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FINSLER SPACE SUBJECTED TO A KROPINA CHANGE WITH AN *h*-VECTOR

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Abstract. In this paper, we discuss the Finsler spaces (M^n, L) and $(M^n, {}^*L)$, where ${}^*L(x, y)$ is obtained from $L(x, y)$ by Kropina change ${}^*L(x, y) = \frac{L^2(x,y)}{b_i(x,y)\, y^i}$ and $b_i(x, y)$ is an h-vector in (M^n, L). We find the necessary and sufficient condition when the Cartan connection coefficients for both spaces (M^n, L) and $(M^n, {}^*L)$ are the same. We also find the necessary and sufficient condition for Kropina change with an h-vector to be projective.
Keywords: Finsler space, Kropina change, h-vector.

1. Introduction

In 1984, C. Shibata [16] dealt with a change of Finsler metric which is called a β-change of metric. A remarkable class of β-change is Kropina change ${}^*L(x, y) = \frac{L^2(x,y)}{b_i(x)\, y^i}$. If $L(x, y)$ is a metric function of a Riemannian space then ${}^*L(x, y)$ reduces to the metric function of a Kropina space. Kropina metric was first introduced by L. Berwald in connection with a two-dimensional Finsler space with rectilinear extremals and was investigated by V.K. Kropina [8, 9]. Kropina metric is the simplest non-trivial Finsler metric having many interesting applications in physics, electron optics with a magnetic field, plants study for fungal fusion hypothesis, dissipative mechanics and irreversible thermodynamics [1, 2, 3, 6]. In 1978, C. Shibata [15] studied some basic local geometric properties of Kropina spaces. In 1991, M. Matsumoto obtained a set of necessary and sufficient conditions for a Kropina space to be of constant curvature [12].

H. Izumi [7], while studying the conformal transformation of Finsler spaces, introduced the concept of h-vector b_i, which is v-covariant constant with respect to the Cartan connection and satisfies $L\, C^h_{ij}\, b_h = \rho\, h_{ij}$, where ρ is a non-zero scalar function, C^h_{ij} are components of Cartan tensor and h_{ij} are components of angular metric tensor. Thus if b_i is an h-vector then

$$(1.1) \qquad (i)\ b_i|_k = 0\,, \qquad\qquad (ii)\ L\, C^h_{ij}\, b_h = \rho\, h_{ij}\,.$$

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This gives

$$L\,\dot{\partial}_j b_i = \rho\, h_{ij}\,. \tag{1.2}$$

Since $\rho \neq 0$ and $h_{ij} \neq 0$, the h-vector b_i depends not only on positional coordinates but also on directional arguments. Izumi [7] showed that ρ is independent of directional arguments. M. Matsumoto [11] discussed the Cartan connection of Randers change of Finsler metric, while B. N. Prasad [14] obtained the Cartan connection of $(M^n, {}^*\!L)$ where ${}^*\!L(x,y)$ is given by ${}^*\!L(x,y) = L(x,y) + b_i(x,y)\,y^i$, and $b_i(x,y)$ is an h-vector. Present authors [4, 5] discussed the hypersurface of a Finsler space whose metric is given by certain transformations with an h-vector. In this paper we obtain the relation between the Cartan connections of $F^n = (M^n, L)$ and ${}^*\!F^n = (M^n, {}^*\!L)$ where ${}^*\!L(x,y)$ is obtained by the transformation

$${}^*\!L(x,y) = \frac{L^2(x,y)}{b_i(x,y)\,y^i}\,, \tag{1.3}$$

and $b_i(x,y)$ is an h-vector in (M^n, L).

The paper is organized as follows: In Section 2, we study how the fundamental metric tensor and the Cartan tensor change by Kropina change with an h-vector. The relation between the Cartan connection coefficients of both spaces is obtained in the Section 3 and we find the necessary and sufficient condition when these connection coefficients are the same. In Section 4, we find the necessary and sufficient condition for the Kropina change with an h-vector to be projective.

2. The Finsler space ${}^*F^n = (M^n, {}^*\!L)$

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with the fundamental function $L(x,y)$. We consider a change of Finsler metric ${}^*\!L(x,y)$ which is defined by (1.3) and have another Finsler space ${}^*\!F^n = (M^n, {}^*\!L)$. If we denote $b_i\,y^i$ by β then the indicatory property of h_{ij} yields $\dot{\partial}_i\beta = b_i$. Throughout this paper, the geometric objects associated with ${}^*\!F^n$ will be marked by the asterisk. We shall use the notation

$$L_i = \dot{\partial}_i L = l_i\,,\ L_{ij} = \dot{\partial}_i\,\dot{\partial}_j L\,,\ L_{ijk} = \dot{\partial}_k\,L_{ij}\,,\ldots$$

etc. From (1.3), we get

$${}^*\!L_i = 2\,\tau\,L_i - \tau^2\,b_i\,, \tag{2.1}$$

$${}^*\!L_{ij} = (2\tau - \rho\tau^2)\,L_{ij} + \frac{2\,\tau^2}{\beta}\,m_i\,m_j\,, \tag{2.2}$$

