FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 30, No 3 (2015), 295–324

GROWTH ANALYSIS OF ENTIRE FUNCTIONS CONCERNING GENERALIZED RELATIVE TYPE AND GENERALIZED RELATIVE WEAK TYPE

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Abstract. In the paper we study some comparative growth properties of composite entire functions on the basis of generalized relative order, generalized relative type and generalized relative weak type with respect to other entire functions.

Keywords: Entire function, maximum term, maximum modulus, composition, growth, generalized relative order, generalized relative type, generalized relative weak type

1. Introduction, Definitions and Notations

Let f be an entire function defined in the open complex plane $\mathbb{C}$. For entire $f = \sum\limits_{n=0}^{\infty} a_n z^n$ on $|z| = r$, the maximum term denoted as $\mu_f(r)$ and the maximum modulus symbolized as $M_f(r)$ are respectively defined as $\max\limits_{n\geq 0}(|a_n| r^n)$ and $\max\limits_{|z|=r}\left|f(z)\right|$. If f is non-constant entire then $M_f(r)$ is strictly increasing and continuous and therefore there exists its inverse function $M_f^{-1} : \left(\left|f(0)\right|, \infty\right) \rightarrow (0, \infty)$ with $\lim\limits_{s\rightarrow\infty} M_f^{-1}(s) = \infty$. Similarly, $\mu_f^{-1}(r)$ is also a increasing function of r. Moreover for another entire function g, $M_g(r)$ along with $\mu_g(r)$ are too defined and the ratios $\frac{M_f(r)}{M_g(r)}$ when $r \rightarrow \infty$ as well as $\frac{\mu_f(r)}{\mu_g(r)}$ as $r \rightarrow \infty$ are called the comparative growth of f with respect to g in terms of their maximum moduli and the maximum term, respectively. This study of comparative growth properties of entire functions under some different directions is the prime concern of the paper. Our notations are standard within the theory of Nevanlinna's value distribution of entire functions and therefore we do not explain those in detail as available in [14]. In the sequel the following two

Received February 01, 2014; Accepted April 08, 2015
2010 *Mathematics Subject Classification.* Primary 30D35; Secondary 30D30, 30D20

notations are used:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots;$$
$$\log^{[0]} x = x$$

and

$$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots;$$
$$\exp^{[0]} x = x.$$

To start our paper we just recall the following definitions.

Definition 1.1. [10] The *generalized order* $\rho_f^{[l]}$ (respectively, *generalized lower order* $\lambda_f^{[l]}$) of an entire function f is defined as

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log\log M_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r}$$

$$\left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log\log M_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r}\right)$$

where $l \geq 1$.

These definitions extend the definitions of *order* ρ_f and *lower order* λ_f of an entire function f which are classical in complex analysis for integer $l = 2$ since these correspond to the particular cases $\rho_f^{[2]} = \rho_f(2,1) = \rho_f$ and $\lambda_f^{[2]} = \lambda_f(2,1) = \lambda_f$.

Using the inequality

$$\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R) \ \ \{cf. [12]\} \text{ for } 0 \leq r < R,$$

the growth marker ρ_f (respectively λ_f) and consequently $\rho_f^{[l]}$ (respectively $\lambda_f^{[l]}$) are reformulated as:

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log r} \left(\text{respectively } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log r}\right)$$

and

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} \mu_f(r)}{\log r} \left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} \mu_f(r)}{\log r}\right),$$

where $l \geq 1$.

Definition 1.2. The *generalized type* $\sigma_f^{[l]}$ and *generalized lower type* $\overline{\sigma}_f^{[l]}$ of an entire function f are defined as

$$\sigma_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\rho_f}},\ 0 < \rho_f^{[l]} < \infty,$$

where $l \geq 1$. Moreover, when $l = 2$ then $\sigma_f^{[2]}$ and $\overline{\sigma}_f^{[2]}$ are correspondingly denoted as σ_f and $\overline{\sigma}_f$ which are respectively known as *type* and *lower type* of entire f.

Similarly, extending the notion of *weak type* as introduced by Datta and Jha [4], one can define *generalized weak type* to determine the relative growth of two entire functions having same non zero finite *generalized lower order* in the following manner:

Definition 1.3. The *generalized weak type* $\tau_f^{[l]}$ for $l \geq 1$ of an entire function f of finite positive *generalized lower order* $\lambda_f^{[l]}$ are defined by

$$\tau_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\lambda_f}},\ 0 < \lambda_f^{[l]} < \infty.$$

Also one may define the growth indicator $\overline{\tau}_f^{[l]}$ of an entire function f in the following way :

$$\overline{\tau}_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\lambda_f}}\ ,\ 0 < \lambda_f^{[l]} < \infty.$$

For $l = 2$, the above definition reduces to the classical definition as established by Datta and Jha [4]. Also τ_f and $\overline{\tau}_f$ are stand for $\tau_f^{[2]}$ and $\overline{\tau}_f^{[2]}$.

For any two entire functions f and g, Bernal [1], [2] initiated the definition of relative order of f with respect to g, indicated by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r},\end{aligned}$$

which keeps away from comparing growth just with $\exp z$ to find out *order* of entire functions as we see in the earlier and of course this definition corresponds with the classical one [13] for $g = \exp z$.

Analogously, one may define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}\ .$$

In the case of relative order, it therefore seems reasonable to state suitably an alternative definition of relative order of entire function in terms of its maximum terms. Datta and Maji [6] introduced such a definition in the following approach:

Definition 1.4. [5] The relative order $\rho_g(f)$ and the relative lower order $\lambda_g(f)$ of an entire function f with respect to another entire function g are defined as follows:

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r} \text{ and } \lambda_g(f) = \liminf_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r} .$$

To compare the relative growth of two entire functions having same non zero finite relative order with respect to another entire function, Roy [9] recently introduced the notion of relative type of two entire functions in the following manner:

Definition 1.5. [9] Let f and g be any two entire functions such that $0 < \rho_g(f) < \infty$. Then the *relative type* $\sigma_g(f)$ of f with respect to g is defined as:

$$\begin{aligned}\sigma_g(f) &= \inf\left\{k > 0 : M_f(r) < M_g\left(kr^{\rho_g(f)}\right) \text{ for all sufficiently large values of } r\right\}\\ &= \limsup_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}}.\end{aligned}$$

Likewise one can define the *relative lower type* of an entire function f with respect to an entire function g denoted by $\overline{\sigma}_g(f)$ as follows :

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}},\ 0 < \rho_g(f) < \infty .$$

Analogously to determine the relative growth of two entire functions having same non zero finite relative lower order with respect to another entire function, Datta and Biswas [6] introduced the definition of relative weak type of an entire function f with respect to another entire function g of finite positive relative lower order $\lambda_g(f)$ in the following way:

Definition 1.6. [6] The relative weak type $\tau_g(f)$ of an entire function f with respect to another entire function g having finite positive relative lower order $\lambda_g(f)$ is defined as:

$$\tau_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\lambda_g(f)}} .$$

Also one may define the growth indicator $\overline{\tau}_g(f)$ of an entire function f with respect to an entire function g in the following way :

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{M_g^{-1}M_f(r)}{r^{\lambda_g(f)}},\ 0 < \lambda_g(f) < \infty .$$

Considering $g = \exp z$ one may easily verify that Definition 1.4 and Definition 1.5 coincide with the classical type (lower type) and weak type respectively.

Lahiri and Banerjee [8] gave a more generalized concept of relative order in the following way:

Definition 1.7. [8] If $l \geq 1$ is a positive integer, then the l- th *generalized relative order* of f with respect to g, denoted by $\rho_g^{[l]}(f)$ is defined by

$$\begin{aligned}\rho_g^{[l]}(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g\left(\exp^{[l-1]} r^{\mu}\right) \text{ for all } r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r}.\end{aligned}$$

Clearly $\rho_g^{[1]}(f) = \rho_g(f)$ and $\rho_{\exp z}^{[1]}(f) = \rho_f$.

Likewise one can define the *generalized relative lower* order of f with respect to g denoted by $\lambda_g^{[l]}(f)$ as

$$\lambda_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r}.$$

In terms of maximum terms of entire functions Definition 1.2 can be reformulated as:

Definition 1.8. For any positive integer $l \geq 1$, the growth indicators $\rho_g^{[l]}(f)$ and $\lambda_g^{[l]}(f)$ for an entire function f are defined as:

$$\rho_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l]} \mu_g^{-1} \mu_f(r)}{\log r} \text{ and } \lambda_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l]} \mu_g^{-1} \mu_f(r)}{\log r}.$$

In fact, Lemma 2.7 states the equivalence of Definition 1.7 and Definition 1.8.

Now to compare the relative growth of two entire functions having same non zero finite *generalized relative order* with respect to another entire function, we intend to give the definition of *generalized relative type* and *generalized relative lower type* of an entire function with respect to another entire function which are as follows :

Definition 1.9. The *generalized relative type* $\sigma_f^{[l]}$ and *generalized relative lower type* $\overline{\sigma}_f^{[l]}$ of an entire function f are defined as

$$\begin{aligned}\sigma_g^{[l]}(f) &= \limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}} \text{ and} \\ \overline{\sigma}_g^{[l]}(f) &= \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\rho_g^{[l]}(f)}},\ 0 < \rho_g^{[l]}(f) < \infty.\end{aligned}$$

For $l = 2$, Definition 1.9 reduces to Definition 1.5.

