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DIFFERENTIAL CALCULUS ON LIE ALGEBRAS

Valiollah Khalili

Abstract. We state the notion of differential calculus based on derivation for Lie algebras. We also construct graded differential algebra and investigate differential calculus based on derivation for semi-simple Lie algebra $sl(n, \mathbb{C})$. So, we provide the notion of matrix geometry of a Lie algebra in noncommutative differential geometry.
Keywords: differential calculus, Lie algebra, differential algebra

1 Introduction

Lie algebras were originally introduced by Sophus Lie, as an algebraic structure used for the study of linear transformation groups that are now named "Lie groups". Both Lie groups and Lie algebras have become fundamental tools to many branches of mathematics and theoretical physics. Finite dimensional Lie algebras were investigated independently by E. Cartan and W. Killing during the period 1800-1900(see [12]). In 1967 V. G. Kac and R. V. Moody independently discovered a class of infinite dimensional Lie algebras which is called "Kac-Moody Lie algebras" and includes finite dimensional simple Lie algebras (see [13]). In 1990 Hoegh-Kron and Torresani [11] initiated "irreducible quasi-simple Lie algebras" which were investigated systematically in 1997 by Alison, Azam, Berman, Gao and Pianzola in Memoirs AMS [1]. They called these Lie algebras "extended affine Lie algebras". Various classes of these Lie algebras have been investigated in many articles (for example see [2, 3, 14, 15, 16, 21, 22, 23]). There are some applications of Lie algebras which are non-commutative and non-associative algebras in mathematical physics, statistical physics, conformal field theory, string theory and quantum groups.

As indicated in [5], the generalization of differential calculus from classical differential geometry to non-commutative differential geometry is not unique. Namely, according to the various applications in both mathematics and physics, one can

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consider a notion of differential calculus. There are several approaches to non-commutative generalizations of the construction of the deRham forms on the algebra $C^{\infty}(M)$, of smooth functions on a smooth paracompact manifold M as an abstract commutative $*-$algebra.

There are two notions of differential calculus on non-commutative algebra.

One approach of differential calculus was given by A. Conns [6] in 1986. He investigated differential calculus based on the concept of spectral triples. A spectral triple basically consists of a non-commutative algebra $\mathcal{A}$, a representation of this algebra on a Hilbert space $\mathcal{H}$ on which $\mathcal{A}$ is realized as an algebra of bounded operators, an operator $\mathcal{D}$ on $\mathcal{H}$ which is responsible for generating differential calculus, which is named the Dirac operator. This approach is indicated by the term "non-commutative Riemannian geometry", because it emphasizes the metric structure.

The second notion of differential calculus on non-commutative algebra which focuses on the differential objects, is introduced by Dubois-Violette [7] in 1988. This non-commutative differential geometry is encoded into a purely algebraic definition of differential calculus on associative (commutative, non-commutative) algebra which is called "derivation based differential calculus". More precisely, let $\mathcal{A}$ be an associative algebra with a unit. The algebra $\mathcal{A}$ is considered as the generalization of the algebra of smooth functions and the Lie algebra $Der(\mathcal{A})$ of all derivations of $\mathcal{A}$ is considered as the generalization of the Lie algebra of smooth vector fields. The notions of differential forms can be extracted from the graded differential algebra $C(Der(\mathcal{A}), \mathcal{A})$ of Chevalley- Eilenberg cochains of Lie algebra $Der(\mathcal{A})$ with values in the $Der(\mathcal{A})-$module $\mathcal{A}$.

In the present article, we directly use this construction for the graded differential algebra of Lie algebra. More precisely, we state the notion of differential calculus based on derivation for Lie algebra in general, which is not necessarily finite dimensional. We also provide some examples.

To close this introduction, we outline the contents of the paper. In Section 2, we recall the definition of graded differential algebra and some facts that will be needed in the sequel. In Section 3, we first review an introduction on Lie algebra cohomology, which is an essential tool for the concept of graded differential algebra and differential calculus of Lie algebra. Next, we define differential calculus based on derivation for Lie algebra. In Section 4, we provide two examples. The first example indicates the characterization of the differential algebra of Matrix geometry which is already investigated in [9]. The second example provides the realization of the concept of differential calculus based on derivation for a semi-simple Lie algebra $sl(n, \mathbb{C})$.

2 Preliminaries

All vector spaces and algebras in this note are considered over a fixed field $\mathbb{F}$ of characteristic zero. If otherwise, it will be specified. This section is devoted to the study of some concepts and definitions that will be extensively used in the last section.

We first review the definition of graded differential algebra.

Definition 2.1. *(Graded vector spaces)*

Let $\mathcal{V}$ be a vector space (or $\mathbb{F}$-space). By a $\mathbb{Z}-$grading of $\mathcal{V}$ we will denote a family $\{\mathcal{V}^i\}_{i\in\mathbb{Z}}$ of subspaces of $\mathcal{V}$ such that $\mathcal{V} = \bigoplus_{i\in\mathbb{Z}} \mathcal{V}^i$. Given such a grading, we call $\mathcal{V}^i$ a degree subspace and an element belonging to a degree subspace $\mathcal{V}^i$ is said to be homogeneous of degree i.