$$\begin{aligned}{}^*\!L_{ijk} =& (2\tau - \rho\tau^2)\,L_{ijk} + \frac{2\,\tau}{\beta}\,(\rho\tau - 1)\,(m_iL_{jk} + m_jL_{ik} + m_kL_{ij})\\ &- \frac{2\tau^2}{L\beta}(m_im_jl_k + m_jm_kl_i + m_km_il_j) - \frac{6\tau^2}{\beta^2}m_im_jm_k\,,\end{aligned} \tag{2.3}$$

where $\tau = L/\beta$, $m_i = b_i - \frac{1}{\tau} l_i$. The normalized supporting element, the metric tensor and Cartan tensor of ${}^*F^n$ are obtained as

$$ {}^*l_i = 2\,\tau l_i - \tau^2\, b_i\,, \tag{2.4} $$

$$ {}^*g_{ij} = (2\tau^2 - \rho\tau^3)\, g_{ij} + 3\tau^4\, b_i b_j - 4\tau^3 (l_i b_j + b_i l_j) + (4\tau^2 + \rho\tau^3) l_i l_j\,, \tag{2.5} $$

$$ {}^*C_{ijk} = (2\tau^2 - \rho\tau^3)\, C_{ijk} - \frac{\tau^2}{2\,\beta}(4 - 3\rho\tau)(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) - \frac{6\tau^2}{\beta} m_i m_j m_k\,. \tag{2.6} $$

For the computation of the inverse metric tensor, we use the following lemma [10]:

Lemma 2.1. *Let (m_{ij}) be a non-singular matrix and $l_{ij} = m_{ij} + n_i n_j$. The elements l^{ij} of the inverse matrix and the determinant of the matrix (l_{ij}) are given by*

$$ l^{ij} = m^{ij} - (1 + n_k n^k)^{-1}\, n^i n^j, \quad det(l_{ij}) = (1 + n_k n^k)\, det(m_{ij}) $$

respectively, where m^{ij} are elements of the inverse matrix of (m_{ij}) and $n^k = m^{ki} n_i$.

The inverse metric tensor of ${}^*F^n$ is derived as follows:

$$ \begin{aligned} {}^*g^{ij} = (2\tau^2 - \rho\tau^3)^{-1}\Big[g^{ij} - \frac{2\tau}{2b^2\tau - \rho} b^i b^j + \frac{4 - \rho\tau}{2b^2\tau - \rho}(l^i b^j + b^i l^j) \\ - \frac{3\rho b^2\tau^3 - \rho^2\tau^2 - 4b^2\tau^2 - 2\rho\tau + 8}{\tau(2b^2\tau - \rho)} l^i l^j \Big] \end{aligned} \tag{2.7} $$

where b is the magnitude of the vector $b^i = g^{ij} b_j$.

From (2.6) and (2.7), we get

$$ \begin{aligned} {}^*C^h_{ij} = C^h_{ij} - \frac{(4 - 3\rho\tau)\tau}{2L(2 - \rho\tau)}\Big(h_{ij}\, m^h + h^h_j\, m_i + h^h_i\, m_j\Big) - \frac{6\tau}{L(2 - \rho\tau)} m_i m_j m^h \\ + \frac{2\tau\, b^h - (4 - \rho\tau) l^h}{L(2 - \rho\tau)(2b^2\tau - \rho)}\Big[h_{ij}\Big\{\frac{1}{2} m^2 \tau(4 - 3\rho\tau) - \rho(2 - \rho\tau)\Big\} \\ + m_i m_j \Big\{6\tau\, m^2 + \tau(4 - 3\rho\tau)\Big\}\Big]. \end{aligned} \tag{2.8} $$

3. Cartan connection of the space ${}^*F^n$

Let $C\,{}^*\Gamma = ({}^*F^i_{jk}, {}^*N^i_j, {}^*C^i_{jk})$ be the Cartan connection for the space ${}^*F^n = (M^n, {}^*L)$. Since for a Cartan connection $L_{i|j} = 0$, we obtain

$$ \partial_j L_i = L_{ir} N^r_j + L_r F^r_{ij}\,. \tag{3.1} $$

Differentiating (2.1) with respect to x^j, we get

$$\partial_j {}^*L_i = 2\tau\, \partial_j L_i + 2L_i\, \partial_j\tau - \tau^2\, \partial_j b_i - 2\tau\, b_i\, \partial_j\tau\,. \tag{3.2}$$

This equation may be written in tensorial form as

$$\begin{aligned} {}^*L_{ir}\, {}^*N^r_j + {}^*L_r\, {}^*F^r_{ij} &= 2\tau\,(L_{ir}N^r_j + L_r F^r_{ij}) + \frac{2L_i}{\beta}(N^r_j L_r - \tau\beta_j - \tau N^r_j b_r) \\ &\quad - \tau^2(b_{i|j} + \rho L_{ir}N^r_j + b_r F^r_{ij}) - \frac{2\tau}{\beta}\, b_i(N^r_j L_r - \tau\beta_j - \tau N^r_j b_r)\,, \end{aligned} \tag{3.3}$$

where $\beta_j = \beta_{|j}$. If we put

$${}^*F^i_{jk} = F^i_{jk} + D^i_{jk}\,, \tag{3.4}$$

then in view of (2.2), equation (3.3) may be written as

$$(2\tau l_r - \tau^2 b_r)\, D^r_{ij} + \left\{(2\tau - \rho\tau^2)L_{ir} + \frac{2\tau^2}{\beta}\, m_i m_r\right\} D^r_{0j} = \frac{2\tau^2}{\beta}\, m_i\beta_j - \tau^2 b_{i|j}\,, \tag{3.5}$$

where the subscript ‘0’ denote the contraction by y^i.

