FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 30, No 3 (2015), 341–351

MATRIX LIE GROUPS AS 3-DIMENSIONAL ALMOST CONTACT B-METRIC MANIFOLDS

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Dedicated to my doctoral advisor Prof. Dimitar Mekerov on the occasion of his 70th birthday

Abstract. The object of investigation are Lie groups considered as almost contact B-metric manifolds of the lowest dimension three. A correspondence is established of all basic-class-manifolds of the Ganchev-Mihova-Gribachev classification of the studied manifolds and the explicit matrix representation of Lie groups. Some known Lie groups are equipped with almost contact B-metric structure of different types.

Introduction

Differential geometry of almost contact manifolds is well studied (e.g. [3]). In [5], almost contact manifolds with B-metric are introduced and classified. These manifolds can be studied as the odd-dimensional counterpart of almost complex manifolds with the Norden metric [4, 7].

As it is known from [6], every representation of a Lie algebra corresponds uniquely to a representation of the simply connected Lie group. This relation is one-to-one. Then, knowing the representation of a certain Lie algebra settles the question of the representation of its Lie group.

In [11], an arbitrary Lie group considered as a manifold is equipped with an almost contact B-metric structure. For the lowest dimension three, an equivalence is deduced between the classification of considered manifolds given in [5] and the corresponding Lie algebra determined by commutators.

In the present work, an explicit correspondence is found between all basic-class-manifolds of the classification in [5] and Lie groups given by their explicit matrix representation.

The paper is organized as follows. In Sect. 1. we recall some preliminary facts about almost contact B-metric manifolds and Lie algebras related to them. In Sect. 2. we give the correspondence of all basic-class-manifolds and matrix Lie groups. Sect. 3. is devoted to some examples connected to the previous investigations.

Received March 05, 2014; Accepted March 20, 2015
2010 *Mathematics Subject Classification.* Primary 53C15, 53C50, 53D15; Secondary 22E60

1. Preliminaries

1.1. Almost contact B-metric manifolds

Let us denote an almost contact B-metric manifold by (M,φ,ξ,η,g), i.e. M is an $(2n+1)$-dimensional differentiable manifold, φ is an endomorphism of the tangent bundle, ξ is a Reeb vector field, η is its dual contact 1-form and g is a pseudo-Riemannian metric (called a *B-metric*) of signature $(n+1,n)$ such that [5]

$$\varphi\xi = 0, \qquad \varphi^2 = -\mathrm{Id} + \eta\otimes\xi, \qquad \eta\circ\varphi = 0, \qquad \eta(\xi) = 1,$$
$$g(\varphi x,\varphi y) = -g(x,y) + \eta(x)\eta(y),$$

where Id is the identity. In the latter equalities and further, x, y, z, w will stand for arbitrary elements of the algebra of the smooth vector fields on M or vectors in the tangent space T_pM of M at an arbitrary point p in M.

In [5], a classification is given of almost contact B-metric manifolds, consisting of eleven basic classes $\mathcal{F}_1,\mathcal{F}_2,\ldots,\mathcal{F}_{11}$. It is made with respect to the tensor F of type (0,3) defined by

$$F(x,y,z) = g\Big((\nabla_x\varphi)\,y,z\Big),$$

where ∇ stands for the Levi-Civita connection of g.

The special class $\mathcal{F}_0$, determined by the condition $F(x,y,z)=0$, is the intersection of all basic classes and it is known as the class of the *cosymplectic B-metric manifolds*.

Let $\{\xi;e_i\}$ $(i=1,2,\ldots,2n)$ be a basis of T_pM and let $\left(g^{ij}\right)$ be the inverse matrix of $\left(g_{ij}\right)$. Then the Lee 1-forms θ, θ^*, ω associated with F are defined by

$$\theta(z) = g^{ij}F(e_i,e_j,z), \quad \theta^*(z) = g^{ij}F(e_i,\varphi e_j,z), \quad \omega(z) = F(\xi,\xi,z).$$

In this paper, we consider the case of the lowest dimension of the considered manifolds, i.e. $\dim M = 3$.

