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CONSTANT RATIO CURVES ACCORDING TO BISHOP FRAME IN MINKOWSKI 3-SPACE $\mathbb{E}_1^3$

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Abstract. In the present paper, we consider a curve in Minkowski 3-space $\mathbb{E}_1^3$ as a curve whose position vector can be written as linear combination of its Bishop frame vectors. In particular, we study the non-null curves in $\mathbb{E}_1^3$ and characterize such curves in terms of their Bishop curvatures. Further, we obtain some results of $T\mathbb{E}_1^3$.

Keywords: Position vector, Bishop frame, constant ratio curve.

1. Introduction

A curve $x : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ in Euclidean 3curvatures $\kappa(s)$ and $\tau(s)$. From the elementary differential geometry it is well known that a curve $x(s)$ in $\mathbb{E}^3$ lies on a sphere if its position vector (denoted also by x) lies on its normal plane at each point. If the position vector x lies on its rectifying plane then $x(s)$ is called *rectifying curve* [6]. Rectifying curves characterized by the simple equation

$$x(s) = \lambda(s)T(s) + \mu(s)N_2(s),$$

where $\lambda(s)$ and $\mu(s)$ are smooth functions and $T(s)$ and $N_2(s)$ are tangent and binormal vector fields of x respectively [6]. In the same paper B. Y. Chen gave a simple characterization of rectifying curves. In particular, it is shown in [10], that there exists a simple relation between rectifying curves and centrodes, which play an important role in mechanics kinematics as well as in differential geometry in defining the curves of constant procession. It is also provided that a twisted curve is congruent to a non constant linear function of s [7]. Further, in the Minkowski 3-space $\mathbb{E}_1^3$, the rectifying curves are investigated in ([12],[15],[16],[17]). In [17] a characterization of the space-like, the time-like and the null rectifying curves in the Minkowski 3-space in terms of centrodes is given. For a study on rectifying curves in the dual Lorentzian space D_1^3 see [22]. For the characterization of rectifying

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curves in three dimensional compact Lie groups or in dual spaces see [24] or [1], respectively.

For any curve $x(s)$ in $\mathbb{E}_t^n$ with index t, the position vector x can be decompose into its tangential and normal components at each point:

$$x = x^T + x^N. \tag{1.1}$$

A non-null curve $x(s)$ in $\mathbb{E}_t^n$ is said to be of *constant ratio* if the ratio $\left\|x^T\right\| : \left\|x^N\right\|$ is constant on $x(I)$ where $\left\|x^T\right\|$ and $\left\|x^N\right\|$ denote the length of x^T and x^N, respectively [5].

Moreover, a curve in $\mathbb{E}_t^n$ is called *T-constant* (resp. *N-constant*) if the tangential component x^T (resp. the normal component x^N) of its position vector x is of constant length [5]. Recently in [13] the authors give the necessary and sufficient conditions for twisted curves in Euclidean 3-space $\mathbb{E}^3$ to become T-constant or N-constant. See also [14] for the results of T-constant or N-constant curves in Euclidean 4-space $\mathbb{E}^4$.

The Frenet frame of a three times continuously differentiable non-degenerate space-like (time-like) curve invariant under semi-Euclidean space has long been the standard vehicle for analysing properties of the space-like (time-like) curve invariant under semi-Euclidean motions. For arbitrary moving frames that is, orthonormal basis fields, we can express the derivatives of the frame with respect to the space-like (time-like) curve parameter in terms of the frame its self, and due to semi-orthonormality the coefficient matrix is always semi-skew symmetric. Thus it generally has three nonzero entries. The Frenet frame gains part of its special significance from the fact that one of the three derivatives is zero. Another feature of the Frenet frame is that it is adapted to the space-like (time-like) curve: the members are either tangent to or perpendicular to the space-like (time-like) curve. In [2, 18], the authors gave new frames (Bishop frames) of a non-null curve in the Minkowski 3-space $\mathbb{E}_1^3$. Recently, many works related to Bishop frame have been done by several authors. In [20], authors studied space-like biharmonic slant helices according to the Bishop frame in the Lorentzian group of rigid motions. In [19, 23], the authors gave some characterizations of space-like and time-like curves according to Bishop frame in Minkowski 3-space.

