\left(w^{\beta}\left[\theta'',f\right]\right)\subset\Delta^{\alpha}\left(S_{\theta''}^{f,\beta}\right)$.*

Proof. Let $x\in\Delta^{\alpha}\left(w^{\beta}\left[\theta'',f\right]\right)$ and $\lim_{u\to\infty}\frac{f(u)^{\beta}}{u^{\beta}}>0$. For $\varepsilon>0$, we have

$$\frac{1}{\left[h_{r,s}\right]^{\beta}} \sum_{(k,l)\in I_{r,s}} f\left(\left|\Delta^{\alpha}x_{k,l}-\ell\right|\right)$$

$$\begin{aligned}
&\geq \frac{1}{[h_{r,s}]^{\beta}} f\left(\sum_{(k,l)\in I_{r,s}} |\Delta^{\alpha} x_{k,l}-\ell|\right) \\
&\geq \frac{1}{[h_{r,s}]^{\beta}} f\left(\sum_{\substack{(k,l)\in I_{r,s} \\ |\Delta^{\alpha} x_{k,l}-\ell|\geq\varepsilon}} |\Delta^{\alpha} x_{k,l}-\ell|\right) \\
&\geq \frac{1}{[h_{r,s}]^{\beta}} f\left(|\{(k,l)\in I_{r,s} : |\Delta^{\alpha} x_{k,l}-\ell|\geq\varepsilon\}|\,\varepsilon\right) \\
&\geq \frac{c}{[h_{r,s}]^{\beta}} f\left(|\{(k,l)\in I_{r,s} : |\Delta^{\alpha} x_{k,l}-\ell|\geq\varepsilon\}|\right) f(\varepsilon) \\
&= \frac{c}{[h_{r,s}]^{\beta}} \frac{f\left(|\{(k,l)\in I_{r,s} : |\Delta^{\alpha} x_{k,l}-\ell|\geq\varepsilon\}|\right)}{[f(h_{r,s})]^{\beta}} [f(h_{r,s})]^{\beta} f(\varepsilon).
\end{aligned}$$

Therefore, $\Delta^{\alpha}\left(w^{\beta}\left[\theta'', f\right]\right)-\lim x_{k,l}=\ell$ implies $\Delta^{\alpha}\left(S_{\theta''}^{f,\beta}\right)-\lim x_{k,l}=\ell$. □

Theorem 2.4. *Let β_1, β_2 be two real numbers such that $0<\beta_1\leq\beta_2\leq 1$, f be an unbounded modulus function and let $\theta''=\{(k_r,l_s)\}$ be a double lacunary sequence, then we have $\Delta^{\alpha}\left(w_{\theta''}^{f,\beta_1}(p)\right)\subset\Delta^{\alpha}\left(S_{\theta''}^{f,\beta_2}\right)$.*

Proof. Let $x\in\Delta^{\alpha}\left(w_{\theta''}^{f,\beta_1}(p)\right)$ and $\varepsilon>0$ be given and $\sum_1$, $\sum_2$ denote the sums over $(k,l)\in I_{r,s}$, $|\Delta^{\alpha} x_{k,l}-\ell|\geq\varepsilon$ and $(k,l)\in I_{r,s}$, $|\Delta^{\alpha} x_{k,l}-\ell|<\varepsilon$ respectively. Since $f(h_{r,s})^{\beta_1}\leq f(h_{r,s})^{\beta_2}$ for each r and s, we may write

