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COMMON FIXED POINT THEOREMS FOR MULTI-VALUED CONTRACTIONS SATISFYING GENERALIZED CONDITION(B) ON PARTIAL METRIC SPACES

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Abstract. In this paper we prove two common fixed point theorems for two pairs of single and set valued mappings which satisfy the generalized contractive condition in complete partial metric spaces. Our results generalize and improve some previous results.

keywords: generalized condition (B), partial Hausdorff metric, weakly compatible maps, partial metric space.

1. Introduction and Preliminaries

After the famous Banach contraction principle some generalizations and extensions were introduced with different conditions. Nadler [31] developed this principle to the setting of multi valued mappings and he gave some results for this type. Later many authors improved and generalized some fixed point results in various spaces as generalized metric spaces, fuzzy metric spaces and cone metric spaces. Matthews [30] introduced the concept of partial metric space, which is a generalization of the usual metric spaces in which the distance from an object to itself is not necessarily a zero. More recently Aydi et al.[9] introduced the notion of partial Hausdorff metric and generalized the fixed point theorem of Nadler[31] on partial metric spaces.
We recall some basic definitions and properties of the partial metric spaces. Those were given in[30]:

Definition 1.1. [30] Let X be a nonempty set. A function $p : X \times X \to [0, \infty)$ is said to be a partial metric on X if and only if it satisfies the following conditions

1. $p(x, x) = p(y, x) = p(x, y)$ if and only if $x = y$
2. $p(x, x) \le p(x, y)$

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3. $p(x, y) = p(y, x)$
4. $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$,

The space (X, p) is called a partial metric space.

Clearly, if $p(x, y) = 0$ then the first two of the above conditions imply that $x = y$.
Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.
If p is a partial metric on X, then the two functions $p^s, d_p^m : X \times X \to [0, \infty)$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

$$d_p^m(x, y) = \max\{p((x, y) - p(x, x), p(x, y) - p(y, y)\} = p(x, y) - \min\{p(x, x), p(y, y)\}$$

define two equivalent metrics on X.
Furthermore, a sequence $\{x_n\}$ converges in (X, p^s) to a point $x \in X$ if and only if

$$\lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

Definition 1.2. [30] Let (X, p) be a partial metric space

- A sequence x_n in a partial metric space (X, p) converges to a point $x \in X$, with respect to τ_p if and only if

$$p(x, x) = \lim_{n\to\infty} p(x, x_n).$$

- A sequence $\{x_n\}$ in X is said to be a Cauchy sequence, if $\lim\limits_{n\to\infty} p(x_n, x_m$ exists and is finite.
- (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $\lim\limits_{n\to\infty} p(x, x_n) = p(x, x)$.

Lemma 1.1. *[30] Let (X, p) be a partial metric space.*

- *$\{x_n\}$ in (X, p) is said to be a Cauchy sequence in (X, p), if and only if it is as well in (X, p^s).*
- *(X, p) is a complete space if and only if (X, p^s) is a complete space.*

Lemma 1.2. *[3] Let (X, p) be a partial metric space, if a sequence $\{x_n\}$ converges to z in (X, p) such $p(z, z) = 0$, then for every $y \in X$ we have*

$$\lim_{n\to\infty} p(x_n, y) = p(z, y)$$

Aydi et al. [9] defined the partial Hausdorff metric as follows

Definition 1.3. [9] Let (X, p) be a partial metric space and let $CB^p(X)$ be a set of all nonempty, bounded and closed subsets of X, the partial Hausdorff metric is a function $H_p : CB^p(X) \times CB^p(X) \to [0, \infty)$ such that for all $A, B \in CB^p(X)$

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\},$$

$$\text{where } \; p(x, A) = \inf\{p(x, a), a \in A\}, \delta_p(A, B) = \sup\{p(a, B), a \in A\} and$$
$$\delta_p(B, A) = \{p(b, A), b \in B\}.$$

It is clear from the definition of the two functions δ_p and H_p that for $A, B \in CB^p(X)$ and $a \in A$ we have:

$$p(a, B) = \inf_{b \in B} p(a, b) \leq \delta_p(A, B) \leq H_p(A, B).$$

Lemma 1.3. *[6] Let (X, p) be a partial metric space and A is a nonempty subset of X then $a \in \overline{A}$ if and only if $p(a, A) = p(a, a)$, where $\overline{A}$ is the closure of A with respect to the topology τ_p of (X, p).*

Proposition 1.1. *[9]*

1. $\delta_p(A, A) = \sup\{p(a, a), a \in A\}$
2. $\delta_p(A, A) \leq \delta_p(A, B)$
3. $\delta_p(A, B) = 0$ *implies* $A \subseteq B$
4. $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(V, B) - \inf_{c \in C} p(c, c)$

Proposition 1.2. *[9]*

1. $H_p(A, A) \leq H_p(A, B)$
2. $H_p(A, B) = H_p(B, A)$
3. $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(C, C)$

Corollary 1.1. *[9]*
Let (X, p) be a partial metric space for A, B in $CB^p(X)$ if $H_p(A, B) = 0$ then $A = B$

The converse of the last corollary may be not true.

