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ON COMMON FIXED POINT FOR SEQUENCES OF MAPPINGS IN CONE METRIC SPACE *

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Abstract. In this paper the existence of a common fixed point for sequences in cone metric space will be considered. Some recent results of Lj. Gajić, T. Taniguchi, Sh. Rezapour and R. Hamlbarani will be generalized.

Keywords: Common fixed point, Sequences of self-mappings, Cone metric.

1. Introduction

The existence of a common fixed point for several mappings is studied in many papers (see [3], [4], [6], [9], [11]). In this paper we will consider the existence of a common fixed point for sequences in cone metric space.

Huang and Zhang [7] replaced the field of the real numbers by an ordered Banach space and defined cone metric space. They have proved some fixed point theorems for contractive mappings on cone metric spaces.

Let us recall some results and notations from the theory of cone metric spaces (see [1], [2], [5], [8], [10]).

Let E be a real Banach space. A subset P of E is called a *cone* if the following conditions hold:

(i) P is closed, nonempty and $P \neq \{0\}$;

(ii) $a, b \in \mathbb{R}, a, b \geq 0,$ and $x, y \in P$ imply $ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E,$ we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y,$ while $x \ll y$ will stand for $y - x \in \text{int}P$ (interior of P).

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Let E be a normed space and P a cone and $\leq$ the partial ordering defined by P. Then, P is called *normal* if there exists a positive number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \quad \text{implies} \quad \|x\| \leq K \|y\|. \tag{1.1}$$

The least positive number K satisfying (1.1) is called the *normal constant* of P. It is clear that $K \geq 1$.

Definition 1.1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \mapsto E$ satisfies:

(**d1**) $0 \leq d(x, y)$ for all $x, y \in X$;

(**d2**) $d(x, y) = 0$ if and only if $x = y$;

(**d3**) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(**d4**) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X and (X, d) is a *cone metric space*.

It is known that the class of cone metric spaces is bigger than the class of metric spaces [10].

Example 1.1. Let $E = l^1, P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0, \text{ for all } n\}, (X, \rho)$ be a metric space and $d : X \times X \mapsto E$ defined by $d(x, y) = \left\{\frac{\rho(x,y)}{2^n}\right\}_{n\geq 1}$. Then (X, d) is a cone metric space.

Example 1.2. Let $X = \mathbb{R}, E = \mathbb{R}^n$ and $P = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_i \geq 0\}$. it is easy to see that $d : X \times X \mapsto E$ defined by $d(x, y) = (|x - y|, k_1|x - y|, \ldots, k_{n-1}|x - y|)$ is a cone metric on X, where $k_i \geq 0$ for all $i \in \{1, ..., n - 1\}$.

Example 1.3. Let $E = C^1_{\mathbb{R}}([0, 1])$ with norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$. The cone $P = \{f \in E : f \geq 0\}$ is a non-normal cone.

In the following we suppose that E is a Banach space, P is a cone in E with $\text{int}P \neq \emptyset$ and $\leq$ is the partial order on E with respect to P.

Definition 1.2. [7] Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ a sequence in X. Then

(i) $\{x_n\}_{n\geq 1}$ converges to x if for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_n x_n = x$ or $x_n \to x$.

(ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence if for every c in E with $0 \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

2. Main results

In this section we will present main results of this paper.

Theorem 2.1. *Let (X, d) be a complete cone metric space, $S, T : X \mapsto X$ be continuous functions, $A_j : X \mapsto SX \cap TX$, $(j \in \mathbb{N})$ mappings commutative with S and T and for $i, j \in \mathbb{N}$, $i \neq j$, and each $x, y \in X$*

$$d(A_i x, A_j y) \leqslant q \cdot d(Tx, Sy),$$

for some $q \in (0, 1)$. Then there exists a unique common fixed point for family $\{A_j\}$, S and T.