$$\begin{aligned}
&\frac{1}{[f(h_{r,s})]^{\beta_1}} \sum_{(k,l)\in I_{r,s}} [f(|\Delta^{\alpha} x_{k,l}-\ell|)]^{p_k} \\
&= \frac{1}{[f(h_{r,s})]^{\beta_1}} \left[\sum\nolimits_1 [f(|\Delta^{\alpha} x_{k,l}-\ell|)]^{p_k}+\sum\nolimits_2 [f(|\Delta^{\alpha} x_{k,l}-\ell|)]^{p_k}\right] \\
&\geq \frac{1}{[f(h_{r,s})]^{\beta_2}} \left[\sum\nolimits_1 [f(|\Delta^{\alpha} x_{k,l}-\ell|)]^{p_k}+\sum\nolimits_2 [f(|\Delta^{\alpha} x_{k,l}-\ell|)]^{p_k}\right] \\
&\geq \frac{1}{[f(h_{r,s})]^{\beta_2}} \left[\sum\nolimits_1 [f(\varepsilon)]^{p_k}\right] \\
&\geq \frac{1}{H.[f(h_{r,s})]^{\beta_2}} \left[f\left(\sum\nolimits_1 [\varepsilon]^{p_k}\right)\right] \\
&\geq \frac{1}{H.[f(h_{r,s})]^{\beta_2}} \left[f\left(\sum\nolimits_1 \min([\varepsilon]^{h},[\varepsilon]^{H})\right)\right] \\
&\geq \frac{1}{H.[f(h_{r,s})]^{\beta_2}} f\left(|\{(k,l)\in I_{r,s} : |\Delta^{\alpha} x_{k,l}-\ell|\geq\varepsilon\}|\left[\min([\varepsilon]^{h},[\varepsilon]^{H})\right]\right) \\
&\geq \frac{c}{H.[f(h_{r,s})]^{\beta_2}} f\left(|\{(k,l)\in I_{r,s} : |\Delta^{\alpha} x_{k,l}-\ell|\geq\varepsilon\}|\right) f\left(\left[\min([\varepsilon]^{h},[\varepsilon]^{H})\right]\right).
\end{aligned}$$

Hence $x \in \Delta^{\alpha}\left(S_{\theta''}^{f,\beta_2}\right)$. $\square$

Theorem 2.5. *Let $\theta'' = \{(k_r, l_s)\}$ be a double lacunary sequence and β be a fixed real number such that $0 < \beta \leq 1$ and α be a proper fraction. If $\liminf_r q_r > 1$, $\liminf_s q_s > 1$ and $\lim_{u\to\infty} \frac{[f(u)]^{\beta}}{u^{\beta}} > 0$, then $\Delta^{\alpha}\left(S''^{f,\beta}\right) \subset \Delta^{\alpha}\left(S_{\theta''}^{f,\beta}\right)$.*

Proof. Suppose first that $\liminf_r q_r > 1$ and $\liminf_s q_s > 1$; then there exists $a, b > 0$ such that $q_r \geq 1 + a$ and $q_s \geq 1 + b$ for sufficiently large r and s, which implies that

$$\frac{h_r}{k_r} \geq \frac{a}{1+a} \Longrightarrow \left(\frac{h_r}{k_r}\right)^{\beta} \geq \left(\frac{a}{1+a}\right)^{\beta}$$

and

$$\frac{\overline{h}_s}{l_s} \geq \frac{b}{1+b} \Longrightarrow \left(\frac{\overline{h}_s}{l_s}\right)^{\beta} \geq \left(\frac{b}{1+b}\right)^{\beta}.$$

If $\Delta^{\alpha}\left(S''^{f,\beta}\right) - \lim x_{k,l} = \ell$, then for every $\varepsilon > 0$ and for sufficiently large r and s, we have

$$\begin{aligned}
&\frac{1}{[f(k_r l_s)]^{\beta}} f\left(\left|\{k \leq k_r, l \leq l_s : |\Delta^{\alpha} x_{k,l} - \ell| \geq \varepsilon\}\right|\right) \\
\geq\ & \frac{1}{[f(k_r l_s)]^{\beta}} f\left(\left|\{(k,l) \in I_{r,s} : |\Delta^{\alpha} x_{k,l} - \ell| \geq \varepsilon\}\right|\right) \\
=\ & \frac{[f(h_{r,s})]^{\beta}}{[f(k_r l_s)]^{\beta}} \frac{1}{[f(h_{r,s})]^{\beta}} f\left(\left|\{(k,l) \in I_{r,s} : |\Delta^{\alpha} x_{k,l} - \ell| \geq \varepsilon\}\right|\right) \\
=\ & \frac{[f(h_{r,s})]^{\beta}}{[h_{r,s}]^{\beta}} \frac{k_r^{\beta}}{[f(k_r l_s)]^{\beta}} \frac{[h_{r,s}]^{\beta}}{k_r^{\beta}} \frac{f\left(\left|\{(k,l) \in I_{r,s} : |\Delta^{\alpha} x_{k,l} - \ell| \geq \varepsilon\}\right|\right)}{[f(h_{r,s})]^{\beta}} \\
=\ & \frac{[f(h_{r,s})]^{\beta}}{[h_{r,s}]^{\beta}} \frac{k_r^{\beta} l_s^{\beta}}{[f(k_r l_s)]^{\beta}} \frac{h_r^{\beta} \overline{h}_s^{\beta}}{k_r^{\beta} l_s^{\beta}} \frac{f\left(\left|\{(k,l) \in I_{r,s} : |\Delta^{\alpha} x_{k,l} - \ell| \geq \varepsilon\}\right|\right)}{[f(h_{r,s})]^{\beta}} \\
\geq\ & \frac{[f(h_{r,s})]^{\beta}}{[h_{r,s}]^{\beta}} \frac{(k_r l_s)^{\beta}}{[f(k_r l_s)]^{\beta}} \left(\frac{a}{1+a}\right)^{\beta} \left(\frac{b}{1+b}\right)^{\beta} \frac{f\left(\left|\{(k,l) \in I_{r,s} : |\Delta^{\alpha} x_{k,l} - \ell| \geq \varepsilon\}\right|\right)}{[f(h_{r,s})]^{\beta}}.
\end{aligned}$$