Let $\mathcal{V}$ and $\mathcal{W}$ be two graded vector spaces. A linear map of degree n between graded vector spaces is a linear map $f : \mathcal{V} \longrightarrow \mathcal{W}$ such that $f(\mathcal{V}^i) \subseteq \mathcal{W}^{i+n}$ for all $i \in \mathbb{Z}$. Define $Hom^n_{\mathbb{F}}(\mathcal{V},\mathcal{W})$ to be the $\mathbb{F}$-subspace of $Hom_{\mathbb{F}}(\mathcal{V},\mathcal{W})$ consisting of homogeneous elements of degree n. Then

$$Hom_{\mathbb{F}}(\mathcal{V},\mathcal{W}) = \bigoplus_{n\in\mathbb{Z}} Hom^n_{\mathbb{F}}(\mathcal{V},\mathcal{W}),$$

is a graded $\mathbb{F}$-space. If $f \in Hom_{\mathbb{F}}(\mathcal{V},\mathcal{W})$ are homogeneous elements of degree zero, we say that f is a graded homomorphism.

Definition 2.2. *(Differential graded vector space) A differential graded vector space is a graded vector space $\mathcal{V}$ together with $d \in Hom^1_{\mathbb{F}}(\mathcal{V},\mathcal{V})$ such that $d^2 = 0$. The homogeneous element of degree 1 is called a differential of a graded vector space $\mathcal{V}$.*

Lemma 2.3. *Let $(\mathcal{V}, d_{\mathcal{V}})$ and $(\mathcal{W}, d_{\mathcal{W}})$ be two differential graded vector spaces. Then $Hom_{\mathbb{F}}(\mathcal{V},\mathcal{W})$ is a differential graded vector space.*

Proof. Let $f \in Hom^n_{\mathbb{F}}(\mathcal{V},\mathcal{W})$. Define

$$D : Hom^n_{\mathbb{F}}(\mathcal{V},\mathcal{W}) \longrightarrow Hom^{n+1}_{\mathbb{F}}(\mathcal{V},\mathcal{W}),$$

by

$$D(f) = d_{\mathcal{W}} \circ f - (-1)^n f \circ d_{\mathcal{V}}.$$

It is clear that D is a homogeneous element of degree 1. Also

$$\begin{aligned} D^2(f) &= D(d_{\mathcal{W}} \circ f - (-1)^n f \circ d_{\mathcal{V}}) \\ &= d^2_{\mathcal{W}} \circ f - (-1)^{n+1} d_{\mathcal{W}} \circ f \circ d_{\mathcal{V}} - (-1)^n (d_{\mathcal{W}} \circ f \circ d_{\mathcal{V}} \\ &- (-1)^{n+1} f \circ d^2_{\mathcal{V}}) \\ &= 0. \end{aligned}$$

□

Definition 2.4. *A graded $\mathbb{F}-$algebra $\mathcal{A}$ is a graded vector space $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}^n$, together with a multiplication such that $A^m A^n \subseteq A^{m+n}$, for all $m, n \in \mathbb{Z}$. A graded algebra $\mathcal{A}$ is said to be graded commutative if $ab = (-1)^{mn} ba$ for all $a \in \mathcal{A}^m$ and all $b \in \mathcal{A}^n$.*

Definition 2.5. *(Differential graded algebras)*

A differential graded algebra is a graded algebra $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}^n$ together with a graded derivation $d : \mathcal{A} \longrightarrow \mathcal{A}$ of degree 1 (anti-derivation) such that $d^2 = dod = 0$. Thus d satisfies the graded Leibniz rule $d(ab) = d(a)b + (-1)^n ad(b)$, where $a \in \mathcal{A}^n$ and $b \in \mathcal{A}$ and also $d(\mathcal{A}^n) \subset \mathcal{A}^{n+1}$. The anti-derivation d is called a differential of differential graded algebra $\mathcal{A}$.

Corollary 2.6. *Let $(\mathcal{V}, d)$ be a graded vector space. Then $(Hom_{\mathbb{F}}(\mathcal{V}, \mathcal{V}), D)$ with*

$$D(f) = d \circ f - (-1)^n f \circ d \quad , \qquad f \in Hom^n_{\mathbb{F}}(\mathcal{V}, \mathcal{V}),$$

is differential graded algebra.

Proof. By Lemma 2.3 it is enough to show that D satisfies the Leibniz rule.

$$\begin{aligned} D(f \circ g) &= d \circ f \circ g - (-1)^{n+m} f \circ g \circ d) \\ &= d \circ f \circ g - (-1)^n f \circ d \circ g + (-1)^n f \circ d \circ g \\ &- (-1)^{n+m} f \circ g \circ d \\ &= D(f) \circ g + (-1)^n f \circ D(g). \end{aligned}$$

□

Remark 21. *The theory of differential operators on associative algebras is not extended to the non-associative ones [19]. However, there is a notion of differential operators on the commutative ring and its generalization to noncommutative geometry which is not unique. The definition of differential operator of order k on Lie algebras can be find in [18]. Although it is an interesting subject to work on, we shall not discuss it here.*

As an example of graded differential algebra, we now state the graded subspace of differential operators of order $\leq k$. Let $\mathcal{A}$ be a unital graded differential algebra with $1 \in \mathcal{A}^0$. We may consider $\mathcal{A}$ as a graded commutative Lie subalgebra of $Hom^*_{\mathbb{F}}(\mathcal{A}, \mathcal{A})$, where every element a of $\mathcal{A}$ is identified by the operator

$$a \ : \ \mathcal{A} \longrightarrow \mathcal{A} \ ; \quad a(b) = ab.$$

For every integer k we denote, the graded subspace of differential operator of order $\leq k$, by

$$Diff_k(\mathcal{A}) = \bigoplus_{n\in\mathbb{Z}} Diff_k^n(\mathcal{A}) \subset Hom_{\mathbb{F}}^*(\mathcal{A},\mathcal{A}),$$

It is defined recursively by $Diff_k(\mathcal{A}) = 0$ for $k < 0$ and for $k \geq 0$:

$$Diff_k(\mathcal{A}) = \{\ f \in Hom_{\mathbb{F}}^*(\mathcal{A},\mathcal{A})\ :\ [f,a] \in Diff_{k-1}(\mathcal{A}), \quad \forall a \in \mathcal{A}\}.$$

We note that $f \in Diff_0(\mathcal{A})$ if and only if $f(a) = f(1)a$ and every derivation on $\mathcal{A}$ lies in $Diff_1(\mathcal{A})$.