Proof. For $x_0 \in X$ let us define a sequence $\{x_n\}$ in X such that for all $n \in \mathbb{N}$

$$\begin{aligned} Tx_{2n-1} &= A_{2n-1}x_{2n-2}, \\ Sx_{2n} &= A_{2n}x_{2n-1}. \end{aligned}$$

We shall prove that the sequence

$$y_n = \begin{cases} Tx_n & , \ n = 2k-1 \\ Sx_n & , \ n = 2k \end{cases}$$

is a Cauchy sequence.

For $n = 2k$, we get

$$\begin{aligned} d(y_n, y_{n+1}) &= d(Sx_{2k}, Tx_{2k+1}) = d(A_{2k}x_{2k-1}, A_{2k+1}x_{2k}) \\ &\leqslant q \cdot d(Tx_{2k-1}, Sx_{2k}) = q \cdot d(A_{2k-1}x_{2k-2}, A_{2k}x_{2k-1}) \\ &\leqslant q^2 \cdot d(Sx_{2k-2}, Tx_{2k-1}) \\ &\leqslant \ldots \leqslant q^{2k} \cdot d(Sx_0, Tx_1) = q^n \cdot d(Sx_0, A_1x_0) \end{aligned}$$

Evidently, same estimation is valid for $n = 2k - 1$.

So, for any $n, p \in \mathbb{N}$ we have that

$$d(y_n, y_{n+p}) \leqslant \sum_{k=n}^{n+p-1} d(y_k, y_{k+1}) \leqslant \sum_{k=n}^{n+p-1} q^k \cdot d(Sx_0, A_1x_0) \leqslant \frac{q^n}{1-q} d(Sx_0, A_1x_0).$$

Let $0 \ll c$ be given. Choose $\delta > 0$ such that $c + N_\delta(0) \subset P$, where $N_\delta(0) = \{y \in E, \ \|y\| < \delta\}$. Also, choose a natural number n_0 such that $\frac{q^n}{1-q} d(Sx_0, Tx_1) \in N_\delta(0)$, for all natural $n \geqslant n_0$.

Then, $d(y_n, y_{n+p}) \ll c$ for all $n, p \in \mathbb{N}$ and $n \geqslant n_0$. Hence $\{y_n\}$ is a Cauchy sequence and it is true even when P is non-normal cone. Since (X, d) is a complete cone metric space there exists $z \in X$ such that $\lim_n y_n = z$, and consequently $\lim_n Tx_{2n-1} = \lim_n Sx_{2n} = z$. We shall prove that $Sz = Tz$.

Using commutativity it follows

$$\begin{aligned} d(TA_{2n}x_{2n-1}, SA_{2n+1}x_{2n}) &= d(A_{2n}Tx_{2n-1}, A_{2n+1}Sx_{2n}) \\ &= d(A_{2n}A_{2n-1}x_{2n-2}, A_{2n+1}A_{2n}x_{2n-1}) \\ &\leqslant q \cdot d(SA_{2n-1}x_{2n-2}, TA_{2n}x_{2n-1}) \\ &\leqslant \ldots \leqslant q^{2n}\, d(TSx_0, SA_1x_0). \end{aligned}$$

So there exists $n_0^1 \in \mathbb{N}$ such that

$$d(TA_{2n}x_{2n-1}, SA_{2n+1}x_{2n}) \ll \frac{c}{2}, \text{ for } n \geqslant n_0.$$

On the other side, since S is continuous, there exists $n_0^2 \in \mathbb{N}$ such that

$$d(SA_{2n+1}x_{2n}, Sz) \ll \frac{c}{2}, \text{ for } n \geqslant n_0^2.$$

Now, for $n \geqslant n_0 = \max\{n_0^1, n_0^2\}$

$$\begin{aligned} d(TA_{2n}x_{2n-1}, Sz) &\leqslant d(TA_{2n}x_{2n-1}, SA_{2n+1}x_{2n}) \\ &\quad + d(SA_{2n+1}x_{2n}, Sz) \ll \frac{c}{2} + \frac{c}{2} = c. \end{aligned}$$