This proves the sufficiency. $\square$

Theorem 2.6. *Let f be an unbounded modulus, $\theta = (k_r)$ and $\theta' = (l_s)$ be two lacunary sequences, $\theta'' = \{(k_r, l_s)\}$ be a double lacunary sequence, $0 < \beta \leq 1$ and α be a proper fraction. If $\Delta^{\alpha}\left(S_{f,\theta}^{\beta}\right) - \lim x_k = \ell$ and $\Delta^{\alpha}\left(S_{f,\theta'}^{\beta}\right) - \lim x_l = \ell$, then $\Delta^{\alpha}\left(S_{f,\theta''}^{\beta}\right) - \lim x_{k,l} = \ell$.*

Proof. Suppose $\Delta^{\alpha}\left(S_{f,\theta}^{\beta}\right)-\lim x_k=\ell$ and $\Delta^{\alpha}\left(S_{f,\theta'}^{\beta}\right)-\lim x_l=\ell$. Then for $\varepsilon>0$ we can write

$$\lim_r \frac{1}{\left[f\left(h_r\right)\right]^{\beta}}\left|\left\{k\in I_r:\left|\Delta^{\alpha}x_k-\ell\right|\geq\varepsilon\right\}\right|=0$$

and

$$\lim_s \frac{1}{\left[f\left(\overline{h}_s\right)\right]^{\beta}}\left|\left\{l\in I_s:\left|\Delta^{\alpha}x_l-\ell\right|\geq\varepsilon\right\}\right|=0.$$

So we have

$$\begin{aligned}
&\frac{1}{\left[f\left(h_{r,s}\right)\right]^{\beta}}\left|\left\{(k,l)\in I_{r,s}:\left|\Delta^{\alpha}x_{k,l}-\ell\right|\geq\varepsilon\right\}\right|\\
\leq\ &\frac{1}{\left[cf\left(h_r\right)f\left(\overline{h}_s\right)\right]^{\beta}}\left|\left\{(k,l)\in I_{r,s}:\left|\Delta^{\alpha}x_{k,l}-\ell\right|\geq\varepsilon\right\}\right|\\
\leq\ &\frac{1}{c^{\beta}\left[f\left(h_r\right)\right]^{\beta}\left[f\left(\overline{h}_s\right)\right]^{\beta}}\left|\left\{(k,l)\in I_{r,s}:\left|\Delta^{\alpha}x_{k,l}-\ell\right|\geq\varepsilon\right\}\right|\\
\leq\ &\left[\frac{1}{\left[f\left(h_r\right)\right]^{\beta}}\left|\left\{k\in I_r:\left|\Delta^{\alpha}x_k-\ell\right|\geq\varepsilon\right\}\right|\right]\left[\frac{1}{\left[f\left(\overline{h}_s\right)\right]^{\beta}}\left|\left\{l\in I_s:\left|\Delta^{\alpha}x_l-\ell\right|\geq\varepsilon\right\}\right|\right].
\end{aligned}$$

Hence $\Delta^{\alpha}\left(S_{f,\theta''}^{\beta}\right)-\lim x_{k,l}=\ell$. □

Theorem 2.7. *Let f be an unbounded modulus. If $\lim p_k>0$, then $\Delta^{\alpha}\left(w_{\theta''}^{f,\beta}\left(p\right)\right)-\lim x_{k,l}=\ell$ uniquely.*