A simple verification by induction on $m + k$ shows that

$$Diff_m(\mathcal{A})Diff_k(\mathcal{A}) \subset Diff_{m+k}(\mathcal{A}),$$

and

$$[Diff_m(\mathcal{A}), Diff_k(\mathcal{A})] \subset Diff_{m+k-1}(\mathcal{A}).$$

Therefore $Diff(\mathcal{A}) = \bigcup_k Diff_k(\mathcal{A})$ is a graded differential Lie subalgebra of $Hom_{\mathbb{F}}^*(\mathcal{A},\mathcal{A})$.

In the study of differential calculus over an algebra (commutative, noncommutative complex) as the generalization of differential forms and the geometry of fiber bundle through differential forms, the Cartan's operator is one of the main tools [4]. This notion can be considered in the graded differential algebra on Lie algebra.

Definition 2.7. *(Cartan's operator)*

An operation of a Lie algebra $\mathcal{G}$ on a graded differential algebra $\mathcal{A}$ is a linear mapping $x \longmapsto i_x$ of $\mathcal{G}$ into the space of anti-derivations of degree -1 of $\mathcal{A}$ such that

- *(a) for all $x, y \in \mathcal{G}$, we have $i_x i_y + i_y i_x = 0$*
- *(b) for all $x, y \in \mathcal{G}$, we have $L_x i_y - i_y L_x = i_{[x,y]}$,*

where L_x denotes the derivation of degree 0 of $\mathcal{A}$ defined by

$$L_x := i_x \circ d + d \circ i_x.$$

(b) implies that $[L_x, L_y] = L_{[x,y]}$ for all $x, y \in \mathcal{G}$. Therefore, we have a representation of Lie algebra $\mathcal{G}$ to Lie algebra of graded anti-derivations of degree -1 of $\mathcal{A}$. That is, the map $x \longmapsto L_x$ is a Lie algebra homomorphism, since $L_x \circ d = d \circ L_x$ for all $x \in \mathcal{G}$.

We next recall the definition of derivation of a Lie algebra.

Definition 2.8. *(Derivation of Lie algebra)*

Let $\mathcal{G}$ be a Lie algebra over a field $\mathbb{F}$ with center $Z(\mathcal{G})$. A derivation of Lie algebra $\mathcal{G}$ is a linear map $\partial : \mathcal{G} \longrightarrow \mathcal{G}$ such that

$$\partial[x, y] = [\partial(x), y] + [x, \partial(y)], \tag{2.9}$$

for all $x, y \in \mathcal{G}$.

The vector space of all derivations of $\mathcal{G}$ denoted by $Der(\mathcal{G})$ is a Lie algebra for the Lie bracket $[\partial, \delta] = \partial \circ \delta - \delta \circ \partial$ and also a $Z(\mathcal{G})-$module for the product $(z\partial)(x) = z\partial(x)$ where $z \in Z(\mathcal{G})$ and $\partial \in Der(\mathcal{G})$. The subspace $Inn(\mathcal{G}) = \{ad_x : y \longrightarrow [x, y] \mid x \in \mathcal{G}\} \subset Der(\mathcal{G})$ which is called the vector space of inner derivations, is a Lie ideal and also a $Z(\mathcal{G})-$submodule.

Example 2.10. *All derivations of a general linear Lie algebra $gl_n(\mathbb{F})$, a special linear Lie algebra $sl_n(\mathbb{F})$, and orthogonal Lie algebras $o_{2n}(\mathbb{F})$ and $o_{2n+1}(\mathbb{F})$ are inner. Also $Z(gl_n(\mathbb{F})) = \mathbb{F}$ and all other Lie algebras are centerless. Thus $Der(gl_n(\mathbb{F})) = sl_n(\mathbb{F})$ and a derivation of other Lie algebras are itself.*

3 Differential calculus

The Lie algebras involved in this section are not necessarily finite dimensional. First of all, we briefly review an introduction to Lie algebra cohomology which is an essential requirement for the graded differential algebra for Lie algebras. For more details see the excellent source [10]. Secondly, we define the differential calculus based on derivation of a Lie algebra.

Definition 3.1. *(n-cochains on $\mathcal{G}$)*

Let $\mathcal{G}$ be a Lie algebra over the field $\mathbb{F}$ and M be a $\mathcal{G}-$module with the representation $\rho : \mathcal{G} \longrightarrow End(M)$. An $M-$valued $n-$cochain β_n of $\mathcal{G}$ on M is the skew-symmetric $\mathbb{F}-$multilinear mapping

$$\beta_n : \bigwedge^n \mathcal{G} \longrightarrow M \ ; \quad \beta_n(x_1 \wedge x_2 \wedge ... \wedge x_n) = \beta_n(x_1, x_2, ..., x_n),$$

where all $x_1, x_2, ..., x_n \in \mathcal{G}$. The vector space of these $n-$cochains which forms a $\mathcal{G}-$module will be denoted by $C^n(\mathcal{G}, M)$, and is called the Chevalley-Eilenberg cochain of $\mathcal{G}$.