Since T is continuous and limit of convergent sequence is unique, it follows

$$Tz = \lim_n TA_{2n}x_{2n-1} = Sz.$$

Now, we will prove that $A_kz = Tz = Sz$, for any $k \in \mathbb{N}$. For $2n > k$

$$\begin{aligned} d(A_kz, TA_{2n}x_{2n-1}) &= d(A_kz, A_{2n}A_{2n-1}x_{2n-2}) \\ &\leqslant q \cdot d(Tz, SA_{2n-1}x_{2n-2}). \end{aligned}$$

Since, $\lim_n SA_{2n-1}x_{2n-2} = Sz = Tz$, for any $c \in \operatorname{int} P$ there exists $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$

$$d(A_kz, TA_{2n}x_{2n-1}) \ll c.$$

Thus,

$$Tz = \lim_n TA_{2n}x_{2n-1} = A_kz,\ k \in \mathbb{N}.$$

Hence, it follows that $A_kA_kz = A_kz$, for each $k \in \mathbb{N}$ since

$$\begin{aligned} d(A_kA_kz, A_kz) = d(A_kA_kz, A_{k+1}z) &\leqslant q \cdot d(SA_kz, Tz) \\ &= q \cdot d(A_kSz, Tz) = q \cdot d(A_kA_kz, A_kz), \end{aligned}$$

implies that $A_kA_kz = A_kz,\ k \in \mathbb{N}$.

It remains to prove that $y = A_kz = Sz = Tz$ is the unique common fixed point. Suppose that there exists $v \in X$ such that

$$Tv = Sv = A_kv = v,$$

for every $k \in \mathbb{N}$. Then

$$d(y,v) = d(A_k y, A_{k+1} v) \leqslant q \cdot d(Sy, Tv) = q \cdot d(y,v),$$

so $y = v$. □

Remark 2.1. For $S = T = g$, we get $A_n = f$, for all $n \in \mathbb{N}$, and, as corollary, Theorem 2.3 [10].

Corollary 2.1. *Let (X,d) be a complete cone metric space, $g : X \mapsto X$ be a continuous mapping, $f : X \mapsto g(X)$ mapping commutative with g such that for some $q \in (0,1)$*

$$d(fx, fy) \leq q \cdot d(gx, gy),$$

for all $x, y \in X$. Then there exists a unique common fixed point for f and g.

Moreover, if we suppose that $f(X)$ and $g(X)$ are complete subspace in X we can omit continuity of g and relax commutativity for f and g, thus improve Theorem 2.1 from [3]. □

Theorem 2.2. *Let (X,d) be a cone metric space, $g : X \mapsto X$, $f : X \mapsto g(X)$ weakly compatible mappings and let $f(X)$ or $g(X)$ be complete subspaces in X. If for some $q \in (0,1)$*

$$d(fx, fy) \leq q \cdot d(gx, gy)$$

for all $x, y \in X$, then there exists a unique common fixed point for f and g.

Proof. Let $\{gx_n\}$ be defined as $gx_n = fx_{n-1}$, $n \in \mathbb{N}$, for some $x_0 \in X$. As in previous theorem one can prove that it is Cauchy and so a convergent sequence. For $z = \lim gx_n$ there exists $p \in X$ such that $gp = z$. Inequality

$$d(fp, gx_n) = d(fp, fx_{n-1}) \leqslant q \cdot d(gp, gx_{n-1}),$$

implies that $\lim gx_n = fp$ so, by uniqueness of limit point, $fp = gp$.