Similarly, to determine the relative growth of two entire functions having same non zero finite *generalized relative lower order* with respect to another entire function, one may introduce the concepts of *generalized relative weak type* of an entire function with respect to another entire function in the following manner:

Definition 1.10. The *generalized relative weak type* $\tau_g^{[l]}(f)$ of an entire function f with respect to another entire function g having finite positive *generalized relative lower order* $\lambda_g^{[l]}(f)$ is defined as:

$$\tau_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}} .$$

Further one may define the growth indicator $\overline{\tau}_g^{[l]}(f)$ of an entire function f with respect to an entire function g in the following way :

$$\overline{\tau}_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l-1]} M_g^{-1} M_f(r)}{r^{\lambda_g^{[l]}(f)}},\ 0 < \lambda_g^{[l]}(f) < \infty .$$

Definition 1.10 also reduces to Definition 1.6 for particular $l = 2$.

The notions of the growth indicators of entire functions such as *order*, *type* and *weak type* are classical in complex analysis and during the past decades, several researchers have already been exploring their research in the area of comparative growth properties of composite entire functions in different directions using the classical growth indicators. But at that time, the concepts of *relative orders* as well as *generalized relative orders* and consequently the *generalized relative type* and *generalized relative weak type* of entire functions are not at all known to the researchers of this area. In this paper, we establish some newly developed results related to the growth properties of composite entire functions on the basis of their *generalized relative orders*, *generalized relative type* and *generalized relative weak type.*

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [3] *If f and g are any two entire functions then for all sufficiently large values of r,*

$$M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - \left|g(0)\right|\right) \le M_{f\circ g}(r) \le M_f\left(M_g(r)\right).$$

Lemma 2.2. [11] *Let f and g be any two entire functions Then for every $\alpha > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \le \frac{\alpha}{\alpha - 1}\mu_f\left(\frac{\alpha R}{R - r}\mu_g(R)\right).$$

Lemma 2.3. [11] *If f and g are any two entire functions with $g(0) = 0$, then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \ge \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right) - \left|g(0)\right|\right).$$

Lemma 2.4. [2] *Suppose f is an entire function and $\alpha > 1$, $0 < \beta < \alpha$. Then for all sufficiently large r,*

$$M_f(\alpha r) \geq \beta M_f(r).$$

Lemma 2.5. [5] *If f be an entire function and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large r,*

$$\mu_f(\alpha r) \geq \beta \mu_f(r) .$$

Lemma 2.6. [7] *Let f and h be any two entire functions Then for any $\alpha > 1$,*

$$(i) \quad M_h^{-1} M_f(r) \leq \mu_h^{-1}\left[\frac{\alpha}{(\alpha-1)}\mu_f(\alpha r)\right] \text{ and}$$

$$(ii) \quad \mu_h^{-1}\mu_f(r) \leq \alpha M_h^{-1}\left[\frac{\alpha}{(\alpha-1)} M_f(r)\right].$$

Lemma 2.7. *Definition 1.7 and Definition 1.8 are equivalent.*

Proof. For any $\alpha > 1$ and $l \geq 1$, we get from Lemma 2.5 and the first part of Lemma 2.6 that

$$M_h^{-1} M_f(r) \leq \mu_h^{-1}\mu_f\left[\frac{(2\alpha-1)\,\alpha}{(\alpha-1)} r\right] .$$

Thus from above we get that

$$\begin{aligned} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r} &\leq \frac{\log^{[l]} \mu_h^{-1}\mu_f\left[\frac{(2\alpha-1)\alpha}{(\alpha-1)} r\right]}{\log r} \\ i.e., \ \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r} &\leq \frac{\log^{[l]} \mu_h^{-1}\mu_f\left[\frac{(2\alpha-1)\alpha}{(\alpha-1)} r\right]}{\log\left[\frac{(2\alpha-1)\alpha}{(\alpha-1)} r\right] + O(1)} \end{aligned}$$

$$\begin{aligned} i.e.\ \rho_g^{[l]}(f) &= \limsup_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\mu_f\left[\frac{(2\alpha-1)\alpha}{(\alpha-1)} r\right]}{\log[r] + O(1)} \end{aligned}$$

$$i.e., \ \rho_g^{[l]}(f) \leq \limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\mu_f(r)}{\log r} \tag{2.1}$$

and accordingly

$$\lambda_g^{[l]}(f) \leq \liminf_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\mu_f(r)}{\log r} . \tag{2.2}$$

Similarly, in view of Lemma 2.4 it follows from the second part of Lemma 2.5 that

$$\mu_h^{-1}\mu_f(r) \leq \alpha M_h^{-1} M_f\left[\left(\frac{2\alpha-1}{\alpha-1}\right)r\right]$$

and from above we obtain that

$$\begin{aligned} \frac{\log^{[l]}\mu_h^{-1}\mu_f(r)}{\log r} &\leq \frac{\log^{[l]}\alpha M_h^{-1}M_f\left[\left(\frac{2\alpha-1}{\alpha-1}\right)r\right]}{\log r} \\ i.e., \ \frac{\log^{[l]}\mu_h^{-1}\mu_f(r)}{\log r} &\leq \frac{\log^{[l]}M_h^{-1}M_f\left[\left(\frac{2\alpha-1}{\alpha-1}\right)r\right]+O(1)}{\log\left[\left(\frac{2\alpha-1}{\alpha-1}\right)r\right]+O(1)} \end{aligned}$$

$$\begin{aligned} i.e. \ \rho_g^{[l]}(f) &= \limsup_{r\to\infty}\frac{\log^{[l]}M_h^{-1}M_f\left[\left(\frac{2\alpha-1}{\alpha-1}\right)r\right]+O(1)}{\log\left[\left(\frac{2\alpha-1}{\alpha-1}\right)r\right]+O(1)} \\ &\geq \limsup_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\mu_f(r)}{\log r} \end{aligned}$$

$$i.e., \ \rho_g^{[l]}(f) \geq \limsup_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\mu_f(r)}{\log r} \tag{2.3}$$

and consequently

$$\lambda_g^{[l]}(f) \geq \liminf_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\mu_f(r)}{\log r}\,. \tag{2.4}$$

Combining (2.1), (2.3) and (2.2), (2.4) we obtain that

$$\rho_g^{[l]}(f) = \limsup_{r\to\infty}\frac{\log^{[l]}\mu_g^{-1}\mu_f(r)}{\log r} \text{ and } \lambda_g^{[l]}(f) = \liminf_{r\to\infty}\frac{\log^{[l]}\mu_g^{-1}\mu_f(r)}{\log r}\,.$$

This proves the lemma. □

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let f, g and h be any three entire functions such that*
(i) $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$,
(ii) $\rho_h^{[p]}(f) = \rho_g$,

(iii) $\sigma_g < \infty$, *and*

(iv) $\overline{\sigma}_h^{[p]}(f) > 0$ *where p is any positive integer. Then*

$$(i)\ \liminf_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \leq \min\left\{A\frac{\rho_h^{[p]}(f)\,\sigma_g}{\sigma_h^{[p]}(f)}, A\frac{\rho_h^{[p]}(f)\,\overline{\sigma}_g}{\overline{\sigma}_h^{[p]}(f)}, A\frac{\lambda_h^{[p]}(f)\,\sigma_g}{\overline{\sigma}_h^{[p]}(f)}\right\}$$

where $A = (\gamma\alpha\beta)^{\rho_h^{[p]}(f)}$ *for any* $\beta > 1$ *and* $\gamma(\alpha-1) > \alpha > 1$

and

$$(ii)\ \liminf_{r\to\infty} \frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p-1]} M_h^{-1}M_f(r)} \leq \min\left\{\frac{\rho_h^{[p]}(f)\,\sigma_g}{\sigma_h^{[p]}(f)}, \frac{\rho_h^{[p]}(f)\,\overline{\sigma}_g}{\overline{\sigma}_h^{[p]}(f)}, \frac{\lambda_h^{[p]}(f)\,\sigma_g}{\overline{\sigma}_h^{[p]}(f)}\right\}.$$

Proof. As $\mu_h^{-1}(r)$ is an increasing function of r, taking $R = \beta r\,(\beta > 1)$ in Lemma 2.2 and in view of Lemma 2.5 we have for all sufficiently large values of r that

$$\mu_{f\circ g}(r) \leq \left(\frac{\alpha}{\alpha-1}\right)\mu_f\left(\frac{\alpha\beta}{(\beta-1)}\mu_g(\beta r)\right)$$

$$i.e.,\ \mu_{f\circ g}(r) \leq \mu_f\left(\frac{(2\alpha-1)\,\alpha\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right).$$

$$\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \leq \log^{[p]} \mu_h^{-1}\mu_f\left(\frac{(2\alpha-1)\,\alpha\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right) \tag{3.1}$$

$$i.e.,\ \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \leq \left(\rho_h^{[p]}(f)+\varepsilon\right)\log \mu_g(\beta r) + O(1).$$

Now we get from above and in view of the inequality $\mu(r,f) \leq M(r,f)$ $\{cf.\ [12]\}$ and the condition *(ii)* for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \leq \left(\rho_h^{[p]}(f)+\varepsilon\right)\log M_g(\beta r) + O(1) \tag{3.2}$$

$$i.e.,\ \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \leq \left(\rho_h^{[p]}(f)+\varepsilon\right)\left(\sigma_g+\varepsilon\right)[\beta r]^{\rho_h^{[p]}(f)} + O(1). \tag{3.3}$$

