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MULTIVALUED FIXED POINT RESULTS AND STABILITY OF FIXED POINT SETS IN METRIC SPACES

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Abstract. In this paper we establish certain multivalued fixed point results for mappings satisfying rational type almost contractions involving a control function in the framework of metric spaces. The main result is supported with an example. We use the Hausdorff distance in our theorems. We also study the stability of fixed point sets of the above mentioned set valued contractions.
Keywords: Hausdorff metric; Multivalued mapping; Rational type almost contraction; Fixed point; Stability.

1. Introduction

Metric fixed point theory is widely recognized to have been originated in the work of S. Banach in 1922 [5], where he proved the famous contraction mapping principle. Banach's contraction mapping principle has very few parallels in modern science, in terms of the various influences it has had in the developments of different branches of mathematics and of physical science in general. Over the years, metric fixed point theory has developed in different directions. A comprehensive account of this development is provided in the handbook entitled by Kirk and Sims [29]. Further fixed point and some related results are described in [20, 33, 34, 35].

The concept of almost contractions was introduced by Berinde [7, 8]. It was shown in [7] that any strict contraction, the Kannan [27] and Zamfirescu [41] mappings, as well as a large class of quasi-contractions, are all almost contractions. Almost contractions and its generalizations were further considered in several works, such as [1, 4, 13, 14, 16].

Dass and Gupta [19] generalized the Banach's contraction mapping principle by using a contractive condition of rational type. Fixed point theorems for contractive type conditions satisfying rational inequalities in metric spaces have been developed in a number of works [2, 11, 12, 24, 25, 26, 31].

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Multivalued analysis is an important extension of the general concepts studied in mathematical analysis. Several aspects of this study are described by Aubin et al in their book [3]. Fixed point theory for multivalued operators is an important topic of set-valued analysis. Nadler [37] extended the Banach contraction principle to set-valued mappings by using the Hausdorff metric. Inspired by the results of Nadler, the fixed point theory of set-valued contraction using this Hausdorff metric has been further developed in different directions by many authors [17, 18, 22, 23, 28, 39].

Stability is a concept associated with the limiting behaviors of a system. It has been studied in the contexts of both discrete and continuous dynamical systems [38, 40]. The study of the relationship between the convergence of a sequence of mappings and their fixed points, known as the stability of fixed points, has also been widely studied in various settings [6, 9, 10, 15, 21, 30, 32, 36]. The fixed point sets of a sequence of mappings are said to be stable if they converge to the set of fixed points of the limit mapping in the Hausdorff metric. Multivalued mappings often have more fixed points than their single-valued counterparts [21, 30, 32, 36, 37]. Therefore, the set of fixed points of multivalued mappings becomes larger and hence more interesting for the study of stability.

The purpose of this paper is to establish the existence of fixed points of certain multivalued mappings in metric spaces. The mappings are assumed to satisfy certain rational type almost contractive inequalities. In Section 2 we describe some mathematical preliminaries which we use in our results in Sections 3 and 4. In Section 3 we prove a fixed point result for multivalued mapping satisfy rational type almost contractive inequalities. In Section 4 we investigate the stability of fixed point sets of above mentioned set valued contractions.

2. Mathematical Preliminaries

The following are the concepts from set valued analysis which we shall use in this paper. Let (X, d) be a metric space. Then

$N(X) = \{A : A \text{ is a non-empty subset of } X\}$,

$B(X) = \{A : A \text{ is a non-empty bounded subset of } X\}$,

$CB(X) = \{A : A \text{ is a non-empty closed and bounded subset of } X\}$ and

$C(X) = \{A : A \text{ is a non-empty compact subset of } X\}$.

For $x \in X$ and $B \in N(X)$, the function $D(x, B)$, and for $A, B \in CB(X)$, the function $H(A, B)$ are defined as follows:

$$D(x, B) = \inf \{d(x, y) : y \in B\}$$

and

$$H(A, B) = \max \{\sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A)\}.$$

H is known as the Hausdorff metric induced by d on $CB(X)$ [37]. Further, if (X, d) is complete then $(CB(X), H)$ is also complete.