In order to find the difference tensor D^i_{jk}, we construct supplementary equations to (3.5). From (2.2), we obtain

$$\begin{aligned} \partial_k {}^*L_{ij} &= (2\tau - \rho\tau^2)\, \partial_k L_{ij} + L_{ij}(2\partial_k\tau - 2\rho\tau\, \partial_k\tau - \tau^2 \partial_k\rho) \\ &\quad + \frac{2\,\tau^2}{\beta}\, m_i\, \partial_k m_j + \frac{2\,\tau^2}{\beta}\, m_j\, \partial_k m_i + 2m_i m_j \frac{1}{\beta^2}\Big(2\tau\beta\, \partial_k\tau - \tau^2\, \partial_k\beta\Big)\,. \end{aligned} \tag{3.6}$$

From $L_{ij|k} = 0$, equation (3.6) is written in the form

$$\begin{aligned} {}^*L_{ijr}\, {}^*N^r_k + {}^*L_{rj}\, {}^*F^r_{ik} + {}^*L_{ir}\, {}^*F^r_{jk} &= (2\tau - \rho\tau^2)\,\{L_{ijr}N^r_k + L_{rj}F^r_{ik} + L_{ir}F^r_{jk}\} \\ &\quad + L_{ij}\Big\{\frac{2}{\beta}(1 - \rho\tau)(-\tau\beta_k - \tau\, N^r_k m_r) - \tau^2\rho_k\Big\} \\ &\quad + \frac{2\tau^2}{\beta} m_i\Big\{F^r_{jk}m_r + N^r_k L_{jr}\Big(\rho - \frac{1}{\tau}\Big) - \frac{1}{\beta\tau} N^r_k l_j m_r\Big\} \\ &\quad + \frac{2\tau^2}{\beta} m_j\Big\{F^r_{ik}m_r + N^r_k L_{ir}\Big(\rho - \frac{1}{\tau}\Big) - \frac{1}{\beta\tau} N^r_k l_i m_r\Big\} \\ &\quad + \frac{2}{\beta^2} m_i m_j\Big\{2\tau(-\tau\beta_k - \tau\, N^r_k m_r) - \tau^2(\beta_k + N^r_k b_r)\Big\}, \end{aligned}$$

where $\rho_k = \rho_{|k} = \partial_k \rho$. In view of (2.2), (2.3) and (3.4), above equation is written as

$$(2\tau - \rho\tau^2)\{L_{ijr}D^r_{0k} + L_{rj}D^r_{ik} + L_{ir}D^r_{jk}\} + \frac{2\tau^2}{\beta} m_r(m_j D^r_{ik} + m_i D^r_{jk})$$

(3.7) $$- \frac{2\tau^2}{\beta^2}(m_i m_j b_r + m_j m_r b_i + m_r m_i b_j)D^r_{0k} + \frac{2\tau}{\beta}\beta_k L_{ij} - \frac{2\rho\tau^2}{\beta}\beta_k L_{ij}$$

$$+ \frac{2\tau^2}{\beta}(\rho - \frac{1}{\tau})(L_{jr}m_i + L_{ir}m_j + L_{ij}m_r)D^r_{0k} + \frac{6\tau^2}{\beta^2}\beta_k m_i m_j + \tau^2 \rho_k L_{ij} = 0.$$

Now we will prove:

Proposition 3.1. *The difference tensor D^i_{jk} is completely determined by the equations (3.5) and (3.7).*

To prove this, first we will prove a lemma:

Lemma 3.1. *The system of algebraic equations*

$$(i)\ {}^*L_{ir}A^r = B_i\,, \qquad (ii)\ {}^*L_r A^r = B\,,$$

has a unique solution A^r for given B and B_i.

Proof. It follows from (2.2) that (i) is written in the form

(3.8) $$\Big\{(2\tau - \rho\tau^2)\frac{1}{L}(g_{ir} - l_i l_r) + \frac{2\tau^2}{\beta} m_i m_r\Big\} A^r = B_i.$$

Contracting by b^i, we get

$$\Big\{(2\tau - \rho\tau^2)\frac{1}{L}\Big(b_r - \frac{1}{\tau} l_r\Big) + \frac{2\tau^2}{\beta} m^2\, m_r\Big\} A^r = B_\beta,$$

i.e.

(3.9) $$m_r\, A^r = B_\beta \Big(\frac{2\tau^2\, b^2}{\beta} - \frac{\rho\tau^2}{L}\Big)^{-1},$$

where the subscript β denote the contraction by b^i, i.e. $B_\beta = B_i b^i$.
Also from (2.1), equation (ii) is written in the form

$$(2\tau\, l_r - \tau^2 b_r)\, A^r = B$$

i.e.