We introduce an almost contact structure (φ,ξ,η) on M defined by

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_0 = 0, \quad \xi = e_0,$$
$$\eta(e_1) = \eta(e_2) = 0, \quad \eta(e_0) = 1$$

and a B-metric g such that

$$g(e_0,e_0) = g(e_1,e_1) = -g(e_2,e_2) = 1, \quad g(e_i,e_j) = 0, \;\; i\neq j \in \{0,1,2\}.$$

The components of F, θ, θ^*, ω with respect to the *φ-basis* $\{e_0,e_1,e_2\}$ are denoted by $F_{ijk} = F(e_i,e_j,e_k)$, $\theta_k = \theta(e_k)$, $\theta^*_k = \theta^*(e_k)$, $\omega_k = \omega(e_k)$. According to [9], we have:

$$\theta_0 = F_{110} - F_{220}, \qquad \theta_1 = F_{111} - F_{221}, \qquad \theta_2 = F_{112} - F_{211},$$
$$\theta^*_0 = F_{120} + F_{210}, \qquad \theta^*_1 = F_{112} + F_{211}, \qquad \theta^*_2 = F_{111} + F_{221},$$
$$\omega_0 = 0, \qquad \omega_1 = F_{001}, \qquad \omega_2 = F_{002}.$$

If F_s $(s = 1, 2, \dots, 11)$ are the components of F in the corresponding basic classes $\mathcal{F}_s$ then: [9]

$$
\begin{aligned}
&F_1(x,y,z) = \left(x^1\theta_1 - x^2\theta_2\right)\left(y^1z^1 + y^2z^2\right),\\
&\qquad \theta_1 = F_{111} = F_{122}, \qquad \theta_2 = -F_{211} = -F_{222};\\
&F_2(x,y,z) = F_3(x,y,z) = 0;\\
&F_4(x,y,z) = \tfrac{1}{2}\theta_0\Big\{x^1\left(y^0z^1 + y^1z^0\right) - x^2\left(y^0z^2 + y^2z^0\right)\Big\},\\
&\qquad \tfrac{1}{2}\theta_0 = F_{101} = F_{110} = -F_{202} = -F_{220};\\
&F_5(x,y,z) = \tfrac{1}{2}\theta_0^*\Big\{x^1\left(y^0z^2 + y^2z^0\right) + x^2\left(y^0z^1 + y^1z^0\right)\Big\},\\
&\qquad \tfrac{1}{2}\theta_0^* = F_{102} = F_{120} = F_{201} = F_{210};\\
&F_6(x,y,z) = F_7(x,y,z) = 0;\\
&F_8(x,y,z) = \lambda\Big\{x^1\left(y^0z^1 + y^1z^0\right) + x^2\left(y^0z^2 + y^2z^0\right)\Big\},\\
&\qquad \lambda = F_{101} = F_{110} = F_{202} = F_{220};\\
&F_9(x,y,z) = \mu\Big\{x^1\left(y^0z^2 + y^2z^0\right) - x^2\left(y^0z^1 + y^1z^0\right)\Big\},\\
&\qquad \mu = F_{102} = F_{120} = -F_{201} = -F_{210};\\
&F_{10}(x,y,z) = \nu x^0\left(y^1z^1 + y^2z^2\right), \qquad \nu = F_{011} = F_{022};\\
&F_{11}(x,y,z) = x^0\Big\{\left(y^1z^0 + y^0z^1\right)\omega_1 + \left(y^2z^0 + y^0z^2\right)\omega_2\Big\},\\
&\qquad \omega_1 = F_{010} = F_{001}, \qquad \omega_2 = F_{020} = F_{002},
\end{aligned}
\tag{1.1}
$$

where $x = x^ie_i$, $y = y^je_j$, $z = z^ke_k$. Obviously, the class of 3-dimensional almost contact B-metric manifolds is

$$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}.$$

1.2. The Lie algebras corresponding to Lie groups with almost contact B-metric structure

Let L and $\mathfrak{l}$ be a 3-dimensional real connected Lie group and its corresponding Lie algebra. If $\{E_0, E_1, E_2\}$ is a basis of left invariant vector fields then an almost contact structure (φ, ξ, η) and a B-metric g are defined as follows:

$$
\begin{gathered}
\varphi E_0 = 0, \quad \varphi E_1 = E_2, \quad \varphi E_2 = -E_1, \quad \xi = E_0,\\
\eta(E_0) = 1, \quad \eta(E_1) = \eta(E_2) = 0,\\
g(E_0,E_0) = g(E_1,E_1) = -g(E_2,E_2) = 1,\\
g(E_0,E_1) = g(E_0,E_2) = g(E_1,E_2) = 0.
\end{gathered}
$$