Proof. Let $\lim p_k=s>0$. Assume that $\Delta^{\alpha}\left(w_{\theta''}^{f,\beta}\left(p\right)\right)-\lim x_{k,l}=\ell_1$ and $\Delta^{\alpha}\left(w_{\theta''}^{f,\beta}\left(p\right)\right)-\lim x_{k,l}=\ell_2$. Then

$$\lim_{r,s}\frac{1}{\left[f\left(h_{r,s}\right)\right]^{\beta}}\sum_{(k,l)\in I_{r,s}}\left[f\left(\left|\Delta^{\alpha}x_{k,l}-\ell_1\right|\right)\right]^{p_k}=0,$$

and

$$\lim_{r,s}\frac{1}{\left[f\left(h_{r,s}\right)\right]^{\beta}}\sum_{(k,l)\in I_{r,s}}\left[f\left(\left|\Delta^{\alpha}x_{k,l}-\ell_2\right|\right)\right]^{p_k}=0.$$

By definition of f, we have

$$\frac{1}{\left[f\left(h_{r,s}\right)\right]^{\beta}}\sum_{(k,l)\in I_{r,s}}\left[f\left(\left|\ell_1-\ell_2\right|\right)\right]^{p_k}$$

$$\begin{aligned}
&\leq \frac{D}{[f(h_{r,s})]^{\beta}}\left(\sum_{(k,l)\in I_{r,s}}[f(|\Delta^{\alpha}x_{k,l}-\ell_1|)]^{p_k}+\sum_{(k,l)\in I_{r,s}}[f(|\Delta^{\alpha}x_{k,l}-\ell_2|)]^{p_k}\right)\\
&= \frac{D}{[f(h_{r,s})]^{\beta}}\sum_{(k,l)\in I_{r,s}}[f(|\Delta^{\alpha}x_{k,l}-\ell_1|)]^{p_k}\\
&\quad +\frac{D}{[f(h_{r,s})]^{\beta}}\sum_{(k,l)\in I_{r,s}}[f(|\Delta^{\alpha}x_{k,l}-\ell_2|)]^{p_k}
\end{aligned}$$

where $\sup_k p_k = H$ and $D = \max\left(1, 2^{H-1}\right)$. Hence

$$\lim_{r,s}\frac{1}{[f(h_{r,s})]^{\beta}}\sum_{(k,l)\in I_{r,s}}[f(|\ell_1-\ell_2|)]^{p_k}=0.$$

Since $\lim_{k\to\infty} p_k = s$ we have $\ell_1 - \ell_2 = 0$. Thus the limit is unique. □

Theorem 2.8. *Let $\theta_1'' = \{(k_r, l_s)\}$ and $\theta_2'' = \{(s_r, t_s)\}$ be two double lacunary sequences such that $I_{r,s} \subset J_{r,s}$ for all $r, s \in \mathbb{N}$, β_1, β_2 two real numbers such that $0 < \beta_1 \leq \beta_2 \leq 1$ and α be a proper fraction. If*

$$\lim_{r,s\to\infty}\inf\frac{[f(h_{r,s})]^{\beta_1}}{[f(\ell_{r,s})]^{\beta_2}}>0 \tag{2.1}$$

then $\Delta^{\alpha}\left(w_{\theta_2''}^{f,\beta_2}(p)\right) \subset \Delta^{\alpha}\left(w_{\theta_1''}^{f,\beta_1}(p)\right)$, where
$I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\}$, $k_{r,s} = k_r l_s$, $h_{r,s} = h_r\overline{h}_s$ and $J_{r,s} = \{(s,t) : s_{r-1} < s \leq s_r \text{ and } t_{s-1} < l \leq t_s\}$, $s_{r,s} = s_r t_s$, $\ell_{r,s} = \ell_r\overline{\ell}_s$.