(3.10) $$\tau^2 m_r A^r - \tau\, l_r A^r = -B.$$

Using (3.9) in (3.10), we get

$$l_r A^r = \tau^{-1}B + \tau B_\beta \Big(\frac{2\tau^2 b^2}{\beta} - \frac{\rho\tau^2}{L}\Big)^{-1}.$$

Then (3.8) is written as

$$g_{ir} A^r = \frac{L}{2\tau - \rho\tau^2} B_i + l_i \Big\{\tau^{-1}B + \tau B_\beta \Big(\frac{2\tau^2 b^2}{\beta} - \frac{\rho\tau^2}{L}\Big)^{-1}\Big\} - \frac{2\tau^2}{2-\rho\tau} m_i B_\beta \Big(\frac{2\tau^2 b^2}{\beta} - \frac{\rho\tau^2}{L}\Big)^{-1}.$$

This gives

(3.11)
$$A^i = \frac{L}{2\tau - \rho\tau^2} B^i + l^i \Big\{\tau^{-1}B + \tau B_\beta \Big(\frac{2\tau^2 b^2}{\beta} - \frac{\rho\tau^2}{L}\Big)^{-1}\Big\} - \frac{2\tau^2}{2-\rho\tau} m^i B_\beta \Big(\frac{2\tau^2 b^2}{\beta} - \frac{\rho\tau^2}{L}\Big)^{-1},$$

which is the concrete form of the solution A^i. □

We are now in a position to prove the proposition.
Taking the symmetric and anti-symmetric parts of (3.5), we get

(3.12)
$$\begin{aligned} &2(2\tau l_r - \tau^2 b_r)\, D^r_{ij} + \Big\{(2\tau - \rho\tau^2)L_{ir} + \frac{2\tau^2}{\beta} m_i m_r\Big\} D^r_{0j} \\ &\quad + \Big\{(2\tau - \rho\tau^2)L_{jr} + \frac{2\tau^2}{\beta} m_j m_r\Big\} D^r_{0i} = \frac{2\tau^2}{\beta}(m_i\beta_j + m_j\beta_i) - 2\tau^2 E_{ij}, \end{aligned}$$

and

(3.13)
$$\begin{aligned} &\Big\{(2\tau - \rho\tau^2)L_{ir} + \frac{2\tau^2}{\beta} m_i m_r\Big\} D^r_{0j} - \Big\{(2\tau - \rho\tau^2)L_{jr} + \frac{2\tau^2}{\beta} m_j m_r\Big\} D^r_{0i} \\ &\qquad = \frac{2\tau^2}{\beta}(m_i\beta_j - m_j\beta_i) - 2\tau^2 F_{ij}, \end{aligned}$$

where we put $2E_{ij} = b_{i|j} + b_{j|i}$ and $2F_{ij} = b_{i|j} - b_{j|i}$.

On the other hand, applying Christoffel process with respected to indices i, j, k in equation (3.7), we get

(3.14)
$$\begin{aligned} &(2\tau - \rho\tau^2)\Big\{L_{ijr}D^r_{0k} + L_{jkr}D^r_{0i} - L_{kir}D^r_{0j}\Big\} + 2D^r_{ik}\Big\{(2\tau - \rho\tau^2)L_{rj} + \frac{2\tau^2}{\beta} m_r m_j\Big\} \\ &- \frac{2\tau}{\beta}\Big\{\beta_k\Big((\rho\tau - 1)L_{ij} - \frac{3\tau}{\beta} m_i m_j\Big) + \beta_i\Big((\rho\tau - 1)L_{jk} - \frac{3\tau}{\beta} m_j m_k\Big) - \beta_j\Big((\rho\tau - 1)L_{ki} - \frac{3\tau}{\beta} m_k m_i\Big)\Big\} \\ &+ \frac{2\tau}{\beta} D^r_{0k}\, \mathfrak{S}_{(ijr)} m_i\Big\{(\rho\tau - 1)L_{jr} - \frac{\tau}{\beta} m_j b_r\Big\} + \frac{2\tau}{\beta} D^r_{0i}\, \mathfrak{S}_{(jkr)} m_j\Big\{(\rho\tau - 1)L_{kr} - \frac{\tau}{\beta} m_k b_r\Big\} \\ &- \frac{2\tau}{\beta} D^r_{0j}\, \mathfrak{S}_{(kir)} m_k\Big\{(\rho\tau - 1)L_{ir} - \frac{\tau}{\beta} m_i b_r\Big\} + \tau^2(\rho_k L_{ij} + \rho_i L_{jk} - \rho_j L_{ki}) = 0, \end{aligned}$$

where $\mathfrak{S}_{(ijk)}$ denote cyclic interchange of indices i, j, k and summation. Contracting (3.12) and (3.13) by y^j, we get

$$(4\tau\, l_r - 2\tau^2 b_r)D^r_{0i} + \Big\{(2\tau - \rho\tau^2)L_{ir} + \frac{2\tau^2}{\beta} m_i m_r\Big\}D^r_{00} = \frac{2\tau^2}{\beta}\beta_0 m_i - 2\tau^2 E_{i0}, \tag{3.15}$$

and

$$\Big\{(2\tau - \rho\tau^2)L_{ir} + \frac{2\tau^2}{\beta} m_i m_r\Big\}D^r_{00} = \frac{2\tau^2}{\beta}\beta_0 m_i - 2\tau^2 F_{i0},$$

i.e.