Nadler [37] established the following Lemma.

Lemma 2.1. [37] Let (X, d) be a metric space and $A, B \in CB(X)$. Let $q > 1$. Then for each $x \in A$, there exists $y \in B$ such that $d(x, y) \leq q\, H(A, B)$.

In [15, 37] it is shown that the above Lemma is also valid for $q \geq 1$, if $A, B \in C(X)$.

Lemma 2.2. [15, 37] Let (X, d) be a metric space and $A, B \in C(X)$. Let $q \geq 1$. Then for each $x \in A$, there exists $y \in B$ such that $d(x, y) \leq q\, H(A, B)$.

The following is a consequence of Lemma 2.2.

Lemma 2.3. Let A and B be two nonempty compact subsets of a metric space (X, d) and $T : A \to C(B)$ be a multivalued mapping. Let $q \geq 1$. Then for $a, b \in A$ and $x \in Ta$, there exists a $y \in Tb$ such that $d(x, y) \leq q\, H(Ta, Tb)$.

Definition 2.1. Let X be a nonempty set, $f : X \longrightarrow X$ a single-valued mapping and $T : X \longrightarrow N(X)$ a multivalued mapping. A point $x \in X$ is a fixed point of f (resp. T) iff $x = fx$ (resp. $x \in Tx$).

The set of all fixed points of f and T are denoted respectively by $F(f)$ and $F(T)$.

3. Main Results

Theorem 3.1. *Let (X, d) be a complete metric space and $T : X \longrightarrow C(X)$ a multivalued mapping. Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing and continuous function with $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ and $\psi(t) < t$ for each $t > 0$. Suppose that there exists a real number $L \geq 0$ such that, for all $x, y \in X$,*

$$(3.1)\quad \begin{aligned} H(Tx, Ty) \leq \psi(max\,\{&d(x, y), D(x, Tx), D(y, Ty), \frac{D(y, Tx) + D(x, Ty)}{2}, \\ &\frac{D(y, Ty)\,[1 + D(x, Tx)]}{1 + d(x, y)}, \frac{D(y, Tx)\,[1 + D(x, Ty)]}{1 + d(x, y)}\}) \\ &+ L\, min\,\{D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\}. \end{aligned}$$

Then T has a fixed point in X.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. Then, by Lemma **??**, there exists an $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq H(Tx_0, Tx_1).$$

Applying (3.1) and using the monotone property of ψ, we have

$$\begin{aligned} d(x_1, x_2) &\leq H(Tx_0, Tx_1) \\ &\leq \psi(\max\,\{d(x_0, x_1), D(x_0, Tx_0), D(x_1, Tx_1), \frac{D(x_1, Tx_0) + D(x_0, Tx_1)}{2}, \end{aligned}$$

$$\frac{D(x_1,\ Tx_1)\,[1+D(x_0,\ Tx_0)]}{1+d(x_0,\ x_1)},\ \frac{D(x_1,\ Tx_0)\,[1+D(x_0,\ Tx_1)]}{1+d(x_0,\ x_1)}\})$$
$$+L\min\{D(x_0,\ Tx_0),\ D(x_1,\ Tx_1),\ D(x_0,\ Tx_1),\ D(x_1,\ Tx_0)\}$$
$$\le \psi(\max\{d(x_0,\ x_1),\ d(x_0,\ x_1),\ d(x_1,\ x_2),\ \frac{d(x_1,\ x_1)+d(x_0,\ x_2)}{2},$$
$$\frac{d(x_1,\ x_2)\,[1+d(x_0,\ x_1)]}{1+d(x_0,\ x_1)},\ \frac{d(x_1,\ x_1)\,[1+d(x_0,\ x_2)]}{1+d(x_0,\ x_1)}\})$$
$$+L\min\{d(x_0,\ x_1),\ d(x_1,\ x_2),\ d(x_0,\ x_2),\ d(x_1,\ x_1)\}$$
$$=\psi(\max\{d(x_0,\ x_1),\ d(x_1,\ x_2),\ \frac{d(x_0,\ x_2)}{2}\}).$$