Since $\omega = fp = gp$ is a unique point of coincidence, by Proposition 1.4 [3], ω is a unique common fixed point for f and g. □

Theorem 2.3. *Let (X,d) be a complete cone metric space, and $\{A_s\}$, $\{B_p\}$, $s, p \in \mathbb{N}$ be two sequences of mappings from X to X. Suppose that following conditions are satisfied:*

a) there exists a $q \in (0,1)$ such that

$$d(A_{2n-1}x, A_{2n}y) \leqslant q \cdot d(B_{2n-1}x, B_{2n}y),$$
$$d(A_{2n}x, A_{2m+1}y) \leqslant q \cdot d(B_{2n}x, B_{2m+1}y)$$

for all $m \geqslant n$, $m, n \in \mathbb{N}$ and all $x, y \in X$,

b) $A_{2n}B_{2m} = B_{2m}A_{2n}$ and $A_{2n-1}B_{2m-1} = B_{2m-1}A_{2n-1}$, *c)* $B_{2n}B_{2m} = B_{2m}B_{2n}$ *and* $B_{2n-1}B_{2m-1} = B_{2m-1}B_{2n-1}$,

d) $A_{2n-1}(X) \subseteq B_{2n}(X)$ *and* $A_{2n}(X) \subseteq B_{2n+1}(X)$.

If each B_q, $q \in \mathbb{N}$, *is continuous, then there exists a unique common fixed point for two sequences* $\{A_s\}$ *and* $\{B_p\}$, $s, p \in \mathbb{N}$.

Proof. Let x_0 be an arbitrary point in X. By the condition (d), there exists a point $x_1 \in X$ such that $A_1x_0 = B_2x_1$. Inductively, we can define the sequence $\{x_n\}$ such that

(2.1) $$A_nx_{n-1} = B_{n+1}x_n, \ n \in \mathbb{N}.$$

Let us show that $\{B_nx_{n-1}\}$ is a Cauchy sequence. By (2.1) and condition (a), we obtain that for every $n \in \mathbb{N}$, $n \geq 3$,

$$\begin{aligned} d(B_{2n-1}x_{2n-2}, B_{2n}x_{2n-1}) &= d(A_{2n-2}x_{2n-3}, A_{2n-1}x_{2n-2}) \\ &\leqslant q \cdot d(B_{2n-2}x_{2n-3}, B_{2n-1}x_{2n-2}) \\ &= q \cdot d(A_{2n-3}x_{2n-4}, A_{2n-2}x_{2n-3}) \\ &\leq q^2 \cdot d(B_{2n-3}x_{2n-4}, B_{2n-2}x_{2n-3}) \\ &\leqslant \ldots \leqslant q^{2n-2}d(B_1x_0, A_1x_0) \end{aligned}$$

So we have that

$$d(B_nx_{n-1}, B_mx_{m-1}) \leq \frac{q^{n-1}}{1-q}\, d(B_1x_0, A_1x_0)$$

so $\{B_nx_{n-1}\}$ is a Cauchy sequence.

Let $z = \lim_n B_nx_{n-1}$. Now, since B_n, $n \in \mathbb{N}$, is a continuous, we obtain that

$$\begin{aligned} B_{2m}z = B_{2m}(\lim_n B_{2n+1}x_{2n}) &= \lim_n B_{2m}B_{2n+1}x_{2n} \\ = \lim_n B_{2m}A_{2n}x_{2n-1} &= \lim_n A_{2n}B_{2m}x_{2n-1} \end{aligned}$$

and $B_{2m+1}z = \lim_n A_{2n+1}B_{2m+1}x_{2n}$.

We are going to prove that $B_{2m}z = B_{2m+1}z$.

$$\begin{aligned} &d(B_{2m}z, B_{2m+1}z) \\ &\quad \leqslant d(B_{2m}z, A_{2n}B_{2m}x_{2n-1}) + d(A_{2n}B_{2m}x_{2n-1}, A_{2n+1}B_{2m+1}x_{2n}) \\ &\qquad + d(A_{2n+1}B_{2m+1}x_{2n}, B_{2m+1}z) \\ &\quad \leqslant d(B_{2m}z, B_{2m}B_{2n+1}x_{2n}) + d(B_{2m+1}B_{2n+2}x_{2n+1}, B_{2m+1}z) \\ &\qquad + q \cdot d(B_{2m}B_{2n}x_{2n-1}, B_{2m+1}B_{2n+1}x_{2n}) \\ &\quad \leqslant d(B_{2m}z, B_{2m}B_{2n+1}x_{2n}) + d(B_{2m+1}B_{2n+2}x_{2n+1}, B_{2m+1}z) \\ &\qquad + q \cdot d(B_{2m}B_{2n}x_{2n-1}, B_{2m}z) + q \cdot d(B_{2m}z, B_{2m+1}z) \\ &\qquad + q \cdot d(B_{2m+1}z, B_{2m+1}B_{2n+1}x_{2n}). \end{aligned}$$