$$\overset{*}{L}_{ir} D^r_{00} = \frac{2\tau^2}{\beta}\beta_0 m_i - 2\tau^2 F_{i0}, \tag{3.16}$$

which on contraction by b^i gives

$$m_r\, D^r_{00} = \Big(\frac{2}{\beta}\beta_0 m^2 - 2F_{\beta 0}\Big)\Big(\frac{2}{\beta}b^2 - \frac{\rho}{L}\Big)^{-1},$$

where $\beta_0 = \beta_j y^j$. Similarly contraction of (3.14) by y^k gives

$$\begin{aligned}
&(2\tau - \rho\tau^2)\Big\{L_{ijr}D^r_{00} - L_{jr}D^r_{0i} + L_{ir}D^r_{0j}\Big\} + 2D^r_{0i}\Big\{(2\tau - \rho\tau^2)L_{rj} + \frac{2\tau^2}{\beta} m_r m_j\Big\} \\
&+ \frac{2\tau}{\beta} D^r_{00}\, \mathfrak{S}_{(ijr)} m_i \Big\{(\rho\tau - 1)L_{jr} - \frac{\tau}{\beta} m_j b_r\Big\} - \frac{2\tau^2}{\beta} D^r_{0i}\, m_j m_r + \frac{2\tau^2}{\beta} D^r_{0j}\, m_i m_r \\
&- \frac{2\tau}{\beta}\beta_0\Big((\rho\tau - 1)L_{ij} - \frac{3\tau}{\beta} m_i m_j\Big) + \tau^2 \rho_0 L_{ij} = 0\,,
\end{aligned} \tag{3.17}$$

Contraction of (3.15) by y^i gives

$$(2\tau\, l_r - \tau^2 b_r)D^r_{00} = -\tau^2 E_{00},$$

i.e.

$$\overset{*}{L}_r D^r_{00} = E_{00}. \tag{3.18}$$

We can apply Lemma 3.1 to equations (3.16) and (3.18) to obtain

$$\begin{aligned}
D^i_{00} =& l^i\, \tau \Big\{\Big(\frac{2}{\beta}\beta_0 m^2 - 2F_{\beta 0}\Big)\Big(\frac{2}{\beta}b^2 - \frac{\rho}{L}\Big)^{-1} - E_{00}\Big\} \\
&- \frac{2\tau^2}{2 - \rho\tau}\Big(\frac{2}{\beta}\beta_0 m^2 - 2F_{\beta 0}\Big)\Big(\frac{2}{\beta}b^2 - \frac{\rho}{L}\Big)^{-1} m^i + \frac{2L\tau}{2 - \rho\tau}\Big(\frac{1}{\beta}\,\beta_0\, m^i - F^i_0\Big),
\end{aligned} \tag{3.19}$$

where $F^i_0 = g^{ij}F_{j0}$. Also note that

$$E_{00} = E_{ij}\, y^i\, y^j = b_{i|j}\, y^i\, y^j = (b_i\, y^i)_{|j} y^j = \beta_{|0} = \beta_0.$$

Now adding (3.13) and (3.17), we obtain

$$D^r_{0j}\Big\{(2\tau-\rho\tau^2)L_{ir}+\frac{2\tau^2}{\beta}m_im_r\Big\}=G_{ij}\,,$$

i.e.

(3.20) $$\overset{*}{L}_{ir}\,D^r_{0j}=G_{ij}\,,$$

where we put

(3.21) $$\begin{aligned}G_{ij}=&\frac{\tau^2}{\beta}(m_i\beta_j-m_j\beta_i)-\tau^2F_{ij}-\frac{1}{2}(2\tau-\rho\tau^2)L_{ijr}D^r_{00}-\frac{\tau^2}{2}\rho_0L_{ij}\\&-\frac{\tau}{\beta}D^r_{00}\,\mathfrak{S}_{(ijr)}m_i\Big\{(\rho\tau-1)L_{jr}-\frac{\tau}{\beta}m_jb_r\Big\}+\frac{\tau}{\beta}\beta_0\Big\{(\rho\tau-1)L_{ij}-\frac{3\tau}{\beta}m_im_j\Big\}.\end{aligned}$$

The equation (3.15) is written in the form

$$(2\tau\,l_r-\tau^2\,b_r)D^r_{0j}=G_j\,,$$

i.e.

(3.22) $$\overset{*}{L}_rD^r_{0j}=-\frac{1}{\tau^2}\,G_j\,,$$

where

$$G_j=\frac{\tau^2}{\beta}\,\beta_0m_j-\tau^2E_{j0}-\Big\{\frac{1}{2}(2\tau-\rho\tau^2)L_{jr}+\frac{\tau^2}{\beta}m_jm_r\Big\}\,D^r_{00}.$$

In view of (3.16), G_j are written as

(3.23) $$G_j=\tau^2(F_{j0}-E_{j0})\,.$$

Applying Lemma 3.1 to equations (3.20) and (3.22) to obtain

(3.24) $$D^i_{0j}=l^i\Big\{\frac{1}{\tau}\Big(\frac{2}{\beta}b^2-\frac{\rho}{L}\Big)^{-1}G_{\beta j}+\frac{1}{\tau}G_j\Big\}-\frac{2\,m^i}{2-\rho\tau}\Big(\frac{2}{\beta}b^2-\frac{\rho}{L}\Big)^{-1}G_{\beta j}+\frac{L}{2\tau-\rho\tau^2}G^i_j,$$

where $G^i_j=g^{ik}G_{kj}$.