Since $\frac{d(x_0,\ x_2)}{2} \le \frac{d(x_0,\ x_1)+d(x_1,\ x_2)}{2} \le \max\{d(x_0,\ x_1),\ d(x_1,\ x_2)\}$, it follows that

$$d(x_1,\ x_2) \le \psi(\max\{d(x_0,\ x_1),\ d(x_1,\ x_2)\}). \tag{3.2}$$

Suppose that $d(x_0,\ x_1) < d(x_1,\ x_2)$. Then $d(x_1,\ x_2) \ne 0$, and it follows from (3.2) and a property of ψ that

$$d(x_1,\ x_2) \le \psi(d(x_1,\ x_2)) < d(x_1,\ x_2),$$

which is a contradiction. Hence $d(x_1,\ x_2) \le d(x_0,\ x_1)$. Then from (3.2), we have

$$d(x_1,\ x_2) \le \psi(d(x_0,\ x_1)). \tag{3.3}$$

Since $x_2 \in Tx_1$, by Lemma **??**, there exists an $x_3 \in Tx_2$ such that

$$d(x_2,\ x_3) \le H(Tx_1,\ Tx_2).$$

Applying (3.1) and using the monotone property of ψ, we have

$$d(x_2,\ x_3) \le H(Tx_1,\ Tx_2)$$
$$\le \psi(\max\{d(x_1,x_2),\ D(x_1,Tx_1),\ D(x_2,Tx_2),\ \frac{D(x_2,\ Tx_1)+D(x_1,Tx_2)}{2},$$
$$\frac{D(x_2,\ Tx_2)\,[1+D(x_1,\ Tx_1)]}{1+d(x_1,\ x_2)},\ \frac{D(x_2,\ Tx_1)\,[1+D(x_1,\ Tx_2)]}{1+d(x_1,\ x_2)}\})$$
$$+L\min\{D(x_1,\ Tx_1),\ D(x_2,\ Tx_2),\ D(x_1,\ Tx_2),\ D(x_2,\ Tx_1)\}$$
$$\le \psi(\max\{d(x_1,\ x_2),\ d(x_1,\ x_2),\ d(x_2,\ x_3),\ \frac{d(x_2,\ x_2)+d(x_1,\ x_3)}{2},$$
$$\frac{d(x_2,\ x_3)\,[1+d(x_1,\ x_2)]}{1+d(x_1,\ x_2)},\ \frac{d(x_2,\ x_2)\,[1+d(x_1,\ x_3)]}{1+d(x_1,\ x_2)}\})$$
$$+L\min\{d(x_1,\ x_2),\ d(x_2,\ x_3),\ d(x_1,\ x_3),\ d(x_2,\ x_2)\}$$
$$=\psi(\max\{d(x_1,\ x_2),\ d(x_2,\ x_3),\ \frac{d(x_1,\ x_3)}{2}\}).$$

Since

$$\frac{d(x_1,\ x_3)}{2} \le \frac{d(x_1,\ x_2)+d(x_2,\ x_3)}{2} \le \max\{d(x_1,\ x_2),\ d(x_2,\ x_3)\},$$

it follows that

$$d(x_2,\ x_3) \le \psi(\max\{d(x_1,\ x_2),\ d(x_2,\ x_3)\}). \tag{3.4}$$

Suppose that $d(x_1, x_2) < d(x_2, x_3)$. Then $d(x_2, x_3) \neq 0$, and it follows from (3.3) and a property of ψ that

$$d(x_2, x_3) \leq \psi(d(x_2, x_3)) < d(x_2, x_3),$$

which is a contradiction. Hence $d(x_2, x_3) \leq d(x_1, x_2)$. From (3.3), we have

$$d(x_2, x_3) \leq \psi(d(x_1, x_2)). \tag{3.5}$$

Continuing this process we construct a sequence $\{x_n\}$ such that, for all $n \geq 0$

$$x_{n+1} \in Tx_n \tag{3.6}$$

and

$$d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})). \tag{3.7}$$

By repeated application of (**??**) and the monotone property of ψ, we have

$$d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})) \leq \psi^2(d(x_{n-1}, x_n)) \leq ... \leq \psi^{n+1}(d(x_0, x_1)).$$