Then $(L, \varphi, \xi, \eta, g)$ is a 3-dimensional almost contact B-metric manifold. If its Lie algebra $\mathfrak{l}$ is determined by

$$\left[E_i, E_j\right] = C_{ij}^k E_k, \quad i, j, k \in \{0, 1, 2\},$$

then the components of F, θ, θ^*, ω are the following: [11]

$$\begin{array}{ll}
F_{111} = F_{122} = 2C_{12}^1, & F_{211} = F_{222} = 2C_{12}^2, \\
F_{120} = F_{102} = -C_{01}^1, & F_{020} = F_{002} = -C_{01}^0, \\
F_{210} = F_{201} = -C_{02}^2, & F_{010} = F_{001} = C_{02}^0, \\
F_{110} = F_{101} = \frac{1}{2}\left(C_{12}^0 - C_{01}^2 + C_{02}^1\right), & \\
F_{220} = F_{202} = \frac{1}{2}\left(C_{12}^0 + C_{01}^2 - C_{02}^1\right), & \\
F_{011} = F_{022} = C_{12}^0 + C_{01}^2 + C_{02}^1, &
\end{array}$$

$$\begin{array}{lll}
\theta_0 = -C_{01}^2 + C_{02}^1, & \theta_0^* = -C_{01}^1 - C_{02}^2, & \omega_0 = 0, \\
\theta_1 = 2C_{12}^1, & \theta_1^* = 2C_{12}^2, & \omega_1 = C_{02}^0, \\
\theta_2 = -2C_{12}^2, & \theta_2^* = 2C_{12}^1, & \omega_2 = -C_{01}^0.
\end{array}$$

Theorem 1.1. [11] *The manifold $(L, \varphi, \xi, \eta, g)$ belongs to the basic class $\mathcal{F}_s$ ($s \in \{1, 4, 5, 8, 9, 10, 11\}$) if and only if the corresponding Lie algebra $\mathfrak{l}$ is determined by the following commutators:*

$$\begin{array}{ll}
\mathcal{F}_1: & [E_0, E_1] = [E_0, E_2] = 0, \quad [E_1, E_2] = \alpha E_1 + \beta E_2; \\
\mathcal{F}_4: & [E_0, E_1] = \alpha E_2, \quad [E_0, E_2] = -\alpha E_1, \quad [E_1, E_2] = 0; \\
\mathcal{F}_5: & [E_0, E_1] = \alpha E_1, \quad [E_0, E_2] = \alpha E_2, \quad [E_1, E_2] = 0; \\
\mathcal{F}_8: & [E_0, E_1] = \alpha E_2, \quad [E_0, E_2] = \alpha E_1, \quad [E_1, E_2] = -2\alpha E_0; \\
\mathcal{F}_9: & [E_0, E_1] = \alpha E_1, \quad [E_0, E_2] = -\alpha E_2, \quad [E_1, E_2] = 0; \\
\mathcal{F}_{10}: & [E_0, E_1] = \alpha E_2, \quad [E_0, E_2] = \alpha E_1, \quad [E_1, E_2] = 0; \\
\mathcal{F}_{11}: & [E_0, E_1] = \alpha E_0, \quad [E_0, E_2] = \beta E_0, \quad [E_1, E_2] = 0,
\end{array}$$

where α, β are arbitrary real parameters. Moreover, the relations of α and β with the non-zero components F_{ijk} in the different basic classes $\mathcal{F}_s$ from (1.1) are the following:

$$\begin{array}{ll}
\mathcal{F}_1: \quad \alpha = \frac{1}{2}\theta_1, \quad \beta = \frac{1}{2}\theta_2; & \mathcal{F}_4: \quad \alpha = \frac{1}{2}\theta_0; \\
\mathcal{F}_5: \quad \alpha = -\frac{1}{2}\theta_0^*; & \mathcal{F}_8: \quad \alpha = -\lambda; \\
\mathcal{F}_9: \quad \alpha = -\mu; & \mathcal{F}_{10}: \quad \alpha = \frac{1}{2}\nu; \\
\mathcal{F}_{11}: \quad \alpha = -\omega_2, \quad \beta = \omega_1. &
\end{array}$$

2. The main result

Theorem 1.1 yields an equivalence between the manifolds of the classification in [5] and the corresponding Lie algebra.