Finally we solve (3.12) and (3.14) for $D^{\,i}_{jk}$. These equations may be written as

(3.25) $$\overset{*}{L}_{rj}\,D^r_{ik}=H_{jik}\,,$$

and

(3.26) $$\overset{*}{L}_r\,D^r_{ik}=H_{ik}\,,$$

where

(3.27)

$$\begin{aligned}H_{jik} &= \frac{(\rho\tau^2-2\tau)}{2}\left\{L_{ijr}D^r_{0k}+L_{jkr}D^r_{0i}-L_{kir}D^r_{0j}\right\}\\&-\frac{\tau}{\beta}D^r_{0k}\,\mathfrak{S}_{(ijr)}m_i\Big\{(\rho\tau-1)L_{jr}-\frac{\tau}{\beta}m_jb_r\Big\}-\frac{\tau}{\beta}D^r_{0i}\,\mathfrak{S}_{(jkr)}m_j\Big\{(\rho\tau-1)L_{kr}-\frac{\tau}{\beta}m_kb_r\Big\}\\&+\frac{\tau}{\beta}D^r_{0j}\,\mathfrak{S}_{(kir)}m_k\Big\{(\rho\tau-1)L_{ir}-\frac{\tau}{\beta}m_ib_r\Big\}-\frac{\tau^2}{2}(\rho_kL_{ij}+\rho_iL_{jk}-\rho_jL_{ki})\\&+\frac{\tau}{\beta}\Big\{\beta_k\Big((\rho\tau-1)L_{ij}-\frac{3\tau}{\beta}m_im_j\Big)+\beta_i\Big((\rho\tau-1)L_{jk}-\frac{3\tau}{\beta}m_jm_k\Big)\\&-\beta_j\Big((\rho\tau-1)L_{ki}-\frac{3\tau}{\beta}m_km_i\Big)\Big\},\end{aligned}$$

and

$$H_{ik}=\frac{\tau^2}{\beta}(m_i\beta_k+m_k\beta_i)-\tau^2E_{ik}-\frac{1}{2}(G_{ik}+G_{ki})\,. \tag{3.28}$$

Again applying Lemma 3.1 to equations (3.25) and (3.26) to obtain

$$D^j_{ik}=l^j\Big\{\frac{1}{\tau}\Big(\frac{2}{\beta}b^2-\frac{\rho}{L}\Big)^{-1}H_{\beta ik}+\frac{1}{\tau}H_{ik}\Big\}-\frac{2\,m^j}{2-\rho\tau}\Big(\frac{2}{\beta}b^2-\frac{\rho}{L}\Big)^{-1}H_{\beta ik}+\frac{L}{2\tau-\rho\tau^2}H^{\ j}_{ik}\,, \tag{3.29}$$

where we put $H^{\ j}_{ik}=g^{jm}H_{mik}$. This completes the Proposition 3.1.

We now propose a lemma:

Lemma 3.2. *If the h-vector is gradient then the scalar ρ is constant.*

Proof. Taking h-covariant derivative of (1.2) and using $L_{|k}=0$ and $h_{ij|k}=0$, we get

$$L\,(\dot\partial_jb_i)_{|k}=\rho_{|k}\,h_{ij}\,.$$

Utilizing the commutation formula exhibited by

$$\dot\partial_k(T^i_{j|h})-(\dot\partial_k\,T^i_j)_{|h}=T^r_j\,\dot\partial_kF^i_{rh}-T^i_r\,\dot\partial_kF^r_{jh}-(\dot\partial_rT^i_j)C^r_{hk|0}\,;$$

we get

$$2L\,\dot\partial_jF_{ik}=\rho_{|k}\,h_{ij}-\rho_{|i}\,h_{jk}\,.$$

If b_i is a gradient vector, i.e. $2F_{ij}=b_{i|j}-b_{j|i}=0$. Then above equation becomes

$$\rho_{|k}\,h_{ij}-\rho_{|i}\,h_{jk}=0$$

which after contraction by y^k gives $\rho_{|k}\,y^k=0$. Differentiating $\rho_{|k}\,y^k=0$ partially with respect to y^j, and using the commutation formula $\dot\partial_j(\rho_{|k})-(\dot\partial_j\rho)_{|k}=-(\dot\partial_r\rho)C^r_{jk|0}$ and the fact that ρ is a function of position only, we get $\rho_{|j}=0$ and therefore $\partial_j\rho=0$. This completes the proof. □

Now, we find the condition for which the Cartan connection coefficients for both spaces F^n and $^*F^n$ are the same, i.e. $^*F^i_{jk} = F^i_{jk}$ then $D^i_{jk} = 0$. Therefore (3.15) and (3.16) gives $E_{i0} = F_{i0}$. This will give

$$b_{0|i} = 0\,, \tag{3.30}$$

i.e. $\beta_{|i} = 0$. Differentiating $\beta_{|i} = 0$ partially with respect to y^j, and using the commutation formula $\dot\partial_j(\beta_{|i}) - (\dot\partial_j\beta)_{|i} = -(\dot\partial_r\beta)C^r_{ij|0}$, we get

$$b_{j|i} = b_r\, C^r_{ij|0}. \tag{3.31}$$

This gives $F_{ij} = 0$ and then in view of Lemma 3.2, $F_{ij} = 0$ implies $\rho_i = \rho_{|i} = 0$. Taking h-covariant derivative of (1.1)(ii) and using $L_{|k} = 0$, $\rho_{|k} = 0$ and $h_{ij|k} = 0$, we get $(b_r\, C^r_{ij})_{|k} = \left(\frac{\rho}{L}\, h_{ij}\right)_{|k} = 0$. This gives

$$b_{r|k}\, C^r_{ij} + b_r\, C^r_{ij|k} = 0\,.$$

From (3.31), we get $b_{r|k} = b_{k|r}$, then above equation becomes

$$b_{k|r}\, C^r_{ij} + b_r\, C^r_{ij|k} = 0\,.$$

Contracting by y^k, we get $b_{0|r}\, C^r_{ij} + b_r\, C^r_{ij|0} = 0$. Using (3.30) and (3.31), this gives $b_{i|j} = 0$, i.e. the h-vector b_i is parallel with respect to the Cartan connection of F^n.