Then, by a property of ψ, we have

$$\sum_n d(x_n, x_{n+1}) \leq \sum_n \psi^n(d(x_0, x_1)) < \infty.$$

This shows that $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists a $z \in X$ such that

$$x_n \longrightarrow z \ \text{ as } \ n \longrightarrow \infty. \tag{3.8}$$

Since $x_{n+1} \in Tx_n$, for all $n \geq 1$, applying (3.1) and using the monotone property of ψ, we get

$$\begin{aligned}
D(x_{n+1}, Tz) &\leq H(Tx_n, Tz) \\
&\leq \psi(\max\{d(x_n, z), D(x_n, Tx_n), D(z, Tz), \frac{D(z, Tx_n) + D(x_n, Tz)}{2}, \\
&\qquad \frac{D(z, Tz)\,[1 + D(x_n, Tx_n)]}{1 + d(x_n, z)}, \frac{D(z, Tx_n)\,[1 + D(x_n, Tz)]}{1 + d(x_n, z)}\}) \\
&\qquad + L \min\{D(x_n, Tx_n), D(z, Tz), D(x_n, Tz), D(z, Tx_n)\} \\
&\leq \psi(\max\{d(x_n, z), d(x_n, x_{n+1}), D(z, Tz), \frac{d(z, x_{n+1}) + D(x_n, Tz)}{2}, \\
&\qquad \frac{D(z, Tz)\,[1 + d(x_n, x_{n+1})]}{1 + d(x_n, z)}, \frac{d(z, x_{n+1})\,[1 + D(x_n, Tz)]}{1 + d(x_n, z)}\}) \\
&\qquad + L \min\{d(x_n, x_{n+1}), D(z, Tz), D(x_n, Tz), d(z, x_{n+1})\}.
\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$ in the above inequality, using (**??**), and the continuity of ψ, we have

$$D(z, Tz) \leq \psi(\max\{0, 0, D(z, Tz), \frac{D(z, Tz)}{2}, D(z, Tz), 0\}) \leq \psi(D(z, Tz)).$$

Suppose that $D(z, Tz) \neq 0$. Then, from the above inequality, it follows by a property of ψ that

$$D(z, Tz) \leq \psi(D(z, Tz)) < D(z, Tz),$$

which is a contradiction. Hence $D(z, Tz) = 0$. Since $Tz \in C(X)$, Tz is compact and hence Tz is closed; that is, $Tz = \overline{Tz}$, where $\overline{Tz}$ denotes the closure of Tz. Now $D(z, Tz) = 0$ implies that $z \in \overline{Tz} = Tz$; that is, z is a fixed point of T.

Example 3.1. Let $X = [a, b]$, where $a, b \in \mathbb{R}$ with $1 < a < b$ and "d" is usual metric on X. Let $T : X \longrightarrow C(X)$ be defined as follows:

$$Tx = [x + \frac{1}{x} - \frac{1}{b}, b], \text{ for } x \in X.$$

Let ψ: $[0, \infty) \longrightarrow [0, \infty)$ be defined by:

$$\psi(t) = k\,t, \text{ where } t \in [0, \infty) \text{ and } 1 - \frac{1}{b^2} \leq k < 1.$$

Let $L \geq 0$ any real number.

Then all of the conditions of Theorem 3.1 are satisfied and it is seen that "b" is a fixed point of T in X.

Using $\psi(t) = k\,t$, where $0 < k < 1$, in Theorem 3.1, we have the following corollary.