In the present study, we consider a non-null curve in $\mathbb{E}_1^3$ according to Bishop frame whose position vector satisfies the parametric equation

$$x(s) = m_0(s)T(s) + m_1(s)M_1(s) + m_2(s)M_2(s), \tag{1.2}$$

for some differentiable functions, $m_i(s)$, $0 \leq i \leq 2$. We characterize the twisted non-null curves in terms of their curvature functions $m_i(s)$ and give the necessary and sufficient conditions for non-null curves to become T-constant or N-constant.

2. Basic Concepts

Let $\mathbb{E}_t^n$ denote the pseudo-Euclidean n-space with index t. Then the pseudo-Euclidean metric on $\mathbb{E}_t^n$ is given by

$$g = -\sum_{i=1}^{t} dx_i^2 + \sum_{j=t+1}^{n} dx_j^2,$$

where $(x_1, ..., x_n)$ is a rectangular coordinate system of $\mathbb{E}_t^n$. In particular $\mathbb{E}_1^n$ is known as the Lorentzian-Minkowski space-time.

For given positive number r, we put

$$\mathbb{S}_t^{n-1}(r^2) = \left\{x \in \mathbb{E}_t^n : g(x,x) = r^2\right\},$$

and

$$\mathbb{H}_{t-1}^{n-1}(-r^2) = \left\{x \in \mathbb{E}_t^n : g(x,x) = -r^2\right\}.$$

It is known that $\mathbb{S}_t^{n-1}(r^2)$ and $\mathbb{H}_{t-1}^{n-1}(-r^2)$ are called pseudo-Riemannian and Pseudo-hyperbolic spaces respectively. In particular, $\mathbb{S}_1^{n-1}(r^2)$ is called a de Sitter space-time and $\mathbb{H}_1^{n-1}(-r^2)$ is an anti-de Sitter space-time [11]. The hyperbolic space $\mathbb{H}^{n-1}(-r^2)$ is defined by

$$\mathbb{H}^{n-1}(-r^2) = \left\{x \in \mathbb{E}_t^n : g(x,x) = -r^2 \text{ and } x_1 > 0\right\}.$$

Recall that an arbitrary vector $v \in \mathbb{E}_t^n$ is called space-like if $g(v,v) > 0$ or $v = 0$, time-like $g(v,v) < 0$, and null (light-like) if $g(v,v) = 0$ and $v \neq 0$. The norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$ and two vectors v and w are said to be orthonormal, if $g(v,w) = 0$. Further, an arbitrary curve $x(s)$ of $\mathbb{E}_t^n$ is called space-like, time-like or null it its velocity vector $x^{'}(s)$ is space-like, time-like or null, respectively [21]. A space-like or time-like curve (i.e., non-null curve) has unit speed, if $g(x^{'}(s), x^{'}(s)) = \pm 1$. The light cone $\mathcal{LC}$ of $\mathbb{E}_t^n$ defined to be

$$\mathcal{LC} = \{x \in \mathbb{E}_t^n,\ g(x,x) = 0\}.$$

Denoted the moving Frenet frame along a space curve $x(s)$ by $\{T, N_1, N_2\}$ where T, N_1 and N_2 are tangent, principal normal and binormal vector of $x(s)$, respectively. For a curve in the Minkowski 3-space $\mathbb{E}_1^3$, the following Frenet formulae are given:

Case 1: If $x(s)$ is a space-like curve, then

$$\begin{aligned} T' &= \kappa N_1 \\ N_1' &= -\epsilon\kappa T + \tau N_2 \\ N_2' &= \tau N_1, \end{aligned}$$

where $\langle T,T\rangle = 1$, $\langle N_1,N_1\rangle = \epsilon$, $\langle N_2,N_2\rangle = -\epsilon$, $(\epsilon = \pm 1)$ [25].

Case 2: If $x(s)$ is a timelike curve, then

$$\begin{aligned} T' &= \kappa N_1 \\ N_1' &= \kappa T + \tau N_2 \\ N_2' &= -\tau N_1, \end{aligned}$$

where $\langle T,T\rangle = -1$, $\langle N_1,N_1\rangle = \langle N_2,N_2\rangle = 1$ [25].