Definition 3.2. *(Coboundary operator on $\mathcal{G}$)*

Let $C(\mathcal{G}, M) = \bigoplus_n C^n(\mathcal{G}, M)$ be the $\mathbb{N}-$graded vector space of all $M-$valued cochains of Lie algebra $\mathcal{G}$ on M. The coboundary operator $d : C^n(\mathcal{G}, M) \longrightarrow C^{n+1}(\mathcal{G}, M)$

is a homogeneous endomorphism of degree 1 of $C(\mathcal{G}, M)$ defined by its action on the cochains:

$$\begin{aligned} d(\beta_n)(x_0, ..., x_n) &= \sum_{k=0}^{n}(-1)^k \rho(x_k)(\beta_n(x_0, ..., \hat{x_k}, ..., x_n) \\ &+ \sum_{0\leq i<j\leq n}(-1)^{i+j}\beta_n([x_i, x_j], x_0, ..., \hat{x_i}, ..., \hat{x_j}, ..., x_n), \end{aligned}$$

where $\beta_n \in C^n(\mathcal{G}, M)$ and $x_1, x_2, ..., x_n \in \mathcal{G}$. Notice that the notation "ˆ" means omission.

Using the Jacobi identity and the fact that ρ is the Lie algebra homomorphism, $\rho([x_1, x_2]) = [\rho(x_1), \rho(x_2)]$, it may be verified that $d^2 = 0$.

Lemma 3.3. *Let $\mathcal{G}$ be a Lie algebra and $\mathcal{A}$ be an algebra which $\mathcal{G}$ acts on $\mathcal{A}$ by derivation. Then the $\mathbb{N}-$graded vector space of all $\mathcal{A}-$valued cochains $C(\mathcal{G}, \mathcal{A}) = \bigoplus_n C^n(\mathcal{G}, \mathcal{A})$ is a graded differential algebra.*

Proof. First, we observe that the multiplication on $C(\mathcal{G}, \mathcal{A})$ is obtained by taking the product in $\mathcal{A}$ after evaluation. Next, suppose that $\beta_n \in C^n(\mathcal{G}, \mathcal{A})$ be an $\mathcal{A}-$valued $n-$cochain of $\mathcal{G}$ on $\mathcal{A}$ and that β_n is an anti-symmetric $\mathbb{F}-$multilinear map. Since d is a coboundary operator, we have $d(C^n(\mathcal{G}, \mathcal{A})) \subseteq C^{n+1}(\mathcal{G}, \mathcal{A})$, and the derivation property of the action of $\mathcal{A}$ on $\mathcal{G}$ implies that d is a graded derivation of degree 1 on $C(\mathcal{G}, \mathcal{A})$. Therefore $(C(\mathcal{G}, \mathcal{A}), d)$ is a graded differential algebra. □

Definition 3.4. *(Cochain complex)*

Let $C^n(\mathcal{G}, M)$ be the vector space of all $M-$valued $n-$cochains of $\mathcal{G}$ on M. Let $\beta_n \in C^n(\mathcal{G}, M)$ and $d(\beta_n) \in C^{n+1}(\mathcal{G}, M)$ be as defined in the Definition 3.2, then we obtain the cochain complex

$$0 \longrightarrow M \xrightarrow{d_1} C^1(\mathcal{G}, M) \xrightarrow{d_2} C^2(\mathcal{G}, M) \longrightarrow ... \xrightarrow{d_n} C^n(\mathcal{G}, M) \longrightarrow ...,$$

which is called the Chevalley-Eilenberg complex and is denoted by $C^*(\mathcal{G}, M)$.

Definition 3.5. *(Lie algebra $n-$cocycle, $n-$coboundary, cohomology)*

Let $C^*(\mathcal{G}, M)$ be the Chevalley-Eilenberg complex with the coefficients in $\mathcal{G}-$module M. As always, we denote the space of $n-$cocycles by

$$Z^n(\mathcal{G}, M) := \{\ \beta \in C^n(\mathcal{G}, M)\ :\ d_n(\beta) = 0\} = \ker d_n,$$

and the space of $n-$coboundaries by

$$B^n(\mathcal{G},M) := \{\ \beta \in C^n(\mathcal{G},M)\ :\ \exists \beta' \in C^{n-1}(\mathcal{G},M), \quad d_{n-1}(\beta') = \beta\}.$$

Then we define the $n-th$ cohomology space of Lie algebra $\mathcal{G}$ with a value in M as the quotient vector space

$$H^n(\mathcal{G},M) = Z^n(\mathcal{G},M)/B^n(\mathcal{G},M).$$

By the convention $B^0(\mathcal{G},M) = 0$, the zero-cochain is defined as constant from $\mathcal{G}$ to M. Thus, a zero-cochain is a vector in M.