Proof. Let $x \in \Delta^{\alpha}\left(w_{\theta_2''}^{f,\beta_2}(p)\right)$. We can write

$$\begin{aligned}
&\frac{1}{[f(\ell_{r,s})]^{\beta_2}}\sum_{(k,l)\in J_{r,s}}[f(|\Delta^{\alpha}x_{k,l}-\ell|)]^{p_k}\\
&= \frac{1}{[f(\ell_{r,s})]^{\beta_2}}\sum_{(k,l)\in J_{r,s}-I_{r,s}}[f(|\Delta^{\alpha}x_{k,l}-\ell|)]^{p_k}\\
&\quad +\frac{1}{[f(\ell_{r,s})]^{\beta_2}}\sum_{(k,l)\in I_{r,s}}[f(|\Delta^{\alpha}x_{k,l}-\ell|)]^{p_k}\\
&\geq \frac{1}{[f(\ell_{r,s})]^{\beta_2}}\sum_{(k,l)\in I_{r,s}}[f(|\Delta^{\alpha}x_{k,l}-\ell|)]^{p_k}\\
&\geq \frac{[f(h_{r,s})]^{\beta_1}}{[f(\ell_{r,s})]^{\beta_2}}\frac{1}{[f(h_{r,s})]^{\beta_1}}\sum_{(k,l)\in I_{r,s}}[f(|\Delta^{\alpha}x_{k,l}-\ell|)]^{p_k}.
\end{aligned}$$

Thus if $x \in \Delta^{\alpha}\left(w_{\theta_2''}^{f,\beta_2}(p)\right)$, then $x \in \Delta^{\alpha}\left(w_{\theta_1''}^{f,\beta_1}(p)\right)$. □

From Theorem 2.8. we have the following results.

Corollary 2.1. *Let* $\theta_1'' = \{(k_r, l_s)\}$ *and* $\theta_2'' = \{(s_r, t_s)\}$ *be two double lacunary sequences such that* $I_{r,s} \subset J_{r,s}$ *for all* $r, s \in \mathbb{N}$, β_1, β_2 *two real numbers such that* $0 < \beta_1 \leq \beta_2 \leq 1$ *and* α *be a proper fraction. If* (2.1) *holds then*

(i) $\Delta^\alpha \left(w_{\theta_2''}^{f,\beta}(p) \right) \subset \Delta^\alpha \left(w_{\theta_1''}^{f,\beta}(p) \right)$, *if* $\beta_1 = \beta_2 = \beta$,

(ii) $\Delta^\alpha \left(w_{\theta_2''}^{f}(p) \right) \subset \Delta^\alpha \left(w_{\theta_1''}^{f,\beta_1}(p) \right)$, *if* $\beta_2 = 1$,

(iii) $\Delta^\alpha \left(w_{\theta_2''}^{f}(p) \right) \subset \Delta^\alpha \left(w_{\theta_1''}^{f}(p) \right)$, *if* $\beta_1 = \beta_2 = 1$.

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REFERENCES

1. A. Aizpuru, M. C. Listán-García and F. Rambla-Barreno: *Density by moduli and statistical convergence.* Quaest. Math., **37(4)** (2014), 525–530.
2. Y. Altin and M. Et: *Generalized difference sequence spaces defined by a modulus function in a locally convex space.* Soochow J. Math. **31(2)** (2005), 233–243.
3. H. Altinok, M. Et and R. Çolak: *Some remarks on generalized sequence space of bounded variation of sequences of fuzzy numbers.* Iran. J. Fuzzy Syst. **11(5)** (2014), 39–46, 109.
4. N. D. Aral and M. Et: *On lacunary statistical convergence of order* β *of difference sequences of fractional order.* International Conference of Mathematical Sciences, (ICMS 2019), Maltepe University, Istanbul, Turkey.
5. N. D. Aral and M. Et: *Generalized difference sequence spaces of fractional order defined by Orlicz functions.* Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat. **69(1)** (2020), 941–951.
6. N. D. Aral and H. Şengül: $I-$*Lacunary Statistical Convergence of Order* β *of Difference Sequences of Fractional Order.* Facta Universitatis, Series: Mathematics and Informatics, **36(1)** (2021), 43–55.
7. N. D. Aral: $\rho-$*Statistical Convergence Defined by a Modulus Function of order* (α, β). Maltepe Journal of Mathematics, **4(1)** (2022), 15–23.
8. P. Baliarsingh: *Some new difference sequence spaces of fractional order and their dual spaces.* Appl. Math. Comput. **219(18)** (2013), 9737–9742.
9. P. Baliarsingh, U. Kadak and M. Mursaleen: *On statistical convergence of difference sequences of fractional order and related Korovkin type approximation theorems.* Quaest. Math. **41(8)** (2018), 1117–1133.
10. C. Belen and S. A. Mohiuddine: *Generalized weighted statistical convergence and application.* Applied Mathematics and Computation **219** (2013), 9821–9826.