It is known (e.g. [6]) that for a real Lie algebra of finite dimension there is a corresponding connected simply connected Lie group, which is determined uniquely up to isomorphism.

It is arose the problem for determination of the Lie group which is isomorphic to the given Lie group L equipped with a structure (φ, ξ, η, g) in the class $\mathcal{F}_s$.

Theorem 2.1. *Let $(L, \varphi, \xi, \eta, g)$ be an almost contact B-metric manifold belonging to the class $\mathcal{F}_s$ ($s \in \{1, 4, 5, 8, 9, 10, 11\}$). Then the compact simply connected Lie group G isomorphic to L, both with one and the same Lie algebra, has the form*

$$e^A = E + tA + uA^2,$$

where E is the identity matrix and A is the matrix representation of the corresponding Lie algebra. The matrix form of A as well as the real parameters t and u for the different classes $\mathcal{F}_s$ are given in Table 2.1, where $a, b, c \in \mathbb{R}$ and α, β are introduced in Theorem 1.1.

2.1. Proof of the theorem

2.1.1. Lie groups as manifolds from the classes $\mathcal{F}_1$, $\mathcal{F}_5$, $\mathcal{F}_{11}$

Firstly, let us consider the case when $(L, \varphi, \xi, \eta, g)$ is a $\mathcal{F}_1$-manifold. Then, according to Theorem 1.1, the corresponding Lie algebra is determined by

$$[E_0, E_1] = [E_0, E_2] = 0, \quad [E_1, E_2] = \alpha E_1 + \beta E_2, \tag{2.1}$$

where $\alpha = \frac{1}{2}\theta_1$, $\beta = \frac{1}{2}\theta_2$.

¿From (2.1) we have the nonzero values of the commutation coefficients:

$$C_{12}^1 = -C_{21}^1 = \alpha, \quad C_{12}^2 = -C_{21}^2 = -\beta. \tag{2.2}$$

According to [6], the commutation coefficients provide a matrix representation of the Lie algebra. This representation is obtained by the basic matrices M_i, which entries are determined by

$$(M_i)_j^k = -C_{ij}^k. \tag{2.3}$$

Using (2.2) and (2.3), we obtain

$$M_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\alpha & -\beta \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us suppose that $(a, b) \neq (0, 0)$. The matrix representation of the considered Lie algebra $\mathfrak{g}_1$ is the matrix A, given in Table 2.1. Then, the characteristic polynomial of A is