For any $c \in \operatorname{int} P$, there exist $n_0^1 \in \mathbb{N}$ such that, for $n \geqslant n_0^1$,

$$d(B_{2m}z, B_{2m}B_n x_{n-1}) \ll \frac{1-q}{1+q}\, c$$

and $n_0^2 \in \mathbb{N}$ such that, for $n \geqslant n_0^2$,

$$d(B_{2m+1}z, B_{2m+1}B_n x_{n-1}) \ll \frac{1-q}{2(1+q)}\, c$$

Obviously, for $n \geqslant n_0 = \max\{n_0^1, n_0^2\}$

$$(1-q)d(B_{2m}z, B_{2m+1}z) \ll (1-q)\, c,$$

so $d(B_{2m}z, B_{2m+1}z) \ll c$ for any $c \in \operatorname{int} P$, which implies that $d(B_{2m}z, B_{2m+1}z) = 0$ e.t. $B_{2m}z = B_{2m+1}z$.

Similarly we can prove that $d(B_{2m+1}z, B_{2m+2}z) = 0,\ m \in \mathbb{N} \cup \{0\}$ and we have that

$$B_m z = B_{m+1} z, \ \text{for } m \in \mathbb{N}.$$

Now, we shall prove that $A_n z = B_n z$, for all $n \in \mathbb{N}$.

If $m \geqslant n$, then

$$\begin{aligned} d(B_{2n+1}B_{2m+2}x_{2m+1}, A_{2n}z) &= d(A_{2m+1}B_{2n+1}x_{2m}, A_{2n}z) \\ &\leqslant q \cdot d(B_{2m+1}B_{2n+1}x_{2n}, B_{2m}z). \end{aligned}$$

Since $\lim_m B_{2n+1}B_m x_{m-1} = B_{2n+1}z = B_{2n}z$, it follows

$$A_{2n}z = \lim_m B_{2n+1}\, B_{2m+2}\, x_{2m+1} = B_{2n+1}z.$$

Similarly, $B_{2n}z = A_{2n-1}z$, for all $n \in \mathbb{N}$, so

$$A_{n+1}z = A_n z = B_n z = B_{n+1}z, \ \text{for } n \in \mathbb{N}.$$

Hence z is a coincidence point for the sequences $\{A_s\}$ and $\{B_p\}$.

Furthermore, for any $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d(A_{2n}z, A_{2n+1}A_{2n+1}z) &\leqslant q \cdot d(B_{2n}z, B_{2n+1}A_{2n+1}z) \\ &\leqslant q \cdot d(A_{2n}z, A_{2n+1}A_{2n+1}z), \end{aligned}$$

which implies $d(A_{2n}z, A_{2n+1}A_{2n+1}z) = 0,\ n \in \mathbb{N}$.

Therefore, we obtain that $u = A_p(u) = B_p(u),\ p \in \mathbb{N}$, setting $u = A_m z$.

Let us prove that u is the unique fixed point of $\{A_s\}$ and $\{B_p\}$. If there exists w, such that $w = A_s w = B_p w$, for all $s, p \in \mathbb{N}$, then

$$\begin{aligned} d(u, w) &= d(A_{2m-1}u, A_{2m}w) \leqslant q \cdot d(B_{2m-1}u, B_{2m}w) \\ &\leqslant q \cdot d(u, w), \end{aligned}$$

so $d(u, w) = 0$, which means that $w = u$, $\square$.

Remark 2.2. If (X, d) is metric space(in usually sense) we obtain T. Takeshi fixed point result [11].

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