Also in view of the inequality $\mu_g(r) \leq M_g(r)$ $\{cf.\ [12]\}$ and the condition *(ii)*,we obtain from (3.1) for a sequence of values of r tending to infinity that

$$\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \leq \left(\lambda_h^{[p]}(f)+\varepsilon\right)\left(\sigma_g+\varepsilon\right)[\beta r]^{\rho_h^{[p]}(f)} + O(1) \tag{3.4}$$

and

$$\log^{[p]} \mu_h^{-1} \mu_{f\circ g}(r) \le \left(\rho_h^{[p]}(f) + \varepsilon\right)\left(\overline{\sigma}_g + \varepsilon\right)[\beta r]^{\rho_h(f)} + O(1)\,. \tag{3.5}$$

Again in view of Lemma 2.5, Lemma 2.6 and the definition of generalized relative type we get for any $\gamma > \frac{\alpha}{(\alpha-1)}$ and for a sequence of values of r tending to infinity that

$$\begin{aligned} \mu_h^{-1}\left[\frac{\alpha}{(\alpha-1)}\mu_f(\alpha r)\right] &\ge M_h^{-1} M_f(r) \\ \text{i.e., } \mu_h^{-1}\mu_f(\gamma\alpha r) &\ge M_h^{-1} M_f(r) \end{aligned}$$

$$\text{i.e., } \log^{[p-1]} \mu_h^{-1}\mu_f(r) \ge \log^{[p-1]} M_h^{-1} M_f\left(\frac{r}{\gamma\alpha}\right)$$

$$\text{i.e., } \log^{[p-1]} \mu_h^{-1}\mu_f(r) \ge \left(\sigma_h^{[p]}(f) - \varepsilon\right)\left[\frac{r}{\gamma\alpha}\right]^{\rho_h^{[p]}(f)}. \tag{3.6}$$

Further from the definition of generalized relative lower type, we obtain for any $\gamma(\alpha-1) > \alpha > 1$ and in view of Lemma 2.5 and Lemma 2.6 for all sufficiently large values of r that

$$\log^{[p-1]} \mu_h^{-1}\mu_f(r) \ge \left(\overline{\sigma}_h^{[p]}(f) - \varepsilon\right)\left[\frac{r}{\gamma\alpha}\right]^{\rho_h^{[p]}(f)}. \tag{3.7}$$

Now from (3.3) and (3.6), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \le \frac{\left(\rho_h^{[p]}(f) + \varepsilon\right)\left(\sigma_g + \varepsilon\right)[\beta r]^{\rho_h^{[p]}(f)} + O(1)}{\left(\sigma_h^{[p]}(f) - \varepsilon\right)\left[\frac{r}{\gamma\alpha}\right]^{\rho_h^{[p]}(f)}}$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \le (\gamma\alpha\beta)^{\rho_h^{[p]}(f)}\, \frac{\rho_h^{[p]}(f)\,\sigma_g}{\sigma_h^{[p]}(f)}. \tag{3.8}$$

Similarly from (3.4) and (3.7), we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \le \frac{\left(\lambda_h^{[p]}(f) + \varepsilon\right)\left(\sigma_g + \varepsilon\right)[\beta r]^{\rho_h^{[p]}(f)} + O(1)}{\left(\overline{\sigma}_h^{[p]}(f) - \varepsilon\right)\left[\frac{r}{\gamma\alpha}\right]^{\rho_h^{[p]}(f)}}$$

i.e.,

$$\liminf_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)}\le(\gamma\alpha\beta)^{\rho_h^{[p]}(f)}\frac{\lambda_h^{[p]}(f)\,\sigma_g}{\overline{\sigma}_h^{[p]}(f)}\,. \tag{3.9}$$

Likewise from (3.5) and (3.7), it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)}\le(\gamma\alpha\beta)^{\rho_h^{[p]}(f)}\frac{\rho_h^{[p]}(f)\,\overline{\sigma}_g}{\overline{\sigma}_h^{[p]}(f)}. \tag{3.10}$$

Thus the first part of the theorem follows from (3.8), (3.9) and (3.10).

Since $M_h^{-1}(r)$ is an increasing function of r, by similar reasoning as above the second part of the theorem follows from the second part of Lemma 2.1 and therefore its proof is omitted. □

In view of Theorem 3.1, the following theorem can be carried out and therefore its proof is omitted:

Theorem 3.2. *Let f, g and h be any three entire functions with*
(i) $0<\rho_h^{[p]}(f)<\infty$,
(ii) $\rho_h^{[p]}(f)=\rho_g$,
(iii) $\sigma_g<\infty$, *and*
(iv) $\overline{\sigma}_h^{[p]}(f)>0$ *where p is any positive integer >1. Then*

$$(i)\ \limsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)}\le(\gamma\alpha\beta)^{\rho_h^{[p]}(f)}\frac{\rho_h^{[p]}(f)\,\sigma_g}{\overline{\sigma}_h^{[p]}(f)}$$

where $\beta>1$ and $\gamma(\alpha-1)>\alpha>1$

and

$$(ii)\ \limsup_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\log^{[p-1]}M_h^{-1}M_f(r)}\le\frac{\rho_h^{[p]}(f)\,\sigma_g}{\overline{\sigma}_h^{[p]}(f)}\,.$$

Using the notion of generalized relative weak type, we may state the following two theorems without their proofs as those can be carried out in the line of Theorem 3.1 and Theorem 3.2 respectively.

Theorem 3.3. *Let f, g and h be any three entire functions such that*
(i) $0<\lambda_h^{[p]}(f)\le\rho_h^{[p]}(f)<\infty$,
(ii) $\lambda_h^{[p]}(f)=\lambda_g$,

(iii) $\overline{\tau}_g < \infty$, *and*
(iv) $\tau_h^{[p]}(f) > 0$ *where p is any positive integer. Then*

$$(i)\ \liminf_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \le$$
$$\min\left\{B\frac{\rho_h^{[p]}(f)\overline{\tau}_g}{\overline{\tau}_h^{[p]}(f)}, B\frac{\rho_h^{[p]}(f)\tau_g}{\tau_h^{[p]}(f)}, B\frac{\lambda_h^{[p]}(f)\overline{\tau}_g}{\tau_h^{[p]}(f)}\right\}$$
$$\text{where } B = (\gamma\alpha\beta)^{\lambda_h^{[p]}(f)} \text{ for } \beta > 1 \text{ and } \gamma(\alpha-1) > \alpha > 1$$

and

$$(ii)\ \liminf_{r\to\infty} \frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p-1]} M_h^{-1}M_f(r)} \le \min\left\{\frac{\rho_h^{[p]}(f)\overline{\tau}_g}{\overline{\tau}_h^{[p]}(f)}, \frac{\rho_h^{[p]}(f)\tau_g}{\tau_h^{[p]}(f)}, \frac{\lambda_h^{[p]}(f)\overline{\tau}_g}{\tau_h^{[p]}(f)}\right\}.$$

Theorem 3.4. *Let f, g and h be any three entire functions with*
(i) $0 < \lambda_h^{[p]}(f) \le \rho_h^{[p]}(f) < \infty$,
(ii) $\lambda_h^{[p]}(f) = \lambda_g$,
(iii) $\overline{\tau}_g < \infty$, *and (iv)* $\tau_h^{[p]}(f) > 0$ *where p is any positive integer* > 1. *Then*

$$(i)\ \limsup_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \le (\gamma\alpha\beta)^{\lambda_h^{[p]}(f)} \frac{\rho_h^{[p]}(f)\overline{\tau}_g}{\tau_h^{[p]}(f)}$$

$$\text{where } \beta > 1 \text{ and } \gamma(\alpha-1) > \alpha > 1$$

and

$$(ii)\ \limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p-1]} M_h^{-1}M_f(r)} \le \frac{\rho_h^{[p]}(f)\overline{\tau}_g}{\tau_h^{[p]}(f)}.$$

Theorem 3.5. *Let f, g and h be any three entire functions such that*
(i) $0 < \lambda_h^{[p]}(f) \le \rho_h^{[p]}(f) < \infty$,
(ii) $\rho_h^{[p]}(f) = \rho_g$,

(iii) $\overline{\sigma}_g > 0$, *and (iv)* $\sigma_h^{[p]}(f) < \infty$ *where p is any positive integer* > 1. *Then*

$$(i)\ \limsup_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \geq \max\left\{ C\frac{\lambda_h^{[p]}(f)\overline{\sigma}_g}{\overline{\sigma}_h^{[p]}(f)}, C\frac{\lambda_h^{[p]}(f)\sigma_g}{\sigma_h^{[p]}(f)}, C\frac{\rho_h^{[p]}(f)\overline{\sigma}_g}{\sigma_h^{[p]}(f)} \right\}$$

$$\text{where } C = \left(\frac{1}{4\gamma\beta}\right)^{\rho_h^{[p]}(f)} \text{ for } \beta > 1,\ \gamma(\alpha-1) > \alpha > 1$$

and

$$(ii)\ \limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p-1]} M_h^{-1}M_f(r)} \geq \max\left\{ \frac{\lambda_h^{[p]}(f)\overline{\sigma}_g}{\overline{\sigma}_h^{[p]}(f)}, \frac{\lambda_h^{[p]}(f)\sigma_g}{\sigma_h^{[p]}(f)}, \frac{\rho_h^{[p]}(f)\overline{\sigma}_g}{\sigma_h^{[p]}(f)} \right\}.$$

Proof. Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.3 and Lemma 2.5 for all sufficiently large values of r that

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right) - |g(0)|\right).$$

$$(3.11)\qquad \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \geq \log^{[p]} \mu_h^{-1}\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right) - \frac{|g(0)|}{3}\right)$$

$$i.e.,\ \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \geq \left(\lambda_h^{[p]}(f) - \varepsilon\right)\log\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right) - \frac{|g(0)|}{3}\right)$$

$$i.e.,\ \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \geq \left(\lambda_h^{[p]}(f) - \varepsilon\right)\log \mu_g\left(\frac{r}{4}\right) + O(1).$$