$$P_A(\lambda) = \lambda^2(\lambda - \alpha b + \beta a).$$

Therefore, the eigenvalues λ_i ($i = 1, 2, 3$) are

$$\lambda_1 = \lambda_2 = 0, \qquad \lambda_3 = \alpha b - \beta a.$$

Table 2.1: The matrix form of A and the values of t and u for $\mathcal{F}_s$

$\mathcal{F}_1:$	$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha b & \beta b \\ 0 & -\alpha a & -\beta a \end{pmatrix}$ $\mathrm{tr}A = \alpha b - \beta a$	$t = \begin{cases} \frac{e^{\mathrm{tr}A}-1}{\mathrm{tr}A}, & \mathrm{tr}A \neq 0 \\ 1, & \mathrm{tr}A = 0 \end{cases}$ $u = 0$
$\mathcal{F}_4:$	$A = \begin{pmatrix} 0 & -\alpha b & \alpha a \\ 0 & 0 & -\alpha c \\ 0 & \alpha c & 0 \end{pmatrix}$ $\mathrm{tr}A^2 = -2\alpha^2 c^2$	$t = \begin{cases} \frac{\sin\sqrt{-\frac{1}{2}\mathrm{tr}A^2}}{\sqrt{-\frac{1}{2}\mathrm{tr}A^2}}, & \mathrm{tr}A^2 \neq 0 \\ 1, & \mathrm{tr}A = 0 \end{cases}$ $u = \begin{cases} \frac{1-\cos\sqrt{-\frac{1}{2}\mathrm{tr}A^2}}{-\frac{1}{2}\mathrm{tr}A^2}, & \mathrm{tr}A^2 \neq 0 \\ 0, & \mathrm{tr}A = 0 \end{cases}$
$\mathcal{F}_5:$	$A = \begin{pmatrix} 0 & \alpha a & \alpha b \\ 0 & -\alpha c & 0 \\ 0 & 0 & -\alpha c \end{pmatrix}$ $\mathrm{tr}A = -2\alpha c$	$t = \begin{cases} \frac{e^{\frac{1}{2}\mathrm{tr}A}-1}{\frac{1}{2}\mathrm{tr}A}, & \mathrm{tr}A \neq 0 \\ 1, & \mathrm{tr}A = 0 \end{cases}$ $u = 0$
$\mathcal{F}_8:$	$A = \begin{pmatrix} 0 & \alpha b & \alpha a \\ -2\alpha b & 0 & -\alpha c \\ 2\alpha a & -\alpha c & 0 \end{pmatrix}$ $\mathrm{tr}A^2 = 2\alpha^2\Delta$ $\Delta = 2a^2 - 2b^2 + c^2$	$t = \begin{cases} -\frac{\sin\alpha\sqrt{\lvert\Delta\rvert}}{\alpha\sqrt{\lvert\Delta\rvert}}, & \mathrm{tr}A^2 < 0 \\ 1, & \mathrm{tr}A^2 = 0 \\ \frac{\sinh\alpha\sqrt{\Delta}}{\alpha\sqrt{\Delta}}, & \mathrm{tr}A^2 > 0 \end{cases}$ $u = \begin{cases} \frac{\cos\alpha\sqrt{\lvert\Delta\rvert}-1}{\alpha^2\Delta}, & \mathrm{tr}A^2 < 0 \\ \frac{1}{2}, & \mathrm{tr}A^2 = 0 \\ \frac{\cosh\alpha\sqrt{\Delta}-1}{\alpha^2\Delta}, & \mathrm{tr}A^2 > 0 \end{cases}$
$\mathcal{F}_9:$	$A = \begin{pmatrix} 0 & \alpha a & -\alpha b \\ 0 & -\alpha c & 0 \\ 0 & 0 & \alpha c \end{pmatrix}$ $\mathrm{tr}A^2 = 2\alpha^2 c^2$	$t = \begin{cases} \frac{\sinh\sqrt{\frac{1}{2}\mathrm{tr}A^2}}{\sqrt{\frac{1}{2}\mathrm{tr}A^2}}, & \mathrm{tr}A^2 \neq 0 \\ 1, & \mathrm{tr}A^2 = 0 \end{cases}$ $u = \begin{cases} \frac{\cosh\frac{1}{2}\mathrm{tr}A^2-1}{\frac{1}{2}\mathrm{tr}A^2}, & \mathrm{tr}A^2 \neq 0 \\ 0, & \mathrm{tr}A^2 = 0 \end{cases}$
$\mathcal{F}_{10}:$	$A = \begin{pmatrix} 0 & \alpha b & \alpha a \\ 0 & 0 & -\alpha c \\ 0 & -\alpha c & 0 \end{pmatrix}$ $\mathrm{tr}A^2 = 2\alpha^2 c^2$	$t = \begin{cases} \frac{\sinh\alpha c}{\alpha c}, & \mathrm{tr}A^2 \neq 0 \\ 1, & \mathrm{tr}A^2 = 0 \end{cases}$ $u = \begin{cases} \frac{\cosh\alpha c}{\alpha^2 c^2}, & \mathrm{tr}A^2 \neq 0 \\ 0, & \mathrm{tr}A^2 = 0 \end{cases}$
$\mathcal{F}_{11}:$	$A = \begin{pmatrix} \alpha a + \beta b & 0 & 0 \\ -\alpha c & 0 & 0 \\ -\beta c & 0 & 0 \end{pmatrix}$ $\mathrm{tr}A = \alpha a + \beta b$	$t = \begin{cases} \frac{e^{-\mathrm{tr}A}-1}{\mathrm{tr}A}, & \mathrm{tr}A \neq 0 \\ 1, & \mathrm{tr}A = 0 \end{cases}$ $u = 0$

Hence, the corresponding linearly independent eigenvectors p_i $(i = 1, 2, 3)$ are

$$p_1(1,0,0)^T, \qquad p_2(0,\beta,-\alpha)^T, \qquad p_3(0,-b,a)^T$$

and their matrix P has the following form

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & -b \\ 0 & -\alpha & a \end{pmatrix}.$$

Let us denote $\Delta = \beta a - \alpha b$. Then, we have $\Delta = \det P = -\mathrm{tr} A$.