Remark 31. *For finite dimensional Lie algebra $\mathcal{G}$, there is an algebraic interpretation of the $n-th$ cohomology space for $n = 1$ which is important. This and more interpretations may be found in [10]. Suppose that M acts on $\mathcal{G}$ by "$\cdot$", define*

$$Der(\mathcal{G},M) := \{f \in Hom_{\mathbb{F}}(\mathcal{G},M)\ :\ f([x,y]) = x \cdot f(y) - y \cdot f(x)\},$$

for all $x, y \in \mathcal{G}$, and also

$$PDer(\mathcal{G},M) := \{f \in Hom_{\mathbb{F}}(\mathcal{G},M)\ :\ f(x) = x \cdot m\},$$

for all $x \in \mathcal{G}$ and for some $m \in M$. Then if one-cochain β_1 is a one-cocycle, we have

$$\begin{aligned} (d_1\beta_1)(x,y) &= \rho(x)\beta_1(y) - \rho(y)\beta_1(x) - \beta_1([x,y]) \\ &= 0, \quad \forall x, y \in \mathcal{G}. \end{aligned} \tag{3.6}$$

A one-cochain β_1 is a one-coboundary if there is a zero-cochain $m \in M$ such that $d_1\ m = \beta_1$, that is, if

$$\beta_1(x) = \rho(x)\ m = x \cdot m, \quad \forall x \in \mathcal{G}. \tag{3.7}$$

Therefore, from (3.6) and (3.7) we have

$$H^1(\mathcal{G},M) = Z^1(\mathcal{G},M)/B^1(\mathcal{G},M) = Der(\mathcal{G},M)/PDer(\mathcal{G},M).$$

In case $M = \mathcal{G}$, with the adjoint action, we get $Der(\mathcal{G},M) = Der(\mathcal{G})$ and $PDer(\mathcal{G},M) = Inn(\mathcal{G})$, thus

$$H^1(\mathcal{G}) = Der(\mathcal{G})/Inn(\mathcal{G}) = Out(\mathcal{G}),$$

the Lie algebra of outer derivation of $\mathcal{G}$. A case of interest corresponds to taking $M = \mathbb{F}$ with the trivial action. Then β is a coboundary if $\beta = 0$, and the cocycle condition gives that $\beta([x,y]) = 0$, for all $x, y \in \mathcal{G}$. This implies that the one-cocycles are linear maps vanishing on $[\mathcal{G},\mathcal{G}]$. Thus we have

$$H^1(\mathcal{G},\mathbb{F}) = (\mathcal{G}/[\mathcal{G},\mathcal{G}])^*.$$

If $\mathcal{G}$ is semi-simple we have $[\mathcal{G},\mathcal{G}] = \mathcal{G}$, then $H^1(\mathcal{G}) = 0$.

Example 3.8. *Consider the finite dimensional complex semi-simple Lie algebra $sl_n(\mathbb{C})$. We have $H^1(sl_n(\mathbb{C})) = \{0\}$, that is, all derivations on $sl_n(\mathbb{C})$ are inner, and $H^1(sl_n(\mathbb{C}), \mathbb{C}) = \{0\}$. There is a proposition (see below) which implies that $H^1(sl_n(\mathbb{C}), M) = \{0\}$ for any finite dimensional module M.*

Proposition 3.9. *(Whitehead Lemma)*

Let $\mathcal{G}$ be a finite dimensional complex semi-simple Lie algebra and M be a finite dimensional $\mathcal{G}-$module. Then

$$H^n(\mathcal{G}, M) = 0 \quad , \qquad n = 1, 2.$$

Proof. The proof may be found in Section 3.12 of [20]. □

As mentioned in the introduction, the notion of differential calculus based on derivation was introduced by M. Dubois-Violette in [7]. He provides a general and purely algebraic definition of differential calculus based on derivation for any associative algebra. In [8], a more general systematic study is proposed which uses the categorical point of view on algebras. Following this, we can also directly use this construction on Lie algebras as follows:

Let $\mathcal{G}$ be any Lie algebra over the field $\mathbb{F}$. Suppose that $Der(\mathcal{G})$ is a Lie algebra of all derivations of $\mathcal{G}$ into itself. Consider $C(Der(\mathcal{G}), \mathcal{G}) = \bigoplus_{n=0}^{\infty} C^n(Der(\mathcal{G}), \mathcal{G})$ the graded differential algebra of all $\mathcal{G}-$valued cochains of $Der(\mathcal{G})$, with a differential d. Moreover, $Der(\mathcal{G})$ is also a module over the center $Z(\mathcal{G})$ of $\mathcal{G}$. By the derivation we then have the property $[\partial, z\delta] = z[\partial, \delta] + \partial(z)\delta$, for all $\partial, \delta \in Der(\mathcal{G})$ and all $z \in Z(\mathcal{G})$. Using this property we can extract by $Z(\mathcal{G})-$multilinearity a graded differential subalgebra $\Omega_{Der}(\mathcal{G})$ of $C(Der(\mathcal{G}), \mathcal{G})$ which consists of $Z(\mathcal{G})-$multilinear Chevalley-Eilenberg cochains of Lie algebra $Der(\mathcal{G})$. Notice that $\Omega_{Der}(\mathcal{G})$ is invariant by the differential d and is therefore a graded differential subalgebra of $C(Der(\mathcal{G}), \mathcal{G})$.

Definition 3.10. *(The graded differential algebra $\Omega_{Der}(\mathcal{G})$)*

Let $\Omega^n_{Der}(\mathcal{G})$ be a set of $Z(\mathcal{G})-$multilinear anti-symmetric maps from $Der(\mathcal{G})^n$ to $\mathcal{G}$, with $\Omega^0_{Der}(\mathcal{G}) = \mathcal{G}$ and let $\Omega_{Der}(\mathcal{G}) = \bigoplus_{n=0}^{\infty} \Omega^n_{Der}(\mathcal{G})$. Then the space $\Omega_{Der}(\mathcal{G})$ equipped with differential d (see Definition 3.2) is a graded differential algebra of $\mathcal{G}$.