Denote by $\{T, M_1, M_2\}$ the moving Bishop frame along the curve $x(s) : I \subset \mathbb{R} \to \mathbb{E}_1^3$ in the Minkowski 3-Space $\mathbb{E}_1^3$.

For an unit speed space-like curve $x(s)$ in the space $\mathbb{E}_1^3$, the following Bishop formulas are given,

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ -\epsilon k_1 & 0 & 0 \\ -\epsilon k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix} \tag{2.1}$$

The relations between κ, τ, θ and k_1, k_2 are given as follows:

$$\begin{aligned} \kappa(s) &= \sqrt{\left|k_1^2 - k_2^2\right|} \\ \theta(s) &= \arg\tanh\left(\frac{k_2}{k_1}\right), k_1 \neq 0. \end{aligned}$$

So that k_1 and k_2 effectively correspond to Cartesian coordinate system for the polar coordinates κ, θ with $\theta = \int \tau(s)ds$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ due to the differentiation [2, 4].

For a unit timelike curve $x(s)$ in the space $\mathbb{E}_1^3$, the following Bishop formulas are given,

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix} \tag{2.2}$$

where $\langle T,T\rangle = -1$, $\langle M_1,M_1\rangle = 1$, $\langle M_2,M_2\rangle = 1$.

One can show that

$$\begin{aligned} \kappa(s) &= \sqrt{k_1^2 + k_2^2} \\ \theta(s) &= \arctan\left(\frac{k_2}{k_1}\right), k_1 \neq 0 \\ \tau(s) &= \frac{d\theta(s)}{ds} \end{aligned}$$

[18].

3. Curves of Constant Ratio According to Bishop Frame

In the present section, we characterize the non-null curves given with the arc length function s in $\mathbb{E}_1^3$ in terms of their curvatures. Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed regular curve with curvatures $k_1(s)$ and $k_2(s)$. The position vector of the curve (also defined by x) satisfies the vectorial equation (1.2), for some differential functions $m_i(s)$, $0 \leq i \leq 2$. Let the curve x is space-like. Differentiating (1.2) with respect to the arc length parameter s and using the Bishop frame equations (2.1), we obtain

$$\begin{aligned} x'(s) &= (m_0'(s) - \epsilon k_1(s)m_1(s) - \epsilon k_2(s)m_2(s))T(s) \\ &\quad +(m_1'(s) + k_1(s)m_0(s))M_1(s) \\ &\quad +(m_2'(s) - k_2(s)m_0(s))M_2(s). \end{aligned} \tag{3.1}$$

It follows that

$$\begin{aligned} m_0' - \epsilon k_1 m_1 - \epsilon k_2 m_2 &= 1, \\ m_1' + k_1 m_0 &= 0, \\ m_2' - k_2 m_0 &= 0. \end{aligned} \tag{3.2}$$

When the curve x is time-like, differentiating (1.2) with respect to the arc length parameter s and using the Bishop frame equations (2.2), we obtain

$$\begin{aligned} x'(s) &= (m_0'(s) + k_1(s)m_1(s) + k_2(s)m_2(s))T(s) \\ &\quad +(m_1'(s) + k_1(s)m_0(s))M_1(s) \\ &\quad +(m_2'(s) + k_2(s)m_0(s))M_2(s). \end{aligned} \tag{3.3}$$

It follows that

$$\begin{aligned} m_0' + k_1 m_1 + k_2 m_2 &= 1, \\ m_1' + k_1 m_0 &= 0, \\ m_2' + k_2 m_0 &= 0. \end{aligned} \tag{3.4}$$

Definition 3.1. Let $x : I \subset \mathbb{R} \to \mathbb{E}_t^n$ be a non-null unit speed curve in pseudo-Riemannian space $\mathbb{E}_t^n$. Then the position vector x can be decompose into its tangential and normal components at each point as in (1.1). If the ratio $\|x^T\| : \|x^N\|$ is constant on $x(I)$ then x is said to be of *constant-ratio* [5].

For a unit speed non-null curve x in $\mathbb{E}_t^n$, the gradient of the distance function $\rho = \|x(s)\|$ is given by

$$grad\rho = \frac{d\rho}{ds}x'(s) = \frac{< x(s), x'(s) >}{\|x(s)\|}x'(s), \tag{3.5}$$

where T is the tangent vector field of x.