Let us consider the first case $\Delta \neq 0$. Then we have

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{a}{\Delta} & \frac{b}{\Delta} \\ 0 & \frac{\alpha}{\Delta} & \frac{\beta}{\Delta} \end{pmatrix}.$$

The Jordan matrix J is the diagonal matrix with elements $J_{ii} = \lambda_i$. It is known that $e^A = Pe^J P^{-1}$. Then we obtain the matrix Lie group representation G_1 of the considered Lie algebra $\mathfrak{g}_1$ in this case as follows

$$G_1 = \left\{ e^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + \frac{\alpha b}{\Delta}\left(1 - e^{-\Delta}\right) & \frac{\beta b}{\Delta}\left(1 - e^{-\Delta}\right) \\ 0 & -\frac{\alpha a}{\Delta}\left(1 - e^{-\Delta}\right) & 1 - \frac{\beta a}{\Delta}\left(1 - e^{-\Delta}\right) \end{pmatrix} \middle| \quad \Delta \neq 0 \right\},$$

which can be written as

$$G_1 = \left\{ e^A = E + \frac{1 - e^{-\Delta}}{\Delta} A \;\middle|\; \Delta \neq 0 \right\}.$$

In the second case $\Delta = 0$, the matrix P is non-invertible. Then A is nilpotent and e^A can be computed directly from

$$e^A = E + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots + + \frac{A^{q-1}}{(q-1)!},$$

where q stands for the degree of A. Then, from the form of A in Table 2.1 we obtain that $q = 2$. Therefore, the matrix representation G_1 of the Lie group for the Lie algebra $\mathfrak{g}_1$ in this case is

$$G_1 = \left\{ e^A = E + A,\ \Delta = 0 \right\},$$

i.e.

$$G_1 = \left\{ e^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + \alpha b & \beta b \\ 0 & -\alpha a & 1 - \beta a \end{pmatrix},\ \Delta = 0 \right\}.$$

Both cases for $\mathcal{F}_1$ can be generalized as it is shown in Table 2.1.

By similar considerations we obtain the results in Table 2.1 for the classes $\mathcal{F}_5$ and $\mathcal{F}_{11}$.

2.1.2. Lie groups as manifolds from the classes $\mathcal{F}_4$, $\mathcal{F}_9$, $\mathcal{F}_{10}$

Firstly, let us consider the case when (L,φ,ξ,η,g) is a $\mathcal{F}_4$-manifold. Then, according to Theorem 1.1, the corresponding Lie algebra is determined by

$$[E_0,E_1]=\alpha E_2,\quad [E_0,E_2]=-\alpha E_1,\quad [E_1,E_2]=0, \tag{2.4}$$

where $\alpha=\frac{1}{2}\theta_0$.

In a way similar to §2.1.1, we obtain the form of A for the corresponding Lie algebra $\mathfrak{g}_4$, given in Table 2.1. In this class $\mathrm{tr}A=0$ and we have two cases according to $\mathrm{tr}A^2=-2\alpha^2c^2$. Then, the matrix representation is respectively

$$G_4=\left\{e^A=\begin{pmatrix} 1 & \frac{a}{c}(1-\cos\alpha c)-\frac{b}{c}\sin\alpha c & \frac{b}{c}(1-\cos\alpha c)+\frac{a}{c}\sin\alpha c\\ 0 & \cos\alpha c & -\sin\alpha c\\ 0 & \sin\alpha c & \cos\alpha c\end{pmatrix},\ c\neq 0\right\},$$

$$G_4=\left\{e^A=\begin{pmatrix} 1 & -\alpha b & \alpha a\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ c=0\right\}.$$

Both cases for $\mathcal{F}_4$ can be generalized as it is shown in Table 2.1.

By similar considerations we obtain the results in Table 2.1 for the classes $\mathcal{F}_9$ and $\mathcal{F}_{10}$.

2.1.3. Lie groups as manifolds from the class $\mathcal{F}_8$

According to Theorem 1.1, we have

$$[E_0,E_1]=\alpha E_2,\quad [E_0,E_2]=\alpha E_1,\quad [E_1,E_2]=0,$$

where $\alpha=-\lambda$.

We get the matrix representation of A for the considered Lie algebra $\mathfrak{g}_8$, given in Table 2.1.