Corollary 3.1. *Let (X, d) be a complete metric space and $T : X \longrightarrow C(X)$ a multivalued mapping. Suppose that there exist two real numbers $L \geq 0$ and $0 < k < 1$ such that, for all $x, y \in X$,*

$$\text{(3.9)} \quad H(Tx, Ty) \leq k\, max\,\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(y, Tx) + D(x, Ty)}{2}, \frac{D(y, Ty)\,[1 + D(x, Tx)]}{1 + d(x, y)}, \frac{D(y, Tx)\,[1 + D(x, Ty)]}{1 + d(x, y)}\} + L\, min\,\{D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\}.$$

Then T has a fixed point in X.

With $L = 0$ and $\psi(t) = k\,t$, where $0 < k < 1$, in Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let (X, d) be a complete metric space and $T : X \longrightarrow C(X)$ a multivalued mapping. Suppose that there exists a real number $0 < k < 1$ such that, for all $x, y \in X$,*

$$\text{(3.10)} \quad H(Tx, Ty) \leq k\, max\,\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(y, Tx) + D(x, Ty)}{2}, \frac{D(y, Ty)\,[1 + D(x, Tx)]}{1 + d(x, y)}, \frac{D(y, Tx)\,[1 + D(x, Ty)]}{1 + d(x, y)}\}.$$

Then T has a fixed point in X.

4. Stability of fixed point sets

Theorem 4.1. *Let (X, d) be a complete metric space and $T_i : X \longrightarrow C(X)$, $i = 1, 2$ be two multivalued mappings. Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing and continuous function with $\Phi(t) = \sum_{n=1}^{\infty} \psi^n(t) < \infty$, $\Phi(t) \to 0$ as $t \to 0$ and $\psi(t) < t$ for each $t > 0$. Suppose that there exists a real number $L \geq 0$ such that the T_i satisfy (3.1) for every $i = 1, 2$; that is, for all $x, y \in X$,*

$$H(T_ix, T_iy) \leq \psi(max\,\{d(x, y), D(x, T_ix), D(y, T_iy), \frac{D(y, T_ix)+D(x, T_iy)}{2}, \frac{D(y, T_iy)\,[1 + D(x, T_ix)]}{1 + d(x, y)}, \frac{D(y, T_ix)\,[1 + D(x, T_iy)]}{1 + d(x, y)}\}) + L\; min\,\{D(x, T_ix), D(y, T_iy), D(x, T_iy), D(y, T_ix)\}.$$

Then $H(F(T_1), F(T_2)) \leq \Phi(k)$ where $k = \sup_{x\in X} H(T_1x, T_2x)$.

Proof. From Theorem 3.1 the set of fixed points of T_i $(i = 1, 2)$ is non-empty; that is, $F(T_i) \neq \emptyset$, for $i = 1, 2$. Let $y_0 \in F(T_1)$; that is $y_0 \in T_1y_0$. Then, by Lemma 2.2, there exists a $y_1 \in T_2y_0$ such that

$$d(y_0, y_1) \leq H(T_1y_0, T_2y_0). \tag{4.1}$$

Since $y_1 \in T_2y_0$, by Lemma **??**, there exists a $y_2 \in T_2y_1$ such that

$$d(y_1, y_2) \leq H(T_2y_0, T_2y_1).$$

Then, arguing as in the proof of Theorem 3.1, we construct a sequence $\{y_n\}$ such that, for all $n \geq 0$

$$y_{n+1} \in T_2y_n, \tag{4.2}$$

$$d(y_{n+1}, y_{n+2}) \leq \psi(d(y_n, y_{n+1})), \tag{4.3}$$

and

$$d(y_{n+1}, y_{n+2}) \leq \psi(d(y_n, y_{n+1})) \leq \psi^2(d(y_{n-1}, y_n)) \leq ... \leq \psi^{n+1}(d(y_0, y_1)). \tag{4.4}$$

Similar to the proof of Theorem 3.1, we prove that $\{y_n\}$ is a Cauchy sequence X and there exists a $u \in X$ such that

$$y_n \longrightarrow u \;\text{ as }\; n \longrightarrow \infty. \tag{4.5}$$

Also, u is a fixed point of T_2; that is, $u \in T_2u$.