Using the condition $\rho_h^{[p]}(f) = \rho_g$ and in view of the inequality $M_g(r) \leq \frac{R}{R-r}\mu_g(R)$ $\{cf.$ **[12]** $\}$ for $R = \beta r\,(\beta > 1)$, we get from above for all sufficiently large values of r that

$$(3.12)\qquad \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \geq \left(\lambda_h^{[p]}(f) - \varepsilon\right)\log M_g\left(\frac{r}{4\beta}\right) + O(1)$$

$$(3.13)\qquad i.e.,\ \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \geq \left(\lambda_h^{[p]}(f) - \varepsilon\right)\left(\overline{\sigma}_g - \varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_h^{[p]}(f)} + O(1).$$

Also, in view of the inequality $M_g(r) \leq \frac{R}{R-r}\mu_g(R)$ $\{cf.$ **[12]** $\}$ for $R = \beta r\,(\beta > 1)$ and the condition (ii), it follows from (3.11) for a sequence of values of r tending to infinity that

$$\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \geq \left(\lambda_h^{[p]}(f) - \varepsilon\right)\left(\sigma_g - \varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_h^{[p]}(f)} + O(1) \tag{3.14}$$

and

$$\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \geq \left(\rho_h^{[p]}(f) - \varepsilon\right)\left(\overline{\sigma}_g - \varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_h^{[p]}(f)} + O(1)\,. \tag{3.15}$$

Again in view of Lemma 2.4 and for any $\gamma(\alpha-1) > \alpha > 1$, we get from Lemma 2.6 for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[p-1]} \mu_h^{-1}\mu_f(r) &\leq \log^{[p-1]} \alpha M_h^{-1} M_f(\gamma r) \\ \text{i.e., } \log^{[p-1]} \mu_h^{-1}\mu_f(r) &\leq \left(\overline{\sigma}_h^{[p]}(f) + \varepsilon\right)[\gamma r]^{\rho_h^{[p]}(f)} + O(1). \end{aligned} \tag{3.16}$$

Also for any $\gamma(\alpha-1) > \alpha > 1$, we obtain from the definition of generalized relative type and in view of Lemma 2.4 and Lemma 2.6 for all sufficiently large values of r that

$$\log^{[p-1]} \mu_h^{-1}\mu_f(r) \leq \left(\sigma_h^{[p]}(f) + \varepsilon\right)[\gamma r]^{\rho_h^{[p]}(f)} + O(1)\,. \tag{3.17}$$

Now from (3.13) and (3.16), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \geq \frac{\left(\lambda_h^{[p]}(f) - \varepsilon\right)\left(\overline{\sigma}_g - \varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_h^{[p]}(f)} + O(1)}{\left(\overline{\sigma}_h^{[p]}(f) + \varepsilon\right)[\gamma r]^{\rho_h^{[p]}(f)} + O(1)}$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \geq \left(\frac{1}{4\gamma\beta}\right)^{\rho_h^{[p]}(f)} \frac{\lambda_h^{[p]}(f)\,\overline{\sigma}_g}{\overline{\sigma}_h^{[p]}(f)}. \tag{3.18}$$

Similarly from (3.14) and (3.17), we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \geq \frac{\left(\lambda_h^{[p]}(f) - \varepsilon\right)\left(\sigma_g - \varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_h^{[p]}(f)} + O(1)}{\left(\sigma_h^{[p]}(f) + \varepsilon\right)[\gamma r]^{\rho_h^{[p]}(f)} + O(1)}$$

i.e.,

$$\limsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)}\geq\left(\frac{1}{4\gamma\beta}\right)^{\rho_h^{[p]}(f)}\frac{\lambda_h^{[p]}(f)\,\sigma_g}{\sigma_h^{[p]}(f)}\ . \tag{3.19}$$

Likewise from (3.15) and (3.17), it follows that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)}\geq\left(\frac{1}{4\gamma\beta}\right)^{\rho_h^{[p]}(f)}\frac{\rho_h^{[p]}(f)\,\overline{\sigma}_g}{\sigma_h^{[p]}(f)}. \tag{3.20}$$

Thus the first part of the theorem follows from (3.18), (3.19) and (3.20).

Since $M_h^{-1}(r)$ is an increasing function of r, by similar reasoning as above the second part of the theorem follows from the first part of Lemma 2.1 and therefore its proof is omitted.

In view of Theorem 3.5 the following theorem can be carried out and therefore its proof is omitted: □

Theorem 3.6. *Let f, g and h be any three entire functions with*
(i) $0<\lambda_h^{[p]}(f)\leq\rho_h^{[p]}(f)<\infty$,
(ii) $\rho_h^{[p]}(f)=\rho_g$,
(iii) $\overline{\sigma}_g>0$ *and*
(iv) $\sigma_h^{[p]}(f)<\infty$ *where p is any positive integer >1. Then*

$$(i)\ \liminf_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)}\geq\left(\frac{1}{4\gamma\beta}\right)^{\rho_h^{[p]}(f)}\frac{\lambda_h^{[p]}(f)\,\overline{\sigma}_g}{\sigma_h^{[p]}(f)}$$

where $\beta>1,\ \gamma(\alpha-1)>\alpha>1$

and

$$(ii)\ \liminf_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\log^{[p-1]}M_h^{-1}M_f(r)}\geq\frac{\lambda_h^{[p]}(f)\,\overline{\sigma}_g}{\sigma_h^{[p]}(f)}.$$

Now using the notion of generalized relative weak type, one may state the following two theorems without their proofs as those can be carried out in the line of Theorem 3.5 and Theorem 3.6 respectively.

Theorem 3.7. *Let f, g and h be any three entire functions such that*
(i) $0<\lambda_h^{[p]}(f)\leq\rho_h^{[p]}(f)<\infty$,
(ii) $\lambda_h^{[p]}(f)=\lambda_g$,

(iii) $\tau_g > 0$, *and*

(iv) $\overline{\tau}_h^{[p]}(f) < \infty$ *where p is any positive integer* > 1. *Then*

$$(i)\ \limsup_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \geq \max\left\{ D\frac{\lambda_h^{[p]}(f)\,\tau_g}{\tau_h^{[p]}(f)}, D\frac{\lambda_h^{[p]}(f)\,\overline{\tau}_g}{\overline{\tau}_h^{[p]}(f)}, D\frac{\rho_h^{[p]}(f)\,\tau_g}{\overline{\tau}_h^{[p]}(f)} \right\}$$

$$\text{where } D = \left(\frac{1}{4\gamma\beta}\right)^{\lambda_h^{[p]}(f)} \quad \text{for } \beta > 1,\ \gamma(\alpha-1) > \alpha > 1$$

and

$$(ii)\ \limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p-1]} M_h^{-1}M_f(r)} \geq \max\left\{ \frac{\lambda_h^{[p]}(f)\,\tau_g}{\tau_h^{[p]}(f)}, \frac{\lambda_h^{[p]}(f)\,\overline{\tau}_g}{\overline{\tau}_h^{[p]}(f)}, \frac{\rho_h^{[p]}(f)\,\tau_g}{\overline{\tau}_h^{[p]}(f)} \right\}.$$

Theorem 3.8. *Let f, g and h be any three entire functions with*

(i) $0 < \lambda_h^{[p]}(f) < \infty$,

(ii) $\lambda_h^{[p]}(f) = \lambda_g$,

(iii) $\tau_g > 0$, *and (iv)* $\overline{\tau}_h^{[p]}(f) < \infty$ *where p is any positive integer* > 1. *Then*

$$(i)\ \liminf_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \geq \left(\frac{1}{4\gamma\beta}\right)^{\lambda_h^{[p]}(f)} \frac{\lambda_h^{[p]}(f)\,\tau_g}{\overline{\tau}_h^{[p]}(f)}$$

$$\text{where } \beta > 1,\ \gamma(\alpha-1) > \alpha > 1$$

and

$$(ii)\ \liminf_{r\to\infty} \frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p-1]} M_h^{-1}M_f(r)} \geq \frac{\lambda_h^{[p]}(f)\,\tau_g}{\overline{\tau}_h^{[p]}(f)}.$$

Theorem 3.9. *Let f, g, h and k be any four entire functions such that*

(i) $0 < \overline{\sigma}_k^{[q]}(f) \leq \sigma_k^{[q]}(f) < \infty$,

(ii) $0 < \overline{\sigma}_h^{[p]}(f\circ g) \leq \sigma_h^{[p]}(f\circ g) < \infty$ *and*