Example 1.1. Let $X = [0, \infty)$ be endowed with the partial metric $p(x, y) = \max\{x, y\}$, for all $a, b \in [0, \infty)$ such that $a < b$ we have

$$H^p([a, b], [a, b]) = b \neq 0$$

Babu et al.[13] defined that a selfmapping g of metric space (X, d) is said to be satisfy the condition (B) if there exists $\delta, \geq 0, L \geq 0$ such that $\delta + L < 1$ and for all $x, y \in X$ we have:

$$d(gx, gy) \leq \delta d(x, y) + L\min(d(x, gx), d(y, gy), d(x, gy), d(y, gx)).$$

Abbas et.al[1] gave a generalization for the last definition to two selfmappings as generalized condition (B):

Definition 1.4. [1] Let (X, d) metric space and let two selfmappings $f, g : X \to X$, the map g satisfies generalized condition (B) associated with f, if there exists $\delta \in (0, 1)$ and $L \geq 0$, such that

$$d(gx, gy) \leq \delta M(x, y) + L\min\{d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\},$$

where $M(x, y) = \max\{d(fx, fy), d(fx, gx), d(fy, gy), \frac{d(fx,gy)+d(fy,gx)}{2}\}$

We find the same above definition in paper[2] as: g generalized almost f-contraction. Jungck and Rhoades [22] generalized the concept of compatibility and $\delta-$ compatibility to the weak compatibility as follows:

Definition 1.5. [22] Let X be nonempty set, the mappings $f : X \to X$, and $S; X \to CB(X)$ are said to be weakly compatible, if they commute at their coincidence point, i.e if $fu \in Su$ for some $u \in X$, then $fSu = Sfu$, where $CB(X)$ is a set of closed and bounded subsets of X.

In sequel of our work, we need the following lemma:

Lemma 1.4. *[9] In partial metric space (X, p), let $CB^p(X)$ be a set of closed and bounded subsets and let $A, B \in CB^p(X)$ and $k > 1$ for any $a \in A$ there exists $b(a) \in B$ such*

$$p(a, b) \leq kH_p(A, B)$$

The aim of this paper is to prove two common fixed point theorems, for two pairs of hybrid mappings satisfying generalized condition (B) in partial metric spaces. Our results extend those in paper [23] to the setting of single- and set-valued mappings. An example is given to illustrate this work.

2. Main results

Let f, g be single self-mappings of a complete partial metric space (X, p), and let $S, T : X \to CB^p(X)$ be set-valued mappings, where $CB^p(X)$ is the set of all nonempty, closed and bounded subset of X such that

$$T(X) \subset f(X) \text{ and } S(X) \subset g(X), \tag{2.1}$$

There exist $\delta \in (0,1)$ and $L \geq 0$ such for all $x, y \in X$ we have

(2.2) $H_p(Sx, Ty) \leq \delta M(x,y) + L \min(d_m^p(fx, Sx), d_m^p(gy, Ty), d_m^p(fx, Ty), d_m^p(gy, Sx)),$

where $M(x,y) = \max(p(fx, gy), p(fx, Sx), p(gy, Ty), \dfrac{p(fx, Ty) + p(gy, Sx)}{2})$.
Let $k > 1$ such $k\delta < 1$ and an arbitrary $x_0 \in X$, since $S(X) \subset g(X)$ there is a point $x_1 \in X$ such that $y_1 = gx_1 \in Sx_0$, for this point y_1 and from lemma4 there exists a point $y_2 \in Tx_1$ and since $T(X) \subseteq f(X)$ there is $x_2 \in X$ such that $y_2 = fx_2 \in Tx_1$ and:

$$p(y_2, y_1) = p(fx_2, gx_1) \leq kH_p(Sx_0, Tx_1)$$

For x_2 we can choose x_3 such $y_3 = gx_3 \in Sx_2$ and

$$p(y_2, y_3) \leq kH_p(Sx_2, Tx_1),$$

so by continuing in this manner, we can construct two sequences $\{x_n\}, \{y_n\}$ in X as follows:

$$\begin{cases} y_{2n+1} = gx_{2n+1} \in Sx_{2n}, \\ y_{2n+2} = fx_{2n+2} \in Tx_{2n+1}, \end{cases} \tag{2.3}$$

and satisfy

$$p(y_{2n+2}, y_{2n+1}) \leq kH_p(Sx_{2n}, Tx_{2n+1})$$

Lemma 2.1. *The sequence $\{y_n\}$ which is defined by (2.3) is a Cauchy one.*

Proof. Firstly, we will show

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = 0,$$

$$\begin{aligned} M(x_{2n}, x_{2n+1}) = \max\{&p(fx_{2n}, gx_{2n+1}), p(fx_{2n}, Sx_{2n}), p(gx_{2n+1}, Tx_{2n+1}), \\ &\frac{1}{2}(p(fx_{2n}, Tx_{2n+1}) + (p(gx_{2n+1}, Sx_{2n}))\} \end{aligned}$$

$$= \max\{p(y_{2n}, y_{2n+1}), p(y_{2n+1}, Tx_{2n+1}), \frac{1}{2}(p(y_{2n}, Tx_{2n+1}) + p(y_{2n+1}, Sx_{2n}))\},$$

$$\leq \max\{p(y_{2n}, y_{2n+1}), p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2}, \frac{p(y_{2n}, y_{2n+2} + p(y_{2n+1}, y_{2n+1})}{2}\}$$

$$\leq \max\{p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2})\}$$

$$\min\{d_m^p(fx_{2n}, Sx_{2n}), d_m^p(gx_{2n+1}, Tx_{2n+1}), d_m^p(fx_{2n}, Tx_{2n+1}), d_m^p(gx_{2n+1}, Sx_{2n})\} = 0$$

If $p(y_{2n}, y_{2n+1}) \leq p(y_{2n+1}, y_{2n+2})$, by using (2.2) we get

$$p(y_{2n+1}, y_{2n+2}) \leq kH_p(Sx_{2n}, Tx_{2n+1}) \leq k\delta p(y_{2n+1}, y_{2n+2}) < p(y_{2n+1}, y_{2n+2})$$

which is a contradiction, so we have

$$p(y_{2n+1}, y_{2n+2}) \leq kH_p(Sx_{2n}, Tx_{2n+1}) \leq k\delta p(y_{2n}, y_{2n+1}),$$

putting $\lambda = k\delta < 1$ and by induction we get

$$p(y_{n+1}, y_{n+2}) \leq \lambda p(y_n, y_{n+1}) \leq \lambda^2 p(y_{n-1}, y_n) \leq ... \leq \lambda^{n+1} p(y_0, y_1).$$

For all $n; m \in \mathbb{N}$ that's $m > n$ we have

$$\begin{aligned}
p^s(x_n, x_m) &= 2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m) \\
&\leq 2\Big(p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + ... + p(x_{m-1}, x_m)\Big) \\
&\leq 2\lambda^n\Big(p(x_0, x_1) + \lambda p(x_0, x_1) + ... + \lambda^{m-1} p(x_0, x_1)\Big) \\
&\leq 2\frac{\lambda^n}{1-\lambda} p(x_0, x_1) \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which implies that $\{y_n\}$ is a Cauchy sequence in (X, p^s), and so it is as well in (X, p), since (X, p) is complete, then (X, p^s) is complete and $\{y_n\}$ converges to $z \in X$, also we have

$$\lim_{n\to\infty} p(y_n, z) = \lim_{n\to\infty} p(y_n, y_m) = p(z, z) = 0.$$

□

Theorem 2.1. *Let (X, p) be a complete partial metric space, $f, g : X \to X$ two single-valued and $S, T : X \to CB^p(X)$ two set-valued mappings satisfying (2.1),(2.2) and $f(X)$ or $g(X)$ is closed, if $\{f, S\}$ is weakly compatible as well as $\{g, T\}$, then f, g, S and T have a common fixed point.*