From (**??**) and the definition of k, it follows that

$$d(y_0, y_1) \leq H(T_1y_0, T_2y_0) \leq k = \sup_{x\in X} H(T_1x, T_2x). \tag{4.6}$$

Again, by the triangle inequality and using (**??**), we have

$$d(y_0,\ u) \le \sum_{i=0}^{n} d(y_i,\ y_{i+1}) + d(y_{n+1},\ u) \le \sum_{i=0}^{n} \psi^i(d(y_0,\ y_1)) + d(y_{n+1},\ u).$$

Taking the limit as $n \longrightarrow \infty$ in the above inequality, using (**??**), (**??**) and the properties of ψ, we have

$$d(y_0,\ u) \le \sum_{i=0}^{\infty} \psi^i(d(y_0,\ y_1)) \le \sum_{i=0}^{\infty} \psi^i(k) = \Phi(k).$$

Thus, given an arbitrary $y_0 \in F(T_1)$, we can find a $u \in F(T_2)$ for which

$$d(y_0,\ u) \le \Phi(k).$$

Similarly, we can prove that, for arbitrary $z_0 \in F(T_2)$, there exists a $w \in F(T_1)$ such that $d(z_0,\ w) \le \Phi(k)$. Hence we conclude that

$$H(F(T_1),\ F(T_2)) \le \Phi(k).$$

Lemma 4.1. *Let $(X,\ d)$ be a complete metric space. Let $\{T_n : X \longrightarrow C(X) : n \in \mathbb{N}\}$ be a sequence of multivalued mappings, uniformly convergent to a multivalued mapping $T : X \longrightarrow C(X)$. If T_n satisfies (3.1) for every $n \in \mathbb{N}$, then T also satisfies (3.1), where the function $\psi : [0,\ \infty) \to [0,\ \infty)$ is continuous and $L \ge 0$ a real number.*

Proof. As T_n satisfies (3.1) for every $n \in \mathbb{N}$, we have

$$\begin{aligned} H(T_n x,\ T_n y) \le \psi(\max\{&d(x,\ y),\ D(x,\ T_n x),\ D(y,\ T_n y),\ \frac{D(y,\ T_n x) + D(x,\ T_n y)}{2}, \\ &\frac{D(y,\ T_n y)\,[1 + D(x,\ T_n x)]}{1 + d(x,\ y)},\ \frac{D(y,\ T_n x)\,[1 + D(x,\ T_n y)]}{1 + d(x,\ y)}\}) \\ &+ L \min\{D(x,\ T_n x),\ D(y,\ T_n y),\ D(x,\ T_n y),\ D(y,\ T_n x)\}. \end{aligned}$$

Since the sequence $\{T_n\}$ is uniformly convergent to T and ψ is continuous, taking the limit $n \to \infty$ in the above inequality, we get

$$\begin{aligned} H(Tx,\ Ty) \le \psi(\max\{&d(x,\ y),\ D(x,\ Tx),\ D(y,\ Ty),\ \frac{D(y,\ Tx) + D(x,\ Ty)}{2}, \\ &\frac{D(y,\ Ty)\,[1 + D(x,\ Tx)]}{1 + d(x,\ y)},\ \frac{D(y,\ Tx)\,[1 + D(x,\ Ty)]}{1 + d(x,\ y)}\}) \\ &+ L \min\{D(x,\ Tx),\ D(y,\ Ty),\ D(x,\ Ty),\ D(y,\ Tx)\}, \end{aligned}$$

which shows that T satisfies (3.1).

We now present our stability result.

Theorem 4.2. *Let (X, d) be a complete metric space. Let $\{T_n : X \longrightarrow C(X) : n \in \mathbb{N}\}$ be a sequence of multivalued mappings, uniformly convergent to a multivalued mapping $T : X \longrightarrow C(X)$. Suppose that T_n satisfies (3.1) for every $n \in \mathbb{N}$, where the conditions upon ψ and L are the same as in Theorem 4.1. Then*

$$\lim_{n\to\infty} H(F(T_n), F(T)) = 0;$$

that is, the fixed point sets of T_n are stable.