(iii) $\rho_h^{[p]}(f\circ g)=\rho_k^{[q]}(f)$ *where* p,q *are any two positive integers* >1. *Then*

$$(i)\ \left(\frac{1}{E}\right)^{\rho_k^{[q]}(f)}\frac{\overline{\sigma}_h^{[p]}(f\circ g)}{\sigma_k^{[q]}(f)}\leq\liminf_{r\to\infty}\frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_f(r)}\leq E^{\rho_k^{[q]}(f)}\frac{\overline{\sigma}_h^{[p]}(f\circ g)}{\overline{\sigma}_k^{[q]}(f)}\ \textit{and}$$

$$\left(\frac{1}{E}\right)^{\rho_k^{[q]}(f)}\frac{\overline{\sigma}_h^{[p]}(f\circ g)}{\overline{\sigma}_k^{[q]}(f)}\leq\limsup_{r\to\infty}\frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_f(r)}\leq E^{\rho_k^{[q]}(f)}\frac{\sigma_h^{[p]}(f\circ g)}{\overline{\sigma}_k^{[q]}(f)}$$

$$\textit{where } E=\gamma^2\alpha \textit{ for } \gamma(\alpha-1)>\alpha>1$$

and

$$(ii)\ \frac{\overline{\sigma}_h^{[p]}(f\circ g)}{\sigma_k^{[q]}(f)}\leq\liminf_{r\to\infty}\frac{\log^{[p-1]}M_h^{-1}M_{f\circ g}(r)}{\log^{[q-1]}M_k^{-1}M_f(r)}\leq\frac{\overline{\sigma}_h^{[p]}(f\circ g)}{\overline{\sigma}_k^{[q]}(f)}$$

$$\leq\limsup_{r\to\infty}\frac{\log^{[p-1]}M_h^{-1}M_{f\circ g}(r)}{\log^{[q-1]}M_k^{-1}M_f(r)}\leq\frac{\sigma_h^{[p]}(f\circ g)}{\overline{\sigma}_k^{[q]}(f)}.$$

Proof. For any $\gamma(\alpha-1)>\alpha>1$ we obtain from the definition of generalized relative type and in view of Lemma 2.5 and Lemma 2.6 for all sufficiently large values of r that

$$\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)\ \geq\ \log^{[p-1]}M_h^{-1}M_f\left(\frac{r}{\gamma\alpha}\right)$$

$$\text{(3.21)}\qquad i.e.,\ \log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)\ \geq\ \left(\overline{\sigma}_h^{[p]}(f\circ g)-\varepsilon\right)\left(\frac{r}{\gamma\alpha}\right)^{\rho_h^{[p]}(f\circ g)}.$$

and

$$\log^{[q-1]}\mu_k^{-1}\mu_f(r)\ \leq\ \log^{[q-1]}\left[\alpha M_k^{-1}\left(\frac{\alpha}{(\alpha-1)}M_f(r)\right)\right]$$

$$i.e.,\ \log^{[q-1]}\mu_k^{-1}\mu_f(r)\ \leq\ \log^{[q-1]}M_k^{-1}M_f(\gamma r)+O(1)$$

$$\text{(3.22)}\qquad i.e.,\ \log^{[q-1]}\mu_k^{-1}\mu_f(r)\ \leq\ \left(\sigma_k^{[q]}(f)+\varepsilon\right)(\gamma r)^{\rho_k^{[q]}(f)}+O(1).$$

Now from (3.21), (3.22) and in view of the condition $\rho_h^{[p]}(f\circ g)=\rho_k^{[q]}(f)$, it follows for all sufficiently large values of r that

$$\frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_f(r)}\geqslant\frac{\left(\overline{\sigma}_h^{[p]}(f\circ g)-\varepsilon\right)\left(\frac{r}{\gamma\alpha}\right)^{\rho_h^{[p]}(f\circ g)}}{\left(\sigma_k^{[q]}(f)+\varepsilon\right)(\gamma r)^{\rho_k^{[q]}(f)}+O(1)}.$$

As $\varepsilon\,(>0)$ is arbitrary , we obtain from above that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_f(r)}\geqslant\left(\frac{1}{\gamma^2\alpha}\right)^{\rho_k^{[q]}(f)}\frac{\overline{\sigma}_h^{[p]}(f\circ g)}{\sigma_k^{[q]}(f)}\,. \tag{3.23}$$

Again we obtain for a sequence of values of r tending to infinity that

$$\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)\leq\left(\overline{\sigma}_h^{[p]}(f\circ g)+\varepsilon\right)(\gamma r)^{\rho_h^{[p]}(f\circ g)}+O(1) \tag{3.24}$$

and for all sufficiently large values of r that

$$\log^{[q-1]}\mu_k^{-1}\mu_f(r)\geqslant\left(\overline{\sigma}_k^{[q]}(f)-\varepsilon\right)\left(\frac{r}{\gamma\alpha}\right)^{\rho_k^{[q]}(f)}\,. \tag{3.25}$$

Combining the condition $\rho_h^{[p]}(f\circ g)=\rho_k^{[q]}(f)$, (3.24) and (3.25) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_f(r)}\leq\frac{\left(\overline{\sigma}_h^{[p]}(f\circ g)+\varepsilon\right)(\gamma r)^{\rho_h^{[p]}(f\circ g)}+O(1)}{\left(\overline{\sigma}_k^{[q]}(f)-\varepsilon\right)\left(\frac{r}{\gamma\alpha}\right)^{\rho_k^{[q]}(f)}}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_f(r)}\leq\left(\gamma^2\alpha\right)^{\rho_k^{[q]}(f)}\frac{\overline{\sigma}_h^{[p]}(f\circ g)}{\overline{\sigma}_k^{[q]}(f)}. \tag{3.26}$$

Also for a sequence of values of r tending to infinity that

$$\log^{[q-1]}\mu_k^{-1}\mu_f(r)\leq\left(\overline{\sigma}_k^{[q]}(f)+\varepsilon\right)(\gamma r)^{\rho_k^{[q]}(f)}+O(1)\,. \tag{3.27}$$

Now from (3.21), (3.27) and the condition $\rho_h^{[p]}(f\circ g)=\rho_k^{[q]}(f)$, we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_f(r)}\geq\frac{\left(\overline{\sigma}_h^{[p]}(f\circ g)-\varepsilon\right)\left(\frac{r}{\gamma\alpha}\right)^{\rho_h^{[p]}(f\circ g)}}{\left(\overline{\sigma}_k^{[q]}(f)+\varepsilon\right)(\gamma r)^{\rho_k^{[q]}(f)}+O(1)}\,.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to\infty}\frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_f(r)}\geq\left(\frac{1}{\gamma^2\alpha}\right)^{\rho_k^{[q]}(f)}\frac{\overline{\sigma}_h^{[p]}(f\circ g)}{\overline{\sigma}_k^{[q]}(f)}\,. \tag{3.28}$$

Also for all sufficiently large values of r,

$$\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)\le\left(\sigma_h^{[p]}(f\circ g)+\varepsilon\right)(\gamma r)^{\rho_h^{[p]}(f\circ g)}+O(1)\,. \tag{3.29}$$

As the condition $\rho_h^{[p]}(f\circ g)=\rho_k^{[q]}(f)$, it follows from (3.25) and (3.29) for all sufficiently large values of r that

$$\frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_f(r)}\le\frac{\left(\sigma_h^{[p]}(f\circ g)+\varepsilon\right)(\gamma r)^{\rho_h^{[p]}(f\circ g)}+O(1)}{\left(\overline{\sigma}_k^{[q]}(f)-\varepsilon\right)\left(\frac{r}{\gamma\alpha}\right)^{\rho_k^{[q]}(f)}}\,.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_f(r)}\le\left(\gamma^2\alpha\right)^{\rho_k^{[q]}(f)}\frac{\sigma_h^{[p]}(f\circ g)}{\overline{\sigma}_k^{[q]}(f)}\,. \tag{3.30}$$

Thus the first part of the theorem follows from (3.23), (3.26), (3.28) and (3.30).

Similarly, the second part of the theorem can be established. □

The following theorem can be proved in the line of Theorem 3.9 and so its proof is omitted.

Theorem 3.10. *Let f, g, h and k be any four entire functions with*
(i) $0<\overline{\sigma}_k^{[q]}(g)\le\sigma_k^{[q]}(g)<\infty$,
(ii) $0<\overline{\sigma}_h^{[p]}(f\circ g)\le\sigma_h^{[p]}(f\circ g)<\infty$ *and*
(iii) $\rho_h^{[p]}(f\circ g)=\rho_k^{[q]}(g)$ *where p, q are any positive integers >1. Then*

$$(i)\quad\left(\frac{1}{E}\right)^{\rho_k^{[q]}(g)}\frac{\overline{\sigma}_h^{[p]}(f\circ g)}{\sigma_k^{[q]}(g)}\le\liminf_{r\to\infty}\frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_g(r)}\le E^{\rho_k^{[q]}(g)}\frac{\overline{\sigma}_h^{[p]}(f\circ g)}{\overline{\sigma}_k^{[q]}(g)}\ \text{and}$$

$$\left(\frac{1}{E}\right)^{\rho_k^{[q]}(g)}\frac{\overline{\sigma}_h^{[p]}(f\circ g)}{\overline{\sigma}_k^{[q]}(g)}\le\limsup_{r\to\infty}\frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_g(r)}\le E^{\rho_k^{[q]}(g)}\frac{\sigma_h^{[p]}(f\circ g)}{\overline{\sigma}_k^{[q]}(g)}$$

where $E=\gamma^2\alpha$ *for* $\gamma(\alpha-1)>\alpha>1$

and

$$(ii)\quad\frac{\overline{\sigma}_h^{[p]}(f\circ g)}{\sigma_k^{[q]}(g)}\le\liminf_{r\to\infty}\frac{\log^{[p-1]}M_h^{-1}M_{f\circ g}(r)}{\log^{[q-1]}M_k^{-1}M_g(r)}\le\frac{\overline{\sigma}_h^{[p]}(f\circ g)}{\overline{\sigma}_k^{[q]}(g)}$$

$$\le\limsup_{r\to\infty}\frac{\log^{[p-1]}M_h^{-1}M_{f\circ g}(r)}{\log^{[q-1]}M_k^{-1}M_g(r)}\le\frac{\sigma_h^{[p]}(f\circ g)}{\overline{\sigma}_k^{[q]}(g)}\,.$$

Using the notion of generalized relative weak type, we may state the following two theorems without their proofs because those can be carried out in the line of Theorem 3.9 and Theorem 3.10 respectively.