Lemma 3.1. *[9] Let $x : I \subset \mathbb{R} \to \mathbb{E}_t^n$ be a unit speed non-null curve in $\mathbb{E}_t^n$ with index t. Then $\|grad\rho\| = c$ holds for a constant c if and only if, up to translation of the arc length function s, we have $\|x(s)\| = cs$.*

Theorem 3.1. *[8] Let $x : I \subset \mathbb{R} \to \mathbb{E}_t^n$ be a unit speed space-like curve in $\mathbb{E}_t^n$ with index t. Then $\|grad\rho\| = c$ holds for a constant c if and only if one of the following eight cases occurs:*

i) x lies in the light-like cone $\mathcal{LC}$.

ii) x lies in a pseudo-Riemannian sphere $\mathbb{S}_t^{n-1}(r^2)$.

iii) x lies in a pseudo-hyperbolic space $\mathbb{H}_{t-1}^{n-1}(-r^2)$.

iv) x lies on open portion of a space-like line through the origin.

v) There exist a real number $b > 1$ and time-like unit speed curve $y = y(u)$ which lies in the unit pseudo-Riemannian sphere $\mathbb{S}_t^{n-1}(1)$ such that x is given by $x(s) = bsy\left(\frac{\sqrt{b^2-1}}{b}\ln s\right)$.

vi) There exist a real number $b \in (0, 1)$ and space-like unit speed curve $y = y(u)$ which lies in the unit pseudo-Riemannian sphere $\mathbb{S}_t^{n-1}(1)$ such that x is given by $x(s) = bsy\left(\frac{\sqrt{1-b^2}}{b}\ln s\right)$.

vii) There exist a null curve $y = y(s)$ lying in the unit pseudo-Riemannian $\mathbb{S}_t^{n-1}(1)$ such that x is given by $x(s) = bsy(s)$.

viii) There exist a real number $b > 0$ and space-like unit speed curve $y = w(u)$ which lies in the unit pseudo-hyperbolic space $\mathbb{H}_{t-1}^{n-1}(1)$ such that x lies given by $x(s) = bsw\left(\frac{\sqrt{1-b^2}}{b}\ln s\right)$.

The following result characterizes constant-ratio curves according to its Bishop frame in $\mathbb{E}_1^3$.

Proposition 3.1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed non-null curve and be of constant-ratio in $\mathbb{E}_1^3$, then the position vector of the curve has one of the following parametrizations;*

i) If x is space-like, then

$$\begin{aligned} x(s) &= c^2 sT(s) \\ &+\frac{\epsilon c^2 k_2\left(k_1^2-k_2^2\right)s-k_2'\left(c^2-1\right)}{\epsilon\left(k_1'k_2-k_1k_2'\right)}M_1(s) \\ &+\frac{\epsilon c^2 k_1\left(k_1^2-k_2^2\right)s-k_1'\left(c^2-1\right)}{\epsilon\left(-k_1'k_2+k_1k_2'\right)}M_2(s). \end{aligned}$$

ii) If x is time-like, then

$$\begin{aligned} x(s) &= -c^2 sT(s) \\ &+\frac{c^2k_2\left(k_1^2+k_2^2\right)s+k_2'\left(c^2+1\right)}{-k_1'k_2+k_1k_2'}M_1(s) \\ &+\frac{c^2k_1\left(k_1^2+k_2^2\right)s+k_1'\left(c^2+1\right)}{k_1'k_2-k_1k_2'}M_2(s), \end{aligned}$$

where c is the real constant.

Proof. Let x be a non-null curve of constant-ratio given with the arc length function s. Then, from the previous result the distance function ρ of x satisfies the equality $\rho = \|x(s)\| = cs$ for some real constant c. Further, using (3.5) we get

$$\left\|grad\rho\right\| = \frac{< x(s), x'(s) >}{\|x(s)\|} = c. \tag{3.6}$$

Since, x is a non-null curve of $\mathbb{E}_1^3$, then it satisfies the equality (3.1). If x is space-like, we get