Proof. By lemma 4.1, T satisfies (3.1). Let $k_n = \sup_{x\in X} H(T_n x, Tx)$. Since the sequence $\{T_n\}$ is uniformly convergent to T on X,

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty} \sup_{x\in X} H(T_n x, Tx) = 0. \tag{4.7}$$

Using Theorem 4.1 we get

$$H(F(T_n), F(T)) \le \Phi(k_n), \quad \text{for every } n \in \mathbb{N}.$$

Since ψ is continuous and $\Phi(t) \to 0 \quad$ as $t \to 0$, using (**??**), we have

$$\lim_{n\to\infty} H(F(T_n), F(T)) \le \lim_{n\to\infty} \Phi(k_n) = 0;$$

that is,

$$\lim_{n\to\infty} H(F(T_n), F(T)) = 0.$$

Hence the proof is complete.

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REFERENCES

1. M. Abbas, G. V. R. Babu and G. N. Alemayehu: On common fixed points of weakly compatible mappings satisfying 'generalized condition (B)'. Filomat 25 (2011), 9–19.
2. M. Arshad, E. Karapinar and J. Ahmad: Some unique fixed point theorems for rational contractions in partially ordered metric spaces. J. Inequal. Appl. 2013 (2013) 248.
3. J. P. Aubin and A. Cellina: Differential inclutions multivalued maps and viability theory. Springer-Verlag, 1984.
4. G. V. R. Babu, M. L. Sandhya and M. V. R. Kameswari: A note on a fixed point theorem of Berinde on weak contractions. Carpathian J. Math. 24 (2008), 08–12.
5. S. Banach: Sur les oprations dans les ensembles abstraits et leurs applications aux quations intgrales. Fund Math. 3 (1922), 133–181.
6. L. Barbet and K. Nachi: Sequences of contractions and convergence of fixed points. Monografias del Seminario Matemático Garćia de Galdeano 33 (2006), 51–58.

7. V. Berinde: Approximating fixed points of weak contractions using the Picard iteration. Nonlinear Anal. Forum 9 (2004), 43–53.

8. V. Berinde: General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces. Carpathian J. Math. 24 (2008), 10–19.

9. I. Bhaumik and B. S. Choudhury: Uniform convergence and sequence of maps on a compact metric space with some chaotic properties. Analysis in theory and Applications 26 (2010), 53–58.

10. R. K. Bose and R. N. Mukherjee: Stability of fixed point sets and common fixed points of families of mappings. Indian J. Pure Appl. Math. 9 (1980), 1130–1138.

11. I. Cabrera, J. Harjani and K. Sadarangani: A fixed point theorem for contractions of rational type in partially ordered metric spaces. Ann. Univ. Ferrara 59 (2013), 251–258.

12. S. Chandok and J. K. Kim: Fixed point theorem in ordered metric spaces for generalized contractions mappings satisfying rational type expressions. J. Nonlinear Functional Anal. Appl. 17 (2012), 301–306.

13. B. S. Choudhury and N. Metiya: Fixed point theorems for almost contractions in partially ordered metric spaces. Ann. Univ. Ferrara 58 (2012), 21–36.

14. B. S. Choudhury and N. Metiya: Coincidence point theorems for a family of multivalued mappings in partially ordered metric spaces. Acta Universitatis Matthiae Belii, series Mathematics 21 (2013), 10–23.

15. B. S. Choudhury, N. Metiya and C. Bandyopadhyay: Fixed points of multivalued α-admissible mappings and stability of fixed point sets in metric spaces. Rend. Circ. Mat. Palermo 64 (2015), 43–55.

16. L. B. Ćirić, M. Abbas, R. Saadati and N. Hussain: Common fixed points of almost generalized contractive mappings in ordered metric spaces. Appl. Math. Comput. 217 (2011), 5784–5789.