Conversely, if $b_{i|j} = 0$ then we get $E_{ij} = 0 = F_{ij}$ and $\beta_i = \beta_{|i} = b_{j|i}y^j = 0$. In view of Lemma 3.2, $F_{ij} = 0$ implies $\rho_i = \rho_{|i} = 0$. Therefore from (3.19) we get $D^i_{00} = 0$ and then $G_{ij} = 0$ and $G_j = 0$. This gives $D^i_{0j} = 0$ and then $H_{jik} = 0$ and $H_{ik} = 0$. Therefore (3.29) implies $D^i_{jk} = 0$ and then $^*F^i_{jk} = F^i_{jk}$. Thus, we have:

Theorem 3.1. *For the Kropina change with an h-vector, the Cartan connection coefficients for both spaces F^n and $^*F^n$ are the same if and only if the h-vector b_i is parallel with respect to the Cartan connection of F^n.*

Transvecting (3.4) by y^j and using $F^i_{jk}\, y^j = G^i_k$, we get

$$^*G^i_k = G^i_k + D^i_{0k}\,. \tag{3.32}$$

Further transvecting (3.32) by y^k and using $G^i_k\, y^k = 2\,G^i$, we get

$$2\,^*G^i = 2\,G^i + D^i_{00}\,. \tag{3.33}$$

Differentiating (3.32) partially with respect to y^h and using $\dot\partial_h G^i_k = G^i_{kh}$, we have

$$^*G^i_{kh} = G^i_{kh} + \dot\partial_h D^i_{0k}\,, \tag{3.34}$$

where G^i_{kh} are the Berwald connection coefficients.

Now, if the h-vector b_i is parallel with respect to the Cartan connection of F^n, then by Theorem 3.1, the Cartan connection coefficients for both spaces F^n and $^*F^n$ are the same, therefore $D^i_{jk} = 0$. Then from (3.34), we get $^*G^i_{kh} = G^i_{kh}$.

Thus, we have:

Theorem 3.2. *For the Kropina change with an h-vector, if the h-vector b_i is parallel with respect to the Cartan connection of F^n. Then the Berwald connection coefficients for both the spaces F^n and $^*F^n$ are the same.*

4. Relation between Projective change and Kropina change with an h-vector

We consider two Finsler spaces $F^n = (M^n, L)$ and $^*F^n = (M^n, {}^*L)$. If any geodesic on F^n is also a geodesic on $^*F^n$ and the inverse is true, the change $L \to {}^*L$ of the metric is called *projective*. A geodesic on F^n is given by

$$\frac{dy^i}{dt} + 2\,G^i(x, y) = \tau\, y^i\,; \qquad \tau = \frac{d^2s/dt^2}{ds/dt}.$$

The change $L \to {}^*L$ is a projective change if and only if there exists a scalar $P(x, y)$ which is positively homogeneous of degree one in y^i and satisfies [13]

$${}^*G^i(x, y) = G^i(x, y) + P(x, y)\, y^i\,.$$

Now, we find condition for the Kropina change (1.3) with h-vector to be projective. From (3.33), it follows that the Kropina change with an h-vector is projective if and only if $D^i_{00} = 2\,P\,y^i$. Then from (3.19), we get

$$\begin{aligned} 2\,P\,y^i =& l^i\,\tau\,\Big\{\Big(\frac{2}{\beta}\beta_0 m^2 - 2F_{\beta 0}\Big)\Big(\frac{2}{\beta}b^2 - \frac{\rho}{L}\Big)^{-1} - E_{00}\Big\} \\ &- \frac{2\tau^2}{2-\rho\tau}\Big(\frac{2}{\beta}\beta_0 m^2 - 2F_{\beta 0}\Big)\Big(\frac{2}{\beta}b^2 - \frac{\rho}{L}\Big)^{-1} m^i + \frac{2L\tau}{2-\rho\tau}\Big(\frac{1}{\beta}\,\beta_0\, m^i - F^i_0\Big). \end{aligned} \tag{4.1}$$

Contracting (4.1) by y_i and using $m^i\,y_i = 0 = F^i_0\,y_i$, we get

$$2\,P\,L^2 = \tau\,\Big\{\Big(\frac{2}{\beta}\beta_0 m^2 - 2F_{\beta 0}\Big)\Big(\frac{2}{\beta}b^2 - \frac{\rho}{L}\Big)^{-1} - E_{00}\Big\}\,L\,, \quad \text{i.e.,}$$

$$P = \frac{\tau}{2\,L}\,\Big\{\Big(\frac{2}{\beta}\beta_0 m^2 - 2F_{\beta 0}\Big)\Big(\frac{2}{\beta}b^2 - \frac{\rho}{L}\Big)^{-1} - E_{00}\Big\}. \tag{4.2}$$