$$\begin{aligned} m_0(s) &= c^2 s \\ m_1(s) &= \frac{\epsilon c^2k_2\left(k_1^2-k_2^2\right)s-k_2'\left(c^2-1\right)}{\epsilon\left(k_1'k_2-k_1k_2'\right)}, \\ m_2(s) &= \frac{\epsilon c^2k_1\left(k_1^2-k_2^2\right)s-k_1'\left(c^2-1\right)}{\epsilon\left(-k_1'k_2+k_1k_2'\right)}, \end{aligned} \tag{3.7}$$

by the use of (3.6) and (3.2) with Lemma 3.2. If x is time-like, we get

$$\begin{aligned} m_0(s) &= -c^2 s \\ m_1(s) &= \frac{c^2k_2\left(k_1^2+k_2^2\right)s+k_2'\left(c^2+1\right)}{-k_1'k_2+k_1k_2'}, \\ m_2(s) &= \frac{c^2k_1\left(k_1^2+k_2^2\right)s+k_1'\left(c^2+1\right)}{k_1'k_2-k_1k_2'}, \end{aligned} \tag{3.8}$$

by the use of (3.6) and (3.4) with Lemma 3.2. Substituting these values into (3.1), we obtain the desired results. □

3.1. T-constant Curves According to Bishop Frame in $\mathbb{E}_1^3$

Definition 3.2. Let $x : I \subset \mathbb{R} \to \mathbb{E}_t^n$ be a unit speed non-null curve in $\mathbb{E}_t^n$. If $\left\|x^T\right\|$ is constant then x is called a *T-constant curve* [9]. Further, a T-constant curve x is called first kind if $\left\|x^T\right\| = 0$, otherwise second kind.

As a consequence of (3.1) with (3.2) and (3.4), we get the following result.

Lemma 3.2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed non-null curve in $\mathbb{E}_1^3$.*

i) x is a T-constant space-like curve if and only if

$$\begin{aligned} 0 &= 1 + \epsilon k_1 m_1 + \epsilon k_2 m_2 \\ 0 &= m_1' + k_1 m_0 \\ 0 &= m_2' - k_2 m_0, \end{aligned} \tag{3.9}$$

ii) x is a T-constant time-like curve if and only if

$$\begin{aligned} 0 &= 1 - k_1 m_1 - k_2 m_2 \\ 0 &= m_1' + k_1 m_0 \\ 0 &= m_2' + k_2 m_0, \end{aligned} \tag{3.10}$$

hold, where $m_0 \in \mathbb{R}$, $m_1(s)$ and $m_2(s)$ are differentiable functions.

As a consequence of (3.9) and (3.10), we get the following result.

Theorem 3.2. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed non-null curve in $\mathbb{E}_1^3$ with the curvatures k_1 and k_2. Then,*

i) x is a T-constant space-like curve of first kind, if and only if

$$-\epsilon k_1 (s + a) - \epsilon k_2 (s + b) = 1.$$

ii) x is a T-constant time-like curve of first kind, if and only if

$$k_1 (s + a) + k_2 (s + b) = 1,$$

where a, b are real constants.

Proof. Let x be a T-constant space-like (time-like) curve of first kind. Then, from the second and third equalities in (3.9) and (3.10), we get $m_1' = m_2' = 0$. Then substituting the solution of the last equation into the first equation of (3.9) and (3.10), we get the desired result. □

Theorem 3.3. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a non-null unit speed curve in $\mathbb{E}_1^3$. Then,*

i) x is a T-constant space-like curve of second kind if and only if

$$\left(\frac{k_1' m_1 + k_2' m_2}{k_1^2 - k_2^2} \right)' = 0.$$

ii) x is a T-constant timelike curve of second kind if and only if

$$\left(\frac{k_1' m_1 + k_2' m_2}{k_1^2 + k_2^2} \right)' = 0.$$

Proof. Suppose that x is a T-constant space-like (time-like) curve of second kind. Differentiating the first equations of (3.9) and (3.10) and substituting the second and the third equations of (3.9) and (3.10) into the differential of the first equation, we get the result. □

Corollary 3.1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a non-null unit speed curve in $\mathbb{E}_1^3$. If x is a T-constant non-null unit speed curve of second kind, then the curvatures functions k_1, k_2 according to Bishop frame cannot be constant functions.*

Proof. If the curvatures k_1, k_2 according to Bishop frame are constant, then m_0 is zero, which means that the curve is T-constant of first kind. Therefore, the curvatures functions k_1, k_2 according to Bishop frame cannot be constant functions. □

For T-constant curves of second kind, we give the following result.