In this class $\mathrm{tr}A=0$ and we have three cases according to the sign of $\mathrm{tr}A^2=2\alpha^2\Delta$, where $\Delta=2a^2-2b^2+c^2$. They can be generalized as it is shown in Table 2.1.

3. Equipping of known Lie groups with almost contact B-metric structure

In this final section we equip some known Lie groups with almost contact B-metric structures of different types. These considerations use the results given in [10].

3.1. Examples 1, 2, 3

In [8], the matrix Lie groups G_{I}, G_{II} and G_{III} of the following form are considered:

$$G_{\mathrm{I}} = \begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{pmatrix}, \quad G_{\mathrm{II}} = \begin{pmatrix} \cos z & -\sin z & x \\ \sin z & \cos z & y \\ 0 & 0 & 1 \end{pmatrix}, \quad G_{\mathrm{III}} = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$. These matrix groups and their automorphisms represent the only three types of generalized affine symmetric spaces denoted as I, II, III. The Lie group G_{I} is the group of hyperbolic motions of the plane $\mathbb{R}^2$, G_{II} is the group of isometries of the plane $\mathbb{R}^2$ and G_{III} is the Heisenberg group. The corresponding Lie algebras are determined by the commutators as follows:

$$\begin{array}{llll} \mathfrak{g}_{\mathrm{I}}: & [X_1, X_3] = X_1, & [X_2, X_3] = -X_2, & [X_1, X_2] = 0, \\ \mathfrak{g}_{\mathrm{II}}: & [X_1, X_3] = -X_2, & [X_2, X_3] = X_1, & [X_1, X_2] = 0, \\ \mathfrak{g}_{\mathrm{III}}: & [X_1, X_3] = X_2, & [X_2, X_3] = 0, & [X_1, X_2] = 0. \end{array}$$

The types of the Lie algebras $\mathfrak{g}_{\mathrm{I}}$, $\mathfrak{g}_{\mathrm{II}}$, $\mathfrak{g}_{\mathrm{III}}$ according to the well-known Bianchi classification in [1, 2] of three-dimensional real Lie algebras are Bia(V), Bia(VII$_0$), Bia(II), respectively.

Bearing in mind Table 2.1 for $\mathcal{F}_9$ and substituting $X_1 = E_1$, $X_2 = E_2$, $X_3 = -E_0$, $\alpha = 1$, we have that G_{I} can be considered as an almost contact B-metric manifold of the class $\mathcal{F}_9$ when the derived algebra is on $\ker(\eta)$, otherwise the manifold belongs to $\mathcal{F}_1 \oplus \mathcal{F}_{11}$. [10]

Considering Table 2.1 for $\mathcal{F}_4$ and substituting $X_1 = E_1$, $X_2 = E_2$, $X_3 = E_0$, $\alpha = 1$, we have that G_{II} can be considered as an almost contact B-metric manifold of the class $\mathcal{F}_4$ when the derived algebra is on $\ker(\eta)$, otherwise the manifold belongs to $\mathcal{F}_4 \oplus \mathcal{F}_8$ or $\mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$. [10]

Depending on the way of equipping with an almost contact B-metric structure we can obtain G_{III} as a manifold in $\mathcal{F}_4 \oplus \mathcal{F}_{10}$ or $\mathcal{F}_8 \oplus \mathcal{F}_{10}$ when the derived algebra is on $\ker(\eta)$ or $\mathrm{span}(\xi)$, respectively. [10]

3.2. Example 4

Let us consider the well-known Lie group $SO(3)$ which is called rotation group. The matrix representation A of the corresponding Lie algebra has the form

$$A = \begin{pmatrix} 0 & -\alpha z & \alpha y \\ \alpha z & 0 & -\alpha x \\ -\alpha y & \alpha x & 0 \end{pmatrix},$$

where α is the angle of rotation and $x, y, z \in \mathbb{R}$.

In terms of the matrix exponential and according to the Rodrigues rotational formula, the Lie group $SO(3)$ consists of the matrices of the following form

$$e^A = E + \sin\alpha\, A + (1 - \cos\alpha)\, A^2.$$

Then, the type of the considered Lie algebra is Bia(IX), according to the Bianchi classification. Hence, bearing in mind [10], this Lie group can be considered as an almost contact B-metric manifold of the class $\mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$.

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