11. V. K. Bhardwaj and S. Dhawan: *Density by moduli and lacunary statistical convergence.* Abstr. Appl. Anal. **2016** (2016), Art. ID 9365037, 11 pp.

12. A. Caserta, G. Di Maio and L. D. R. Kočinac: *Statistical convergence in function spaces.* Abstr. Appl. Anal. **2011** (2011), Art. ID 420419, 11 pp.

13. J. S. Connor: *The statistical and strong p− Cesaro convergence of sequences.* Analysis **8** (1988), 47–63.

14. H. Çakalli: *Lacunary statistical convergence in topological groups.* Indian J. Pure Appl. Math. **26(2)** (1995) 113–119.

15. H. Çakalli and H. Kaplan: *A variation on lacunary statistical quasi Cauchy sequences.* Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. **66(2)** (2017), 71–79.

16. H. Çakalli and H. Kaplan: *A study on N_θ-quasi-Cauchy sequences.* Abstr. Appl. Anal. **2013** (2013), Article ID 836970, 4 pages.

17. H. Çakalli: *A study on statistical convergence.* Funct. Anal. Approx. Comput. **1(2)** (2009), 19–24.

18. H. Çakalli: *A New Approach to Statistically Quasi Cauchy Sequences.* Maltepe Journal of Mathematics, **1(1)** (2019), 1–8.

19. M. Çinar, M. Karakaş and M. Et: *On pointwise and uniform statistical convergence of order α for sequences of functions.* Fixed Point Theory And Applications, **2013(33)** (2013), 11 pages.

20. R. Çolak: *Statistical convergence of order α.* Modern Methods in Analysis and Its Applications, New Delhi, India: Anamaya Pub. **2010** (2010), 121–129.

21. G. Di Maio and L. D. R. Kočinac: *Statistical convergence in topology.* Topology Appl. **156** (2008), 28–45.

22. M. Et and R. Çolak: *On some generalized difference sequence spaces.* Soochow J. Math. **21 (4)** (1995), 377-386 .

23. M. Et, Y. Altin and H. Altinok: *On some generalized difference sequence spaces defined by a modulus function.* Filomat **17** (2003), 23–33.

24. M. Et, B. C. Tripathy and A. J. Dutta: *On pointwise statistical convergence of order α of sequences of fuzzy mappings.* Kuwait J. Sci. **41(3)** (2014), 17–30.

25. M. Et, M. Mursaleen and M. Işik: *On a class of fuzzy sets defined by Orlicz functions.* Filomat **27(5)** (2013), 789–796.

26. M. Et and H. Şengül: *Some Cesaro-type summability spaces of order α and lacunary statistical convergence of order α.* Filomat **28(8)** (2014), 1593–1602.

27. H. Fast: *Sur la convergence statistique.* Colloq. Math. **2** (1951), 241–244.

28. A. R. Freedman, J. J. Sember and M. Raphael: *Some Cesàro-type summability spaces.* Proc. London Math. Soc. *(3)* **37(3)** (1978), 508–520.

29. J. Fridy: *On statistical convergence.* Analysis **5** (1985), 301–313.

30. J. A. Fridy and C. Orhan: *Lacunary statistical convergence.* Pacific J. Math. **160** (1993), 43–51.

31. J. A. Fridy and C. Orhan: *Lacunary statistical summability.* J. Math. Anal. Appl. **173(2)** (1993), 497–504.

32. A. K. Gaur and M. Mursaleen: *Difference sequence spaces defined by a sequence of moduli.* Demonstratio Math. **31(2)** (1998), 275–278.

33. M. Işık and K. E. Et: *On lacunary statistical convergence of order α in probability.* AIP Conference Proceedings **1676** 020045 (2015), doi: http://dx.doi.org/10.1063/1.4930471.

34. M. Isik and K. E. Akbaş: *On $\lambda-$statistical convergence of order α in probability.* J. Inequal. Spec. Funct. **8(4)** (2017), 57–64.

35. M. Işık: *Generalized vector-valued sequence spaces defined by modulus functions.* J. Inequal. Appl. **2010** Art. ID 457892 (2010), 7 pp.

36. U. Kadak: *Generalized lacunary statistical difference sequence spaces of fractional order.* Int. J. Math. Math. Sci. 2015, Art. ID 984283, 6 pp.