Theorem 3.11. *Let f, g, h and k be any four entire functions such that*
(*i*) $0 < \tau_k^{[q]}(f) \leq \overline{\tau}_k^{[q]}(f) < \infty$,
(*ii*) $0 < \tau_h^{[p]}(f \circ g) \leq \overline{\tau}_h^{[p]}(f \circ g) < \infty$ *and*
(*iii*) $\lambda_h^{[p]}(f \circ g) = \lambda_k^{[q]}(f)$ *where p, q are any positive integers > 1. Then*

$$(i)\quad \left(\frac{1}{E}\right)^{\lambda_k^{[q]}(f)} \frac{\tau_h^{[p]}(f \circ g)}{\overline{\tau}_k^{[q]}(f)} \leq \liminf_{r\to\infty} \frac{\log^{[p-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]} \mu_k^{-1}\mu_f(r)} \leq E^{\lambda_k^{[q]}(f)} \frac{\tau_h^{[p]}(f \circ g)}{\tau_k^{[q]}(f)} \text{ and}$$

$$\left(\frac{1}{E}\right)^{\lambda_k^{[q]}(f)} \frac{\tau_h^{[p]}(f \circ g)}{\tau_k^{[q]}(f)} \leq \limsup_{r\to\infty} \frac{\log^{[p-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]} \mu_k^{-1}\mu_f(r)} \leq E^{\lambda_k^{[q]}(f)} \frac{\overline{\tau}_h^{[p]}(f \circ g)}{\tau_k^{[q]}(f)}$$

$$\text{where } E = \gamma^2\alpha \text{ for } \gamma(\alpha - 1) > \alpha > 1$$

and

$$(ii)\quad \frac{\tau_h^{[p]}(f \circ g)}{\overline{\tau}_k^{[q]}(f)} \leq \liminf_{r\to\infty} \frac{\log^{[p-1]} M_h^{-1}M_{f\circ g}(r)}{\log^{[q-1]} M_k^{-1}M_f(r)} \leq \frac{\tau_h^{[p]}(f \circ g)}{\tau_k^{[q]}(f)}$$

$$\leq \limsup_{r\to\infty} \frac{\log^{[p-1]} M_h^{-1}M_{f\circ g}(r)}{\log^{[q-1]} M_k^{-1}M_f(r)} \leq \frac{\overline{\tau}_h^{[p]}(f \circ g)}{\tau_k^{[q]}(f)}.$$

Theorem 3.12. *Let f, g, h and k be any four entire functions with*
(*i*) $0 < \tau_k^{[q]}(g) \leq \overline{\tau}_k^{[q]}(g) < \infty$,
(*ii*) $0 < \tau_h^{[p]}(f \circ g) \leq \overline{\tau}_h^{[p]}(f \circ g) < \infty$ *and*
(*iii*) $\lambda_h^{[p]}(f \circ g) = \lambda_k^{[q]}(g)$ *where p, q are any two positive integers > 1.*
Then

$$(i)\quad \left(\frac{1}{E}\right)^{\lambda_k^{[q]}(g)} \frac{\tau_h^{[p]}(f \circ g)}{\overline{\tau}_k^{[q]}(g)} \leq \liminf_{r\to\infty} \frac{\log^{[p-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]} \mu_k^{-1}\mu_g(r)} \leq E^{\lambda_k^{[q]}(g)} \frac{\tau_h^{[p]}(f \circ g)}{\tau_k^{[q]}(g)} \text{ and}$$

$$\left(\frac{1}{E}\right)^{\lambda_k^{[q]}(g)} \frac{\tau_h^{[p]}(f \circ g)}{\tau_k^{[q]}(g)} \leq \limsup_{r\to\infty} \frac{\log^{[p-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]} \mu_k^{-1}\mu_g(r)} \leq E^{\lambda_k^{[q]}(g)} \frac{\overline{\tau}_h^{[p]}(f \circ g)}{\tau_k^{[q]}(g)}$$

$$\text{where } E = \gamma^2\alpha \text{ for } \gamma(\alpha - 1) > \alpha > 1$$

and

$$(ii)\ \frac{\tau_h^{[p]}(f\circ g)}{\overline{\tau}_k^{[q]}(g)} \le \liminf_{r\to\infty}\frac{\log^{[p-1]} M_h^{-1}M_{f\circ g}(r)}{\log^{[q-1]} M_k^{-1}M_g(r)} \le \frac{\tau_h^{[p]}(f\circ g)}{\tau_k^{[q]}(g)}$$
$$\le \limsup_{r\to\infty}\frac{\log^{[p-1]} M_h^{-1}M_{f\circ g}(r)}{\log^{[q-1]} M_k^{-1}M_g(r)} \le \frac{\overline{\tau}_h^{[p]}(f\circ g)}{\tau_k^{[q]}(g)}.$$

Theorem 3.13. *Let f, g and h be any three entire functions such that*
(i) $0<\lambda_h^{[p]}(f)\le\rho_h^{[p]}(f)<\infty$,
(ii) $0<\overline{\sigma}_g^{[q]}\le\sigma_g^{[q]}<\infty$,
(iii) $0<\overline{\sigma}_h^{[p]}(f)\le\sigma_h^{[p]}(f)<\infty$ and (iv) $\rho_h^{[p]}(f)=\rho_g^{[q]}$ where p, q are any positive integers with $q>2$. Then

$$(i)\ F^{\rho_h^{[p]}(f)}\frac{\overline{\sigma}_g^{[q]}}{\sigma_h^{[p]}(f)} \le \liminf_{r\to\infty}\frac{\log^{[p+q-2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)}$$
$$\le \min\left\{G^{\rho_h^{[p]}(f)}\frac{\sigma_g^{[q]}}{\sigma_h^{[p]}(f)}, G^{\rho_h^{[p]}(f)}\frac{\overline{\sigma}_g^{[q]}}{\overline{\sigma}_h^{[p]}(f)}\right\} \text{ and}$$
$$\max\left\{F^{\rho_h^{[p]}(f)}\frac{\sigma_g^{[q]}}{\sigma_h^{[p]}(f)}, F^{\rho_h^{[p]}(f)}\frac{\overline{\sigma}_g^{[q]}}{\overline{\sigma}_h^{[p]}(f)}\right\} \le$$
$$\limsup_{r\to\infty}\frac{\log^{[p+q-2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)} \le G^{\rho_h^{[p]}(f)}\frac{\sigma_g^{[q]}}{\overline{\sigma}_h^{[p]}(f)}$$

$$\text{where } F=\frac{1}{4\beta\gamma} \text{ and } G=\alpha\beta\gamma \text{ for } \beta>1 \text{ and } \gamma(\alpha-1)>\alpha>1$$

and

$$(ii)\ \frac{\overline{\sigma}_g^{[q]}}{\sigma_h^{[p]}(f)} \le \liminf_{r\to\infty}\frac{\log^{[p+q-2]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p-1]} M_h^{-1}M_f(r)}$$
$$\le \min\left\{\frac{\sigma_g^{[q]}}{\sigma_h^{[p]}(f)}, \frac{\overline{\sigma}_g^{[q]}}{\overline{\sigma}_h^{[p]}(f)}\right\} \le \max\left\{\frac{\sigma_g^{[q]}}{\sigma_h^{[p]}(f)}, \frac{\overline{\sigma}_g^{[q]}}{\overline{\sigma}_h^{[p]}(f)}\right\}$$
$$\le \limsup_{r\to\infty}\frac{\log^{[p+q-2]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p-1]} M_h^{-1}M_f(r)} \le \frac{\sigma_g^{[q]}}{\overline{\sigma}_h^{[p]}(f)}.$$

Proof. For any $\beta > 1$, it follows from (3.2) and in view of the condition $\rho_h^{[p]}(f) = \rho_g^{[q]}$ for all sufficiently large values of r that

$$\log^{[p+q-2]} \mu_h^{-1}\mu_{f\circ g}(r) \leq \log^{[q-1]} M_g(\beta r) + O(1)$$

$$i.e., \ \log^{[p+q-2]} \mu_h^{-1}\mu_{f\circ g}(r) \leq \left(\sigma_g^{[q]} + \varepsilon\right)[\beta r]^{\rho_h^{[p]}(f)} + O(1) \tag{3.31}$$

and for a sequence of values of r that

$$\log^{[p+q-2]} \mu_h^{-1}\mu_{f\circ g}(r) \leq \left(\overline{\sigma}_g^{[q]} + \varepsilon\right)[\beta r]^{\rho_h^{[p]}(f)} + O(1). \tag{3.32}$$

Further in view of the condition $\rho_h^{[p]}(f) = \rho_g^{[q]}$, it follows from (3.12) for a sequence of values of r that

$$\log^{[p+q-2]} \mu_h^{-1}\mu_{f\circ g}(r) \geq \log^{[q-1]} M_g\left(\frac{r}{4\beta}\right) + O(1)$$

$$i.e., \ \log^{[p+q-2]} \mu_h^{-1}\mu_{f\circ g}(r) \geq \left(\sigma_g^{[q]} - \varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_h^{[p]}(f)} + O(1) \tag{3.33}$$

and for all sufficiently large values of r that

$$\log^{[p+q-2]} \mu_h^{-1}\mu_{f\circ g}(r) \geq \left(\overline{\sigma}_g^{[q]} - \varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_h^{[p]}(f)} + O(1). \tag{3.34}$$