Proof. From Lemma 2.1 the sequence $\{y_n\}$ is a Cauchy one and X complete so it converges to $z \in X$, also the subsequence $\{y_{2n+2}\} = \{fx_{2n+2}\}$ converges to z, since $f(X)$ is closed, then $z \in f(X)$ and there exists $u \in X$ such $z = fu$.
we claim $z = fu \in Su$, if not by using (2.2) we get

$$p(Su, y_{2n+2}) = p(Su, gx_{2n+1}) \leq H_p(Su, Tx_{2n+1})$$

$$\leq \delta \max\{p(fu, gx_{2n+1}), p(fu, Su), p(gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}(p(fu, Tx_{2n+1}) + p(gx_{2n+1}, Su))\}$$
$$+L \min\{d_m^p(fu, Su), d_m^p(gx_{2n+1}, Tx_{2n+1}), d_m^p(fu, Tx_{2n+1}), d_m^p(gx_{2n+1}, Su)\},$$

since $y_{2n+2} \in Tx_{2n+1}$ we get

$$p(gx_{2n+1}, Tx_{2n+1}) \leq p(y_{2n+1}, y_{2n+2})$$

and $p(fu, Tx_{2n+1}) \leq p(fu, y_{2n+2})$, so the last inequality becomes

$$p(Su, y_{2n+1}) \leq \delta \max\{p(fu, y_{2n+1}), p(fu, Su), p(y_{2n+1}, y_{2n+1}), \frac{1}{2}(p(fu, y_{2n+1}) + p(y_{2n+1}, Su))\}$$

$$+L\min\{d_m^p(fu,Su),d_m^p(y_{2n+1},y_{2n+2}),d_m^p(fu,y_{2n+2}),d_m^p(y_{2n+1},Su))\},$$

letting $n\to\infty$, we get

$$p(Su,fu)\le\delta p(fu,Su)<p(fu,Su),$$

which is a contradiction, so $p(fu,Su)=0$ and from Lemma 1.3 $z=fu\in\overline{Su}=Su$, then u is a coincidence point for f and S, the hybrid pair $\{f,S\}$ is weakly compatible and $fu\in Su$ implies that $f(fu)=fz\in Sfu=Sz$.
Since $S(X)\subseteq g(X)$ and $z=fu\in Su$ there is $v\in X$ such $z=fu=gv$, we will show $gv\in Tv$, if not by using (2.2) we get

$$p(gv,Tv)\le H_p(Su,Tv)\le\delta\max\{p(fu,gv),p(fu,Su),p(gv,Tv),\frac{1}{2}(p(fu,Tv)+p(gv,Su))\}$$

$$+L\min\{d_m^p(fu,Su),d_m^p(gv,Tv),d_m^p(fu,Tv),d_m^p(gv,Su)\}$$

$$p(gv,Tv)\le\delta p(gv,Tv)<p(gv,Tv),$$

which is a contradiction, then $z=gv\in Tv$.
The weak compatibility of the pair $\{g,T\}$, implies that $g(gv)=gz\in Tgv=Tz$.
Now we prove $z=fz$, if not by using (2.2) we get

$$p(fz,gv)\le kH_p(Sz,Tv)\le k\delta\max\{p(fz,gv),p(fz,Sz),p(gv,Tv),\frac{1}{2}(p(fz,Tv)+p(gv,Sz)\}$$

$$+kL\min\{d_m^p(fz,Sz),d_m^p(gv,Tv),d_m^p(fz,Tv),d_m^p(gv,Sz)\}$$

$$p(fz,z)\le k\delta p(fz,z)<p(fz,z),$$

which is a contradiction, then $z=fz\in Sz$ and similarly we obtain $z=gz\in Tz$, consequently z is a common fixed point for f,g,S and T. □

Theorem 2.1 generalizes Theorem 3.1 in paper[23] and Theorem 3.2 of Aydi et al.[9], also it extends Theorem2.2 in [1] and Corollary 2.6 in [32], to the partial metric spaces.
If $L=0$ and $f=g=id_X$, we obtain Theorem 2.2 in paper[10].

Remark 2.1. In the above Theorem 2.1 we do not have the uniqueness of common fixed point and we will see that in Example 2.1 below. However if S or T is a single-valued mapping, then the fixed point is unique.