Since $\Omega^0_{Der}(\mathcal{G}) = \mathcal{G}$, it follows that there is a smaller graded differential subalgebra of $C(Der(\mathcal{G}), \mathcal{G})$ generated by $\mathcal{G}$. So the graded differential algebra $\Omega_{Der}(\mathcal{G})$ contains a graded differential subalgebra as follows:

Definition 3.11. *(The graded differential algebra $s\Omega_{Der}(\mathcal{G})$)*

We define the $s\Omega_{Der}(\mathcal{G})$, as the smallest graded differential subalgebra of $\Omega_{Der}(\mathcal{G})$ generated in degree zero by $\mathcal{G}$.

Now we are ready to define differential calculus based on derivation for Lie algebra $\mathcal{G}$.

Definition 3.12. *(Differential calculus on Lie algebra $\mathcal{G}$)*

Let $\mathcal{G}$ be a Lie algebra over a fixed field $\mathbb{F}$ with center $Z(\mathcal{G})$. Consider the graded differential algebra $\Omega_{Der}(\mathcal{G}) = \bigoplus_{n=0}^{\infty} \Omega^n_{Der}(\mathcal{G})$ or $s\Omega_{Der}(\mathcal{G})$. Then we obtain the Chevally-Eielinberg subcomplex

$$0 \longrightarrow \mathcal{G} \xrightarrow{d_0} \Omega^1_{Der}(\mathcal{G}) \xrightarrow{d_1} \Omega^2_{Der}(\mathcal{G}) \longrightarrow ... \xrightarrow{d_n} \Omega^n_{Der}(\mathcal{G}) \longrightarrow ...,$$

which is called differential calculus based on derivation for Lie algebra $\mathcal{G}$. It is finite if Lie algebra $\mathcal{G}$ is finite dimensional.

The notion of the Cartan operation of Lie algebra $Der(\mathcal{G})$ on graded differential algebra $\Omega_{Der}(\mathcal{G})$ can be considered in this case, which we will describe below:

Let $(\Omega_{Der}(\mathcal{G}), d)$ be a graded differential algebra on Lie algebra $\mathcal{G}$. We define an inner product which is a graded derivation of degree -1 on $\Omega^n_{Der}(\mathcal{G})$ by

$$i_\chi : \Omega^n_{Der}(\mathcal{G}) \longrightarrow \Omega^{n+1}_{Der}(\mathcal{G}) \quad , \quad i_\chi(\beta_n)(\chi_1, ..., \chi_{n-1}) = \beta_n(\chi, \chi_1, ..., \chi_{n-1}),$$

for all $\chi, \chi_i \in Der(\mathcal{G})$ and $\beta_n \in \Omega^n_{Der}(\mathcal{G})$. By this definition $i_\chi = 0$ on $\Omega^0_{Der}(\mathcal{G}) = \mathcal{G}$. Therefore the associated Lie derivative L_χ is the graded derivation of degree zero on $\Omega_{Der}(\mathcal{G})$ which is defined by

$$L_\chi = i_\chi d + d i_\chi. \tag{3.13}$$

It is easy to cheek that $\chi \longmapsto i_\chi$ is an operation of $Der(\mathcal{G})$ on $\Omega_{Der}(\mathcal{G})$, which satisfy

$$[L_{\chi_1}, i_{\chi_2}] = i_{[\chi_1,\ \chi_2]}, \tag{3.14}$$

for all $\chi_1, \chi_2 \in Der(\mathcal{G})$. It follows from (3.13) and (3.14) that

$$[L_\chi, d] = 0 \quad , \quad [L_{\chi_1}, L_{\chi_2}] = L_{[\chi_1,\ \chi_2]}, \tag{3.15}$$

for all $\chi_1, \chi_2 \in Der(\mathcal{G})$. Relation (3.15) implies that there is a Lie algebra homomorphism of $Der(\mathcal{G})$ into the Lie algebra $\Omega^0_{Der}(\mathcal{G}) = \mathcal{G}$ of all derivations of degree zero of $\Omega_{Der}(\mathcal{G})$.

4 Defferentiall calculus on Matrix Lie algebra

In this section we use the notion of differential calculus based on derivation as stated in the previous section and we provide two examples relative to it. The first example investigates the differential calculus and graded differential algebra of the reductive Lie algebra $gl_n(\mathbb{C})$, which is the Lie algebra of all complex $n\times n$ matrices. The second example describes differential calculus based on derivation and a presentation of the graded differential algebra of a finite dimensional semi-simple Lie algebra $sl_n(\mathbb{C})$.