17. L. B. Ćirić and J. S. Ume: Some common fixed point theorems for weakly compatible mappings. J. Math. Anal. Appl. 314 (2006), 488–499.

18. B. Damjanović, B. Samet and C. Vetro: Common fixed point theorems for multivalued maps. Acta Mathematica Scientia 32B(2) (2012), 818–824.

19. B. K. Dass and S. Gupta: An extension of Banach contraction principle through rational expressions. Inidan J. Pure Appl. Math. 6 (1975), 1455–1458.

20. Deepmala: A study on fixed point theorems for nonlinear contractions and its applications. Ph.D. Thesis, Pt. Ravishankar Shukla University, Raipur(Chhatisgarh), 2014.

21. K. Goebel and W. A. Kirk: Topics in metric fixed point theory. Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1990.

22. M. E. Gordji, H. Baghani, H. Khodaei and M. Ramezani: A generalization of Nadler's fixed point theorem. J. Nonlinear Sci. Appl. 3 (2010), 148–151.

23. A. A. Harandi: End points of setvalued contractions in metric spaces. Nonlinear Anal. 72 (2010), 132–134.

24. J. Harjani, B. López and K. Sadarangani: A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space. Abstract Appl. Anal. 2010 (2010), Article ID 190701.

25. D. S. Jaggi: Some unique fixed point theorems. Indian J. Pure Appl. Math. 8 (1977), 223–230.
26. D. S. Jaggi and B. K. Das: An extension of Banach's fixed point theorem through rational expression. Bull. Cal. Math. Soc. 72 (1980), 261–264.
27. R. Kannan: Some results on fixed points. Bull. Calcutta Math. Soc. 10 (1968), 71–76.
28. M. Kikkawa and T. Suzuki: Three fixed point theorems for generalized contractions with constants in complete metric spaces. Nonlinear Anal. 69 (2008), 2942–2949.
29. W. A. Kirk and B. Sims: Handbook of metric fixed point theory. 2001, XIII, 703 p.
30. T. C. Lim: Fixed point stability for set valued contractive mappings with applications to generalized differential equations. J. Math. Anal. Appl. 110 (1985), 436–441.
31. N. V. Luong and N. X. Thuan: Fixed point theorem for generalized weak contractions satisfying rational expressions in ordered metric spaces. Fixed Point Theory Appl. 46 (2011), 1–10.
32. J. T. Markin: A fixed point stability theorem for nonexpansive set valued mappings. J. Math. Anal. Appl. 54 (1976), 441–443.
33. L. N. Mishra, S. K. Tiwari, V. N. Mishra and I. A. Khan: Unique fixed point theorems for generalized contractive mappings in partial metric spaces. Journal of Function Spaces 2015 (2015), Article ID 960827.
34. V. N. Mishra: Some problems on approximations of functions in banach spaces. Ph.D. Thesis, Indian Institute of Technology, Roorkee, Uttarakhand, India, 2007.
35. V. N. Mishra, M. L. Mittal and U. Singh: On best approximation in locally convex space. Varahmihir Journal of Mathematical Sciences India 6(2006), 43–48.
36. S. B. Nadler Jr.: Sequences of contractions and fixed points. Pacifc J. Math. 27 (1968), 579–585.
37. S. B. Nadler Jr.: Multivalued contraction mapping. Pac. J. Math. 30 (1969), 475–488.
38. C. Robinson: Dynamical Systems: Stability, Symbolic Dynamics, and chaos. CRC Press, 2nd edition, 1998.
39. M. Shen and S. Hong: Common fixed points for generalized contractive multivalued operators in complete metric spaces. Appl. Math. Lett. 22 (2009), 1864–1869.
40. S. Strogatz: Nonlinear dynamics and chaos : With applications to physics, biology, chemistry, and engineering. Westview Press, 2001.
41. T. Zamfirescu: Fixed point theorems in metric spaces. Arch. Mat. (Basel) 23 (1972), 292–298.

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