Therefore from (3.7) and (3.31), we obtain for any $\gamma(\alpha - 1) > \alpha > 1$ and all sufficiently large values of r that

$$\frac{\log^{[p+q-2]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \leq \frac{\left(\sigma_g^{[q]} + \varepsilon\right)[\beta r]^{\rho_h^{[p]}(f)} + O(1)}{\left(\overline{\sigma}_h^{[p]}(f) - \varepsilon\right)\left[\frac{r}{\gamma\alpha}\right]^{\rho_h^{[p]}(f)}}$$

$$i.e., \ \limsup_{r\to\infty} \frac{\log^{[p+q-2]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \leq (\alpha\beta\gamma)^{\rho_h^{[p]}(f)} \frac{\sigma_g^{[q]}}{\overline{\sigma}_h^{[p]}(f)}. \tag{3.35}$$

Similarly, from (3.31) and (3.6) we have for a sequence of values of r tending to infinity that

$$\frac{\log^{[p+q-2]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \leq \frac{\left(\sigma_g^{[q]} + \varepsilon\right)[\beta r]^{\rho_h^{[p]}(f)} + O(1)}{\left(\sigma_h^{[p]}(f) - \varepsilon\right)\left[\frac{r}{\gamma\alpha}\right]^{\rho_h^{[p]}(f)}}$$

$$i.e., \ \liminf_{r\to\infty} \frac{\log^{[p+q-2]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \leq (\alpha\beta\gamma)^{\rho_h^{[p]}(f)} \frac{\sigma_g^{[q]}}{\sigma_h^{[p]}(f)}. \tag{3.36}$$

Analogously we get from (3.32) and (3.7) for a sequence of values of r tending to infinity that

$$\frac{\log^{[p+q-2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)} \leq \frac{\left(\overline{\sigma}_g^{[q]}+\varepsilon\right)[\beta r]^{\rho_h^{[p]}(f)}+O(1)}{\left(\overline{\sigma}_h^{[p]}(f)-\varepsilon\right)\left[\frac{r}{\gamma\alpha}\right]^{\rho_h^{[p]}(f)}}$$

$$\text{i.e., } \liminf_{r\to\infty}\frac{\log^{[p+q-2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)} \leq (\alpha\beta\gamma)^{\rho_h^{[p]}(f)}\frac{\overline{\sigma}_g^{[q]}}{\overline{\sigma}_h^{[p]}(f)}. \tag{3.37}$$

Now from (3.36) and (3.37), it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p+q-2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)}$$

$$\leq \min\left\{(\alpha\beta\gamma)^{\rho_h^{[p]}(f)}\cdot\frac{\sigma_g^{[q]}}{\sigma_h^{[p]}(f)},(\alpha\beta\gamma)^{\rho_h^{[p]}(f)}\frac{\overline{\sigma}_g^{[q]}}{\overline{\sigma}_h^{[p]}(f)}\right\}. \tag{3.38}$$

Further from (3.33) and (3.17), we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[p+q-2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)} \geq \frac{\left(\sigma_g^{[q]}-\varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_h^{[p]}(f)}+O(1)}{\left(\sigma_h^{[p]}(f)+\varepsilon\right)(\gamma r)^{\rho_h^{[p]}(f)}+O(1)}$$

i.e.,

$$\limsup_{r\to\infty}\frac{\log^{[p+q-2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)} \geq \left(\frac{1}{4\beta\gamma}\right)^{\rho_h^{[p]}(f)}\frac{\sigma_g^{[q]}}{\sigma_h^{[p]}(f)}. \tag{3.39}$$

Likewise from (3.34) and (3.16), we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[p+q-2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)} \geq \frac{\left(\overline{\sigma}_g^{[q]}-\varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_h^{[p]}(f)}+O(1)}{\left(\overline{\sigma}_h^{[p]}(f)+\varepsilon\right)(\gamma r)^{\rho_h^{[p]}(f)}+O(1)}$$

i.e.,

$$\limsup_{r\to\infty}\frac{\log^{[p+q-2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)} \geq \left(\frac{1}{4\beta\gamma}\right)^{\rho_h^{[p]}(f)}\frac{\overline{\sigma}_g^{[q]}}{\overline{\sigma}_h^{[p]}(f)}. \tag{3.40}$$

Thus from (3.39) and (3.40), it follows that

$$\limsup_{r\to\infty}\frac{\log^{[p+q-2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)} \geq \max\left\{\left(\frac{1}{4\beta\gamma}\right)^{\rho_h^{[p]}(f)}\frac{\sigma_g^{[q]}}{\sigma_h^{[p]}(f)}, \left(\frac{1}{4\beta\gamma}\right)^{\rho_h^{[p]}(f)}\frac{\overline{\sigma}_g^{[q]}}{\overline{\sigma}_h^{[p]}(f)}\right\}. \tag{3.41}$$

Also from (3.34) and (3.17), we obtain for all sufficiently large values of r that

$$\frac{\log^{[p+q-2]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]}\mu_h^{-1}\mu_f(r)} \geq \frac{\left(\overline{\sigma}_g^{[q]}-\varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_h^{[p]}(f)}+O(1)}{\left(\sigma_h^{[p]}(f)+\varepsilon\right)(\gamma r)^{\rho_h^{[p]}(f)}+O(1)}$$

i.e.,

$$\liminf_{r\to\infty}\frac{\log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f(r^A)} \geq \left(\frac{1}{4\beta\gamma}\right)^{\rho_h^{[p]}(f)}\frac{\overline{\sigma}_g^{[q]}}{\sigma_h^{[p]}(f)}. \tag{3.42}$$

Therefore the first part of the theorem follows from (3.35), (3.38), (3.41) and (3.42). Using the similar technique as above, the second part of the theorem follows from Lemma 2.1 and therefore its proof is omitted. □

In the line of Theorem 3.9, one can easily verify the following theorem and therefore its proof is omitted.

Theorem 3.14. *Let f, g and h be any three entire functions with*
(i) $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$,
(ii) $0 < \tau_g^{[q]} \leq \overline{\tau}_g^{[q]} < \infty$,
(iii) $0 < \tau_h^{[p]}(f) \leq \overline{\tau}_h^{[p]}(f) < \infty$ *and*
(iv) $\lambda_h^{[p]}(f) = \lambda_g^{[q]}$ *where p, q are any two positive integers with $q > 2$.*

Then

$$(i)\ F^{\lambda_h^{[p]}(f)} \frac{\tau_g^{[q]}}{\overline{\tau}_h^{[p]}(f)} \leq \liminf_{r\to\infty} \frac{\log^{[p+q-2]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)}$$

$$\leq \min\left\{G^{\lambda_h^{[p]}(f)} \frac{\overline{\tau}_g^{[q]}}{\overline{\tau}_h^{[p]}(f)}, G^{\lambda_h^{[p]}(f)} \frac{\tau_g^{[q]}}{\tau_h^{[p]}(f)}\right\} \text{ and}$$

$$\max\left\{F^{\lambda_h^{[p]}(f)} \frac{\overline{\tau}_g^{[q]}}{\overline{\tau}_h^{[p]}(f)}, F^{\lambda_h^{[p]}(f)} \frac{\tau_g^{[q]}}{\tau_h^{[p]}(f)}\right\} \leq$$

$$\limsup_{r\to\infty} \frac{\log^{[p+q-2]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-1]} \mu_h^{-1}\mu_f(r)} \leq G^{\lambda_h^{[p]}(f)} \frac{\overline{\tau}_g^{[q]}}{\tau_h^{[p]}(f)}$$

$$\text{where } F = \frac{1}{4\beta\gamma} \text{ and } G = \alpha\beta\gamma \text{ for } \beta > 1 \text{ and } \gamma(\alpha - 1) > \alpha > 1$$

and

$$(ii)\ \frac{\tau_g^{[q]}}{\overline{\tau}_h^{[p]}(f)} \leq \liminf_{r\to\infty} \frac{\log^{[p+q-2]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p-1]} M_h^{-1}M_f(r)}$$

$$\leq \min\left\{\frac{\overline{\tau}_g^{[q]}}{\overline{\tau}_h^{[p]}(f)}, \frac{\tau_g^{[q]}}{\tau_h^{[p]}(f)}\right\} \leq \max\left\{\frac{\overline{\tau}_g^{[q]}}{\overline{\tau}_h^{[p]}(f)}, \frac{\tau_g^{[q]}}{\tau_h^{[p]}(f)}\right\}$$

$$\leq \limsup_{r\to\infty} \frac{\log^{[p+q-2]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p-1]} M_h^{-1}M_f(r)} \leq \frac{\overline{\tau}_g^{[q]}}{\tau_h^{[p]}(f)}.$$

Theorem 3.15. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$ and $\sigma_g < \infty$ where p is any positive integer. Then for any $\beta > 1$,*

$$(i)\ \liminf_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f\left(\exp(\beta r)^{\rho_g}\right)} \leq \min\left\{\frac{\rho_h^{[p]}(f)\,\overline{\sigma}_g}{\lambda_h^{[p]}(f)}, \sigma_g\right\},$$

$$(ii)\ \limsup_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f\left(\exp(\beta r)^{\rho_g}\right)} \leq \frac{\rho_h^{[p]}(f)\,\sigma_g}{\lambda_h^{[p]}(f)},$$

$$(iii)\ \liminf_{r\to\infty} \frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p]} M_h^{-1}M_f\left(\exp r^{\rho_g}\right)} \leq \min\left\{\frac{\rho_h^{[p]}(f)\,\overline{\sigma}_g}{\lambda_h^{[p]}(f)}, \sigma_g\right\}$$

and

$$(iv)\ \limsup_{r\to\infty}\frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p]} M_h^{-1}M_f(\exp r^{\rho_g})}\leq\frac{\rho_h^{[p]}(f)\,\sigma_g}{\lambda_h^{[p]}(f)}.$$