Example 4.1. *The graded differential algebra of $gl_n(\mathbb{C})$.*

The study of this example was initiated in [9]. A complete description can be found in [17]. The main result can be summarized as bellow:

Proposition 4.2. *Suppose that $gl_n(\mathbb{C})$ is the Lie algebra of complex $n \times n$ matrix with $n \geq 2$. One has the following:*

- $Z(gl_n(\mathbb{C})) = \mathbb{C}$.
- $Der(gl_n(\mathbb{C})) = vInn(gl_n(\mathbb{C}) \cong sl_n(\mathbb{C})$. *The explicit isomorphism associates to any $X \in sl_n(\mathbb{C})$ the derivation $ad_X E = [X, E]$ for any $E \in gl_n(\mathbb{C})$ and $Out(gl_n(\mathbb{C})) =$ 0.*
- *The differential of Chevaly-Elinberg complex of $sl_n(\mathbb{C})$ is represented on $gl_n(\mathbb{C})$ by the adjoint representation. Then*

$$s\Omega_{Der}(gl_n(\mathbb{C})) = \Omega_{Der}(gl_n(\mathbb{C})) \cong gl_n(\mathbb{C}) \otimes \bigwedge sl_n(\mathbb{C})^*.$$

- *There exists a one-cocycle $i\theta \in \Omega^1_{Der}(gl_n(\mathbb{C}))$ such that*

$$i\theta(ad_E) = E - \frac{1}{n} Tr(E) I,$$

for all $E \in gl_n(\mathbb{C})$, where $Tr(E)$ is the trace of matrix E. This cocycle makes the isomorphism

$$Inn(gl_n(\mathbb{C})) \cong sl_n(\mathbb{C}).$$

- *For all $E \in gl_n(\mathbb{C})$ we have*

$$dE = [i\theta, E] \in \Omega^1_{Der}(gl_n(\mathbb{C})).$$

This is not true on $\Omega^n_{Der}(gl_n(\mathbb{C}))$ for large n. □

Example 4.3. *The graded differential algebra of $sl_n(\mathbb{C})$.*

We know that $sl_n(\mathbb{C}) = \{X \in gl_n(\mathbb{C}) \ : \ Tr(X) = 0\}$ is a subalgebra of $gl_n(\mathbb{C})$. It is called special linear Lie algebra. The Lie algebra $sl_n(\mathbb{C})$ spanned by all E_{ij} for

$i \neq j$ together with the diagonal matrices $h_i = E_{ii} - E_{1+1\ i+1}$ for $1 \leq i \leq n-1$. Recall that E_{ij} is the complex $n \times n-$matrices with 1 in the $(i,\ j)$ position and 0 elsewhere. Hence the dimension of $sl_n(\mathbb{C})$ is $n^2 - 1$. Here we compute the structure constant of $sl_n(\mathbb{C})$. First of all, we note that

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}.$$

Secondly, since the h_i's are all diagonal matrices, we have $[h_i, h_j] = 0$. Finally,

$$[h_i, E_{kl}] = C_{ikl} E_{kl},$$

where $C_{ikl} = 0, 1, 2, -1, -2$ depending on i, j and k. By convention, we will denote by $\{e_k\}_{k=1}^{n^2-1}$, the basis of $sl_n(\mathbb{C})$ and

$$[e_i, e_j] = \sum_{k=1}^{n} C_{ij}^k e_k,$$

where $C_{ij}^k \in \mathbb{C}$ is the structure constant.

First, we describe the differential calculus of $sl_n(\mathbb{C})$, $n \geq 2$ in a more general case.

We know that $sl_n(\mathbb{C})$ is a centerless Lie algebra with dimension $n^2 - 1$. Any derivation of $sl_n(\mathbb{C})$ is an inner derivation, thus the Lie algebra $Der(sl_n(\mathbb{C}))$ identifies canonically with $sl_n(\mathbb{C})$. That is, $Der(sl_n(\mathbb{C})) = sl_n(\mathbb{C})$ acts on $sl_n(\mathbb{C})$ via inner derivation. In this case, $sl_n(\mathbb{C})$ is an invariant subalgebra of $Der(sl_n(\mathbb{C}))$ and is also a $Der(sl_n(\mathbb{C}))-$module via the adjoint representation. Let $C(Der(sl_n(\mathbb{C})), sl_n(\mathbb{C})$ be the graded differential algebra of all $sl_n(\mathbb{C})-$valued $n-$cochains on $sl_n(\mathbb{C})$. An element $\beta_n \in C^n(Der(sl_n(\mathbb{C})), sl_n(\mathbb{C})$ is an $n-$linear anti-symmetric map of $Der(sl_n(\mathbb{C}))^n$ to $sl_n(\mathbb{C})$ defined by

$$(\partial_1, ..., \partial_n) \longmapsto \beta_n(\partial_1, ..., \partial_n) \in sl_n(\mathbb{C}).$$

Note that $C^n(Der(sl_n(\mathbb{C})), sl_n(\mathbb{C}))$ is a $Der(sl_n(\mathbb{C}))-$module via the adjoint action. Since $sl_n(\mathbb{C})$ is centerless and all derivations of it are inner, the graded differential algebra $\Omega_{Der}(sl_n(\mathbb{C}))$, which is equal to $s\Omega_{Der}(sl_n(\mathbb{C}))$, coincides with $C(Der(sl_n(\mathbb{C})), sl_n(\mathbb{C}))$ itself. Then we obtain the subcomplex

$$0 \longrightarrow sl_n(\mathbb{C}) \xrightarrow{d_0} \Omega^1_{Der}(sl_n(\mathbb{C})) \xrightarrow{d_1} ... \xrightarrow{d_n} \Omega^{n^2-1}_{Der}(sl_n(\mathbb{C})),$$

as the differential calculus based on derivation for $sl_n(\mathbb{C})$, where the coboundary operator is defined by

$$\begin{aligned} d_k(\beta_k)(\partial_0, ..., \partial_k) &= \sum_{i=0}^{k} (-1)^i [\partial_i, \beta_k(\partial_0, ..., \hat{\partial}_i, ..., \partial_k)] \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \beta_k([\partial_i, \partial_j], \partial_0, ..., \hat{\partial}_i, ..., \hat{\partial}_j, ..., \partial_k). \end{aligned} \tag{4.4}$$