Putting the value of P in (4.1), we get

$$-\frac{2\tau^2}{2-\rho\tau}\Big(\frac{2}{\beta}\beta_0 m^2 - 2F_{\beta 0}\Big)\Big(\frac{2}{\beta}b^2 - \frac{\rho}{L}\Big)^{-1} m^i + \frac{2L\tau}{2-\rho\tau}\Big(\frac{1}{\beta}\,\beta_0\, m^i - F^i_0\Big) = 0,$$

i.e.,

$$F_0^i = \frac{\beta_0}{\beta} m^i - \frac{1}{\beta} m_r D_{00}^r m^i.$$

Transvecing by g_{ij}, we get

$$F_{i0} = \frac{\beta_0}{\beta} m_i - \frac{1}{\beta} m_r D_{00}^r m_i \,. \tag{4.3}$$

Using (4.3) in (3.16), and referring $2\tau - \rho\tau^2 \neq 0$, we get $L_{ir} D_{00}^r = 0$, which transvecting by m^i and using $L_{ir} m^i = \frac{1}{L} m_r$, we get $m_r D_{00}^r = 0$, and then (4.3) becomes

$$F_{i0} = \frac{\beta_0}{\beta} m_i \,. \tag{4.4}$$

This equation (4.4) is a necessary condition for the Kropina change with an h-vector to be a projective change.

Conversely, if (4.4) satisfies, then (3.16) gives

$$\left\{(2\tau - \rho\tau^2)L_{ir} + \frac{2\tau^2}{\beta} m_i m_r\right\} D_{00}^r = 0.$$

Transvecting by m^i and referring $\frac{(2\tau-\rho\tau^2)}{L} + \frac{2\tau^2}{\beta} m^2 \neq 0$, we get $m_r D_{00}^r = 0$ and then (3.19) gives $D_{00}^i = -E_{00}\,\tau\, l^i$. Therefore ${}^*F^n$ is projective to F^n. Thus, we have:

Theorem 4.1. *The Kropina change (1.3) with an h-vector is projective if and only if the condition (4.4) is satisfied.*

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REFERENCES

[1] P. L. Antonelli, R. S. Ingarden and M. Matsumoto, *The theory of Sprays and Finsler spaces with applications in Physics and Biology*, Kluwer Acad. Publishers, Dordrecht / Boston / London, 1993.

[2] P. L. Antonelli, S. F. Rutz and K. T. Fonseca, The mathematical theory of endosymbiosis, II: Models of the Fungal Fusion hypothesis, *Nonlinear Analysis RWA*, **13** (2012) 2096-2103.

[3] G. S. Asanov, *Finsler Geometry, Relativity and Gauge Theories* D. Reidel Publishing Company, Dordrecht, Holland, 1985.

[4] M. K. Gupta and P. N. Pandey, On hypersurface of a Finsler space with a special metric, *Acta Math. Hungar.* **120(1-2)** (2008), 165-177.

[5] M. K. Gupta and P. N. Pandey, Hypersurfaces of conformally and h-conformally related Finsler spaces, *Acta Math. Hungar.* **123(3)** (2009), 257-264.

[6] R. S. Ingarden, *Geometry of thermodynamics* Diff. Geom. Methods in Ther. Phys. (ed. H.D. Doebner et al.), XV Intern. Conf. Clausthal 1986, World Scientific, Singapore, 1987.

[7] H. Izumi, Conformal transformations of Finsler spaces II. An h-conformally flat Finsler space, *Tensor N.S.* **34** (1980), 337-359.

[8] V. K. Kropina, On projective Finsler spaces with a certain special form, *Naučn Doklady vyss. Skoly, fiz.-mat. Nauki* 1959(2) (1960), 38-42 (in Russian).

[9] V. K. Kropina, On projective two-dimensional Finsler spaces with special metric, *Trudy Sem. Vektor. Tenzor. Anal.* **11** (1961), 277-292 (in Russian).

[10] M. Matsumoto, On C-reducible Finsler spaces, *Tensor N.S.* **24** (1972), 29-37.

[11] M. Matsumoto, On Finsler spaces with Randers metric and special forms of special tensors, *J. Math. Kyoto Univ.* **14-3** (1974), 477-498.

[12] M. Matsumoto, Finsler spaces of constant curvature with Kropina metric, *Tensor N.S.* **50** (1991), 194-201.

[13] M. Matsumoto, Theory of Finsler spaces with (α,β)-metric, *Rep. Math. Phys.* **31-1** (1992), 43-83.

[14] B.N. Prasad, On the torsion tensors R_{hjk} and P_{hjk} of Finsler spaces with a metric $ds = (g_{ij}(dx)\,dx^i\,dx^j)^{1/2} + b_i(x,y)\,dx^i$, *Indian J. pure appl. Math.* **21-1** (1990), 27-39.

[15] C. Shibata, On Finsler spaces with Kropina metric, *Rep. Math. Phys.* **13** (1978), 117-128.

[16] C. Shibata, On invariant tensors of β-changes of Finsler metric, *J. Math. Kyoto Univ.* **24-1** (1984), 163-188.

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