Proof. In view of Definition 1.8, we get for a sequence of values of r tending to infinity that

$$\log^{[p]}\mu_h^{-1}\mu_f\left(\exp(\beta r)^{\rho_g}\right)\geq\left(\rho_h^{[p]}(f)-\varepsilon\right)[\beta r]^{\rho_g} \tag{3.43}$$

and for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}\mu_f\left(\exp(\beta r)^{\rho_g}\right)\geq\left(\lambda_h^{[p]}(f)-\varepsilon\right)[\beta r]^{\rho_g}\,. \tag{3.44}$$

Now from (3.3) and (3.43), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp(\beta r)^{\rho_g}\right)}\leq\frac{\left(\rho_h^{[p]}(f)+\varepsilon\right)\left(\sigma_g+\varepsilon\right)[\beta r]^{\rho_g}+O(1)}{\left(\rho_h^{[p]}(f)-\varepsilon\right)[\beta r]^{\rho_g}}\,.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp(\beta r)^{\rho_g}\right)}\leq\sigma_g. \tag{3.45}$$

Likewise from (3.5) and (3.44), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp(\beta r)^{\rho_g}\right)}\leq\frac{\left(\rho_h^{[p]}(f)+\varepsilon\right)\left(\overline{\sigma}_g+\varepsilon\right)[\beta r]^{\rho_h(f)}+O(1)}{\left(\lambda_h^{[p]}(f)-\varepsilon\right)[\beta r]^{\rho_g}}$$

i.e.,

$$\liminf_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp(\beta r)^{\rho_g}\right)}\leq\frac{\rho_h^{[p]}(f)\,\overline{\sigma}_g}{\lambda_h^{[p]}(f)}\,. \tag{3.46}$$

Similarly from (3.3) and (3.44), we obtain for all sufficiently large values of r that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp(\beta r)^{\rho_g}\right)}\leq\frac{\left(\rho_h^{[p]}(f)+\varepsilon\right)\left(\sigma_g+\varepsilon\right)[\beta r]^{\rho_g}+O(1)}{\left(\lambda_h^{[p]}(f)-\varepsilon\right)[\beta r]^{\rho_g}}$$

i.e.,

$$\limsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp(\beta r)^{\rho_g}\right)}\leq\frac{\rho_h^{[p]}(f)\,\sigma_g}{\lambda_h^{[p]}(f)}\,. \tag{3.47}$$

Thus the first and second part of the theorem follows from (3.45), (3.46) and (3.47). □

Theorem 3.16. *Let f, g and h be any three entire functions with $0<\lambda_h^{[p]}(f)\leq\rho_h^{[p]}(f)<\infty$ and $\overline{\sigma}_g>0$ where p is any positive integer. Then for any $\beta>1$,*

$$(i)\ \liminf_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp\left(\frac{r}{4\beta}\right)^{\rho_g}\right)}\geq\frac{\lambda_h^{[p]}(f)\,\overline{\sigma}_g}{\rho_h^{[p]}(f)},$$

$$(ii)\ \limsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp\left(\frac{r}{4\beta}\right)^{\rho_g}\right)}\geq\max\left\{\frac{\lambda_h^{[p]}(f)\,\sigma_g}{\rho_h^{[p]}(f)},\overline{\sigma}_g\right\},$$

$$(iii)\ \liminf_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\log^{[p]}M_h^{-1}M_f\left(\exp r^{\rho_g}\right)}\geq\frac{\lambda_h^{[p]}(f)\,\overline{\sigma}_g}{\rho_h^{[p]}(f)}$$

and

$$(iv)\ \limsup_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\log^{[p]}M_h^{-1}M_f\left(\exp r^{\rho_g}\right)}\geq\max\left\{\frac{\lambda_h^{[p]}(f)\,\sigma_g}{\rho_h^{[p]}(f)},\overline{\sigma}_g\right\}.$$

Proof. In view of Definition 1.8, we obtain for a sequence of values of r tending to infinity that

$$\log^{[p]}\mu_h^{-1}\mu_f\left(\exp\left(\frac{r}{4\beta}\right)^{\rho_g}\right)\leq\left(\lambda_h^{[p]}(f)+\varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_g}. \tag{3.48}$$

Also for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}\mu_f\left(\exp\left(\frac{r}{4\beta}\right)^{\rho_g}\right)\leq\left(\rho_h^{[p]}(f)+\varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_g}. \tag{3.49}$$

Now from (3.13) and (3.48), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp\left(\frac{r}{4\beta}\right)^{\rho_g}\right)}\geq\frac{\left(\lambda_h^{[p]}(f)-\varepsilon\right)\left(\overline{\sigma}_g-\varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_g}+O(1)}{\left(\lambda_h^{[p]}(f)+\varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_g}}$$

As ε (> 0) is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f\left(\exp\left(\frac{r}{4\beta}\right)^{\rho_g}\right)} \geq \overline{\sigma}_g. \tag{3.50}$$

Similarly from (3.14) and (3.49), we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f\left(\exp\left(\frac{r}{4\beta}\right)^{\rho_g}\right)} \geq \frac{\left(\lambda_h^{[p]}(f)-\varepsilon\right)\left(\sigma_g-\varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_h^{[p]}(f)}+O(1)}{\left(\rho_h^{[p]}(f)+\varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_g}}$$

$$i.e.,\ \limsup_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f\left(\exp\left(\frac{r}{4\beta}\right)^{\rho_g}\right)} \geq \frac{\lambda_h^{[p]}(f)\,\sigma_g}{\rho_h^{[p]}(f)}. \tag{3.51}$$

Likewise from (3.13) and (3.49), we have for all sufficiently large values of r that

$$\frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f\left(\exp\left(\frac{r}{4\beta}\right)^{\rho_g}\right)} \geq \frac{\left(\lambda_h^{[p]}(f)-\varepsilon\right)\left(\overline{\sigma}_g-\varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_g}+O(1)}{\left(\rho_h^{[p]}(f)+\varepsilon\right)\left[\frac{r}{4\beta}\right]^{\rho_g}}$$

i.e.,

$$\liminf_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f\left(\exp\left(\frac{r}{4\beta}\right)^{\rho_g}\right)} \geq \frac{\lambda_h^{[p]}(f)\,\overline{\sigma}_g}{\rho_h^{[p]}(f)}. \tag{3.52}$$

Thus the first part of the theorem follows from (3.50), (3.51) and (3.52).

Since $M_h^{-1}(r)$ is an increasing function of r, by similar reasoning as above the second part of the theorem follows from the first part of Lemma 2.1 and therefore its proof is omitted.

Using the same technique as above, the third and fourth part of the theorem follows from the second part of Lemma 2.1 and therefore their proofs are omitted. □

Using the notion of weak type, we may state the following theorem without its proof because it can be carried out in the line of Theorem 3.15 and Theorem 3.16 respectively.

Theorem 3.17. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$ and $\overline{\tau}_g < \infty$ where p is any positive integer. Then for any $\beta > 1$,*

$$(i)\ \liminf_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f\left(\exp(\beta r)^{\lambda_g}\right)} \leq \min\left\{\frac{\rho_h^{[p]}(f)\,\tau_g}{\lambda_h^{[p]}(f)}, \overline{\tau}_g\right\},$$

$$(ii)\ \limsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp(\beta r)^{\lambda_g}\right)}\leq\frac{\rho_h^{[p]}(f)\,\overline{\tau}_g}{\lambda_h^{[p]}(f)},$$

$$(iii)\ \liminf_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\log^{[p]}M_h^{-1}M_f\left(\exp r^{\lambda_g}\right)}\leq\min\left\{\frac{\rho_h^{[p]}(f)\,\tau_g}{\lambda_h^{[p]}(f)},\overline{\tau}_g\right\}$$

and

$$(iv)\ \limsup_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\log^{[p]}M_h^{-1}M_f\left(\exp r^{\lambda_g}\right)}\leq\frac{\rho_h^{[p]}(f)\,\overline{\tau}_g}{\lambda_h^{[p]}(f)}.$$

Theorem 3.18. *Let f,g and h be any three entire functions with $0<\lambda_h^{[p]}(f)\leq\rho_h^{[p]}(f)<\infty$ and $\tau_g>0$ where p is any positive integer. Then for any $\beta>1$,*

$$(i)\ \liminf_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp\left(\frac{r}{4\beta}\right)^{\lambda_g}\right)}\geq\frac{\lambda_h^{[p]}(f)\,\tau_g}{\rho_h^{[p]}(f)},$$

$$(ii)\ \limsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp\left(\frac{r}{4\beta}\right)^{\lambda_g}\right)}\geq\max\left\{\frac{\lambda_h^{[p]}(f)\,\overline{\tau}_g}{\rho_h^{[p]}(f)},\tau_g\right\},$$

$$(iii)\ \liminf_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\log^{[p]}M_h^{-1}M_f\left(\exp r^{\lambda_g}\right)}\geq\frac{\lambda_h^{[p]}(f)\,\tau_g}{\rho_h^{[p]}(f)}$$

and

$$(iv)\ \limsup_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\log^{[p]}M_h^{-1}M_f\left(\exp r^{\lambda_g}\right)}\geq\max\left\{\frac{\lambda_h^{[p]}(f)\,\overline{\tau}_g}{\rho_h^{[p]}(f)},\tau_g\right\}.$$

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