37. H. Kaplan and H. Çakalli: *Variations on strong lacunary quasi-Cauchy sequences.* J. Nonlinear Sci. Appl. **9(6)** (2016), 4371–4380.

38. M. Karakaş, M. Et and V. Karakaya: *Some geometric properties of a new difference sequence space involving lacunary sequences.* Acta Math. Sci. Ser. B (Engl. Ed.) **33(6)** (2013), 1711–1720.

39. H. Kizmaz: *On certain sequence spaces.* Canad. Math. Bull. **24(2)** (1981), 169-176.

40. I. J. Maddox: *Sequence spaces defined by a modulus.* Math. Proc. Camb. Philos. Soc. **100** (1986), 161–166.

41. S. A. Mohiuddine, A. Alotaibi and M. Mursaleen: *Statistical convergence of double sequences in locally solid Riesz spaces.* Abstr. Appl. Anal. **2012** Article ID 719729 (2012), 9 pages.

42. M. Mursaleen and O. H. H. Edely: *Statistical convergence of double sequences.* Journal of Mathematical Analysis and Applications **288(1)** (2003), 223–231.

43. M. Mursaleen and S. A. Mohiuddine: *On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space.* J. Comput. Appl. Math. **233(2)** (2009), 142–149.

44. H. Nakano: *Modulared sequence spaces.* Proc. Japan Acad. **27** (1951), 508–512.

45. L. Nayak, M. Et and P. Baliarsingh: *On certain generalized weighted mean fractional difference sequence spaces.* Proc. Nat. Acad. Sci. India Sect. A **89(1)** (2019), 163–170.

46. F. Nuray, E. Savaş: *Some new sequence spaces defined by a modulus function.* Indian J. Pure Appl. Math. **24(11)** (1993), 657-663.

47. R. F. Patterson and E. Savaş: *Lacunary statistical convergence of double sequences.* Math. Commun. **10(1)** (2005), 55–61.

48. S. Pehlivan and B. Fisher: *Lacunary strong convergence with respect to a sequence of modulus functions.* Comment. Math. Univ. Carolin. **36(1)** (1995), 69–76.

49. S. Pehlivan and B. Fisher: *Some sequence spaces defined by a modulus.* Math. Slovaca **45(3)** (1995), 275–280.

50. A. Pringsheim: *Zur Theorie der zweifach unendlichen Zahlenfolgen.* Mathematische Annalen **53(3)** (1900), 289–321.

51. T. Salat: *On statistically convergent sequences of real numbers.* Math. Slovaca **30** (1980), 139–150.

52. E. Savaş and M. Et: *On $(\Delta_\lambda^m, I)-$statistical convergence of order α.* Period. Math. Hungar. **71(2)** (2015), 135–145.

53. I. J. Schoenberg: *The integrability of certain functions and related summability methods.* Amer. Math. Monthly **66** (1959), 361–375.

54. H. M. Srivastava and M. Et: *Lacunary statistical convergence and strongly lacunary summable functions of order α*. Filomat **31(6)** (2017), 1573–1582.

55. H. Steinhaus: *Sur la convergence ordinaire et la convergence asymptotique*. Colloq. Math. **2** (1951), 73–74.

56. H. Şengül: *Some Cesàro-type summability spaces defined by a modulus function of order (α, β)*. Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. **66(2)** (2017), 80–90.

57. H. Şengül and M. Et: *On lacunary statistical convergence of order α*. Acta Math. Sci. Ser. B Engl. Ed. **34(2)** (2014), 473–482.

58. H. Şengül Kandemir: *On $\rho-$ Statistical Convergence in Topological Groups*. Maltepe Journal of Mathematics, **4(1)** (2022), 9–14.

59. B. C. Tripathy and M. Et: *On generalized difference lacunary statistical convergence*. Studia Univ. Babeş-Bolyai Math., **50(1)** (2005), 119–130.

60. B. C. Tripathy, A. Esi and B. K. Tripathy: *On a new type of generalized difference Cesaro Sequence spaces*. Soochow J. Math. **31(3)** (2005), 333-340.

61. T. Yaying, A. Das, B. Hazarika and P. Baliarsingh: *Compactness of binomial difference operator of fractional order and sequence spaces*. Rend. Circ. Mat. Palermo **(2) 68(3)** (2019), 459–476.

62. T. Yaying and B. Hazarika: *On sequence spaces generated by binomial difference operator of fractional order*. Math. Slovaca **69(4)** (2019), 901-918.