Proposition 3.2. *Let $x \in \mathbb{E}_1^3$ be a unit speed non-null T-constant curve of second kind. Then the distance function $\rho = \|x\|$ satisfies*

$$\rho = \pm\sqrt{c_1 s + c_2}, \tag{3.11}$$

where c_1, c_2 are real constants: and if the curve x is space-like, $c_1 = 2m_0$, if the curve x is time-like $c_1 = -2m_0$.

Proof. Let $x \in \mathbb{E}_1^3$ be a T-constant curve of second kind then by definition the function $m_0(s)$ of x is constant. Therefore, differentiating the squared distance function $\rho^2 = \langle x(s), x(s)\rangle$ and using (3.5), we get $\rho\rho' = m_0$, if the curve x is space-like, and $\rho\rho' = -m_0$, if the curve x is time-like. It is an easy calculation to show that, this differential equation has a nontrivial solution (3.11). □

3.2. N-constant Curves According to Bishop Frame in $\mathbb{E}_1^3$

Definition 3.3. Let $x : I \subset \mathbb{R} \to \mathbb{E}_t^n$ be a unit speed non-null curve in $\mathbb{E}_t^n$. If $\left\|x^N\right\|$ is constant then x is called a N-constant curve. For a N-constant curve x, either $\left\|x^N\right\| = 0$ or $\left\|x^N\right\| = \mu$ for some non-zero smooth function μ [9]. Further, a N-constant curve x is called first kind if $\left\|x^N\right\| = 0$, otherwise second kind.

Note that, for a N-constant space-like(time-like) curve x according to Bishop frame in $\mathbb{E}_1^3$, then

$$\left\|x^N(s)\right\|^2 = \varepsilon\left(m_1^2(s) - m_2^2(s)\right), \tag{3.12}$$

$$\left\|x^N(s)\right\|^2 = m_1^2(s) + m_2^2(s), \tag{3.13}$$

become constant functions, respectively.

For the N-constant curves of first kind we give the following result.

Theorem 3.4. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed non-null curve in $\mathbb{E}_1^3$. Then x is a N-constant curve of first kind if and only if either x is a straight line or x is a planar curve and m_0 is a non-constant linear function of an arc length function s, i.e., $m_0(s) = c_1 s + c_2$ for some constants c_1 and c_2.*

Proof. Suppose that x is N-constant curve of first kind in $\mathbb{E}_1^3$, then if x is a time-like curve, $m_1^2 + m_2^2 = 0$, which means $m_1 = m_2 = 0$. Writing the last equation in (3.4), we get $m_0 = s + c$, for some constant $c \in \mathbb{R}$ and k_1, k_2 is zero.This means that x is a straight line. Further, if x is a space-like curve, $\varepsilon\left(m_1^2(s) - m_2^2(s)\right) = 0$. In this case, either $m_1 = m_2 = 0$ or $m_1 = \pm m_2 \neq 0$. In particular, if $m_1 = m_2 = 0$, then from (3.2), x is a straight line. If $m_1 = \pm m_2 \neq 0$, then from (3.2), $k_1 = \pm k_2$, which means x is a planar curve. Moreover, writing $m_1 = \pm m_2 \neq 0$ and $k_1 = \pm k_2$ in (3.2), we obtain $m_0(s) = c_1 s + c_2$ for some constants c_1 and c_2. □

Proposition 3.3. *Let $x(s) \in \mathbb{E}_1^3$ be a non-null curve in $\mathbb{E}_1^3$ and s be its arc-length function. Then x is a N-constant curve of second kind if and only if either x lies in normal plane or $m_0 = s + c$ for some constant $c \in \mathbb{R}$.*

Proof. Let x be a N-constant curve of second kind then the equation $m_1 m_1' - m_2 m_2' = 0$ holds. Hence, by the use of the equations in (3.2) and (3.4), we get

$$m_0\left(m_0' - 1\right) = 0. \tag{3.14}$$

Therefore, there are two possible cases: $m_0 = 0$, or $m_0' - 1 = 0$. If $m_0 = 0$, x lies in normal plane. Otherwise, $m_0 = s + c$ for some constant $c \in \mathbb{R}$. □

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