Obviously, we found in the above proof of Theorem 2.1 $z=fz\in Sz$, if S is a single valued mapping, then $Sz=fz=z$.
Suppose there is another common fixed point w, by using (2.2) we get

$$p(z,w)\le H_p(Sz,Tw)\le\delta\max\{p(fz,gw),p(fz,Sz),p(gw,Tw),\frac{1}{2}(p(fz,Tw),p(gw,Sz))\}$$

$$+L\min\{d_m^p(fz,Sz),d_m^p(gw,Tw),d_m^p(fz,Tw),d_m^p(gw,Sz)\}$$

$$\le\delta p(z,w)<p(z,w),$$

which is a contradiction, so the z is unique.
If $S=T$ and $f=g$ we obtain the following corollary:

Corollary 2.1. *Let $(X, p$ be a complete partial metric space, $f : X \to X$ and $S : X \to CB^p(X)$ two single and set-valued mappings, respectively such that $SX \subseteq fX$ and for all $x, y \in X$ satisfying:*

$$H_p(Sx, Sy) \leq \delta M(x, y) + L \min\{d_m^p(fx, Sx), d_m^p(fy, Sy), d_m^p(fx, Sy), d_m^p(fy, Sx)\},$$

where $M(x, y) = \max\{p(fx, fy), p(fx, Sx), p(fy, Sy), \frac{p(fx,Sy)+p(fy,Sx)}{2}\}$, if the pair $\{f, S\}$ is weakly compatible and $f(X)$ is closed, then S and f have a common fixed point.

Corollary 2.1 generalizes Corollary 2.2 in [23] and Corollary 2.5 of Aydi [8].

Theorem 2.2. *Let (X, p) be a complete partial metric space, $f, g : X \to X$, two single valued and $S, T : X \to CB^p(X)$ two set valued mappings satisfying the following conditions:*

(a) $TX \subseteq fX,\ SX \subseteq gX$

(b) *there exists $\delta > 0$ and $L \geq 0$ such $2\delta < 1$ and for all $x, y \in X$ we have*

$$H_p(Sx, Ty) \leq \delta M(x, y) + L \min(d_m^p(fx, Sx), d_m^p(gy, Ty), d_m^p(fx, Ty), d_m^p(gy, Sx)), \tag{2.4}$$

where $M(x, y) = \max(p(fx, gy), p(fx, Sx), p(gy, Ty), p(fx, Ty), p(gy, Sx))$

(c) $f(X)$*or,* $T(X)$ *is closed.*

(d) *the two pairs $\{f, S\}, \{g, T\}$ are weakly compatible,*

then f, g, S and T have a common fixed point.

Proof. Let $\{y_n\}$ be the sequence which defined in (2.3), we choose k in manner that satisfies $2k\delta < 1$ which implies that $k(\delta + L) < 1$ (condition in theorem 1) and so we have:

$$\begin{aligned}
M(x_{2n}, x_{2n+1}) &= \max\{p(fx_{2n}, gx_{2n+1}), p(fx_{2n}, Sx_{2n}),\\
&\quad p(gx_{2n+1}, Tx_{2n+1}), p(fx_{2n}, Tx_{2n+1}), p(gx_{2n+1}, Sx_{2n})\}\\
&\leq \max\{p(y_{2n}, y_{2n+1}), p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2}), p(y_{2n}, y_{n+2}), p(y_{2n+1}, y_{2n+1})\},\\
&\leq \{p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2}), p(y_{2n}, y_{2n+2})\},\\
&\leq p(y_{2n}, y_{2n+1}) + p(y_{2n+1}, y_{2n+2}) \leq 2\max\{p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2})\}
\end{aligned}$$

$$\begin{aligned}
&\min\{d_m^p(fx_{2n}, Sx_{2n}), d_m^p(gx_{2n+1}, Tx_{2n+1}), d_m^p(fx_{2n}, Tx_{2n+1}), d_m^p(gx_{2n+1}, Sx_{2n})\}\\
&= \min\{d_m^p(y_{2n}, S_{2n}), d_m^p(y_{2n+1}, T_{2n+1}), d_m^p(y_{2n}, T_{2n+1}), d_m^p(y_{2n+1}, S_{2n})\},
\end{aligned}$$

since $y_{2n+1} \in Sx_{2n}$ and Sx_{2n} is closed, then

$$\min\{d_m^p(y_{2n}, A_n), d_m^p(y_{2n+1}, A_{n+1}), d_m^p(y_{2n}, A_{n+1}), d_m^p(y_{2n+1}, A_n)\} = 0,$$

now by using (2.4) we get:

$$p(y_{2n+2}, y_{n+1})\} \leq kH_p(Sx_{2n}, Tx_{2n+1} \leq k\delta(p(y_{2n}, y_{2n+1}) + p(y_{2n+1}, y_{2n+2}))$$

$$p(y_{2n+1}, y_{2n+2}) \leq \frac{k\delta}{1-k\delta} p(y_{2n}, y_{2n+1}),$$

putting $\mu = \frac{k\delta}{1-k\delta} < 1$ and by induction we obtain:

$$p(y_{n+1}, y_{n+2}) \leq \mu^{n+1} p(y_0, y_1),$$

which implies that

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = 0$$

For all $n, m \in N$ such that $m > n$ we have

$$p^s(x_n, x_m) \leq 2p(x_n, x_m) \leq 2p(x_n, x_{n+1}) + 2p(x_{n+1}, x_{n+2}) + ... + 2p(x_{m-1}, x_m)$$

$$\leq 2\mu^n \Big(p(x_0, x_1) + \mu p(x_0, x_1) + ... + \mu^{m-1} p(x_0, x_1) \Big)$$

$$\leq 2 \frac{\mu^n}{1-\mu} p(x_0, x_1) \to 0 \text{ as } n \to \infty,$$

this yields that $\{y_n\}$ is a Cauchy sequence in (X, p^s) and so it is also in (X, p), since (X, p) is complete then (X, p^s) is complete and $\{y_n\}$ }converges to $z \in X$ and we have

$$\lim_{n\to\infty} p(y_n, z) = \lim_{n\to\infty} p(y_n, y_m) = p(z, z) = 0$$

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.2 generalizes and improves Theorem 2 of Altun et al.[6] to the setting of single- and set-valued mappings. It also generalizes and extends Theorem 3.2 of Berinde[16] and Theorem 3.1 in [1] to the partial metric spaces.
If $S = T$ and $f = g$ we obtain the following corollary:

Corollary 2.2. *Let (X, p) be a partial metric space, $f : X \to X$ and $S : X \to CB^p(X)$ two single and set valued mappings respectively satisfying:*

(2.5) $\quad H_p(Sx, Sy) \leq \delta M(x, y) + L \min\{p(fx, Sx), p(fy, Sy), p(fx, Sy), p(fy, Sx)\},$

where $M(x, y) = \max\{p(fx, fy), p(fx, Sx), p(fy, Sy), p(fx, Sy) + p(fy, Sx)\}$, if the following conditions hold:

1. *$SX \subseteq fX$*
2. *fX is closed*
3. *the pair $\{f, S\}$ is weakly compatible,*

then S and f have a common fixed point.

Example 2.1. Let $X = [0,4]$ endowed with the partial metric:

$$p(x,y) = \begin{cases} 0, & x = y = 1 \text{ or } x = y = 2 \\ \max\{x,y\}, & \text{otherwise} \end{cases}$$

Consider the mappings f, g, S and T defined by:

$$fx = \begin{cases} \frac{x}{2}, & 0 \leq x < 2 \\ 2, & x = 2 \\ \frac{7}{2}, & 2 < x \leq 4 \end{cases} \qquad gx = \begin{cases} 2, & x \in [0,1) \cup (1,2] \\ 1, & x = 1 \\ 4, & 2 < x \leq 4 \end{cases}$$

$$Sx = Tx = \begin{cases} [0,1], & 0 \leq x \leq 2 \\ \{2\}, & 2 < x \leq 4 \end{cases}$$

Clearly, X with the metric $p^s(x,y) = \max\{x,y\} - x - y = |x-y|$ is a complete metric space and so (X,p) is a complete partial metric space, also the subspace $f(X) = [\frac{1}{2}, 1] \cup \{0\}$ is closed, also choosing $\delta = \frac{2}{3}$.

1. For $x, y \in [0,2]$, we have

$$H_p(Sx, Ty) = 1 \leq \frac{4}{3} = \frac{2}{3} p(fx, gy)$$

2. For $x \in [0,2]$ and $y \in (2,4]$ we have

$$H_p(Sx, Ty) = 2 \leq \frac{8}{3} = \frac{2}{3} p(fx, gy),$$

3. For $2 < x \leq 4$ and $y \in [0,2]$, we have

$$H_p(Sx, Ty) = 2 \leq \frac{7}{3} = \frac{2}{3} p(fx, gy)$$

4. For $x, y \in (2,4]$, we have

$$H_p(Sx, Ty) = p(2,2) = 0 \leq \frac{8}{3} = \frac{2}{3} p(fx, gy).$$

Also, 1 and 2 are two coincidence points satisfying $fS1 = [0,1] = Sf1$, $fS2 = \{2\} = Sf2$ and $gT1 = [0,1] = Tg1$, $gT2 = \{2\} = Tg2$. Consequently, if all hypotheses of Theorem 1 are satisfied then f, g, S and T have a fixed point and, therefore, 1 and 2 are two common fixed point. This example illustrate Remark 1 of the non-uniqueness of the common fixed point.

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