Let us describe the one-coboundary and one-cocycle on $sl_n(\mathbb{C})$. *By (4.4) a one-cochain* β_1 *is one-coboundary if there exists a zero-cochain* $E \in sl_n(\mathbb{C})$ *such that* $d_0E = \beta_1$. *That is,*

$$(4.5) \qquad (d_0E)(\partial) = [\partial, E] = -ad_E\partial, \quad \text{for all} \quad \partial \in Der(sl_n(\mathbb{C})).$$

If one-cochain β_1 *is a one-cocycle, then*

$$(4.6) \qquad (d_1\beta_1)(\partial, \delta)) = [\partial, \beta_1(\delta)] - [\delta, \beta_1(\partial)] - \beta_1([\partial, \delta]) = 0.$$

It follows from (4.5) that any one-coboundary is an inner derivation on $sl_n(\mathbb{C})$. *Also, (4.6) implies that any one-cocycle on* $sl_n(\mathbb{C})$ *satisfies*

$$\beta_1([\partial, \delta]) = [\beta_1(\partial), \delta] + [\partial, \beta_1(\delta)].$$

Next, as in the general case, there is a Cartan operation of the Lie algebra $Der(sl_n(\mathbb{C}))$ *in the graded differential algebra* $\Omega_{Der}(sl_n(\mathbb{C}))$, *which will be defined as follows:*

For any $\partial \in Der(sl_n(\mathbb{C}))$ *we can define an anti-derivation* i_∂ *of degree* -1 *on* $\Omega_{Der}(sl_n(\mathbb{C}))$ *by*

$$i_\partial\beta_k(\delta_1, ..., \delta_{k-1}) = \beta_k(\partial, \delta_1, ..., \delta_{k-1}), \quad k \geq 1$$

for $\beta_k \in \Omega^k_{Der}(sl_n(\mathbb{C}))$ *and* $\delta_1, \delta_2, ..., \delta_{k-1} \in Der(sl_n(\mathbb{C})$. *In this case the Lie derivative is defined by*

$$L_\partial = d_k \circ i_\partial + i_\partial \circ d_k,$$

which is a derivation of degree zero on $\Omega_{Der}(sl_n(\mathbb{C}))$. *One may verify that*

$$i_{\partial_1}i_{\partial_2} + i_{\partial_2}i_{\partial_1} = 0, \qquad [L_{\partial_1}, L_{\partial_2}] = i_{[\partial_1, \partial_2]},$$

and also $[L_{\partial_1}, L_{\partial_2}] = L_{[\partial_1, \partial_2]}$, *for all* $\partial_1, \partial_2 \in Der(sl_n(\mathbb{C}))$.

Finally, we give a presentation in terms of generators and relations in our investigation of the differential calculus and graded differential algebra of $sl_n(\mathbb{C})$.

Let $\{e_k\}_{k=1}^{n^2-1}$ *be the basis of* $sl_n(\mathbb{C})$ *and* $\partial_k = ade_k, k \in \{1, 2, ..., n^2 - 1\}$ *be the basis element of* $Der(sl_n(\mathbb{C}))$. *Then* $\{\partial_k\}_{k=1}^{n^2-1}$ *forms a basis of* $Der(sl_n(\mathbb{C}))$ *and* $[\partial_k, \partial_l] = C^m_{kl}\partial_m$. *Let* $\{\theta^k\}_{k=1}^{n^2-1}$ *be the dual basis of* $\{\partial_k\}_{k=1}^{n^2-1}$ *in* $sl_n(\mathbb{C})^*$, *that is,* $\theta^k(\partial_l) = \delta_{kl}$. *Since one-cocycles* $\beta \in \Omega^1_{Der}(sl_n(\mathbb{C}))$ *are endomorphisms of* $sl_n(\mathbb{C})$, *so they can be identified in* $\Omega^1_{Der}(sl_n(\mathbb{C}))$ *to* $1 \otimes sl_n(\mathbb{C})$. *We then have* $\Omega_{Der}(sl_n(\mathbb{C})) = sl_n(\mathbb{C}) \otimes sl_n(\mathbb{C})^*$ *together with* $\{e_k \otimes \theta^m\}$ *as a basis for the graded differential algebra of* $sl_n(\mathbb{C})$. *So, from (4.5), we have*

$$(4.7) \qquad d_0(e_l) = C^l_{ml}e_k \otimes \theta^m,$$

and from (4.6)

$$d_1(e_k \otimes \theta^m) = C^l_{nk} e_l \otimes (\theta^n \wedge \theta^m) - \frac{1}{2} C^m_{nr} e_k \otimes (\theta^n \wedge \theta^r). \tag{4.8}$$

Therefore, in the graded differential algebra $\Omega_{Der}(sl_n(\mathbb{C}))$ we have

$$e_k \otimes \theta^m = \theta^m \otimes e_k, \tag{4.9}$$

and

$$\theta^m \wedge \theta^n = -\theta^n \wedge \theta^m. \tag{4.10}$$

Also, the differential d of $\Omega_{Der}(sl_n(\mathbb{C}))$ is given by

$$de_l = C^k_{ml} e_k \otimes \theta^m, \qquad d\theta^k = -\frac{1}{2} C^k_{lm} \theta^l \wedge \theta^m. \tag{4.11}$$

Note that the Jacobian identity implies that $d^2 = 0$. Relations (4.9), (4.10) and (4.11), together with the generators $\{e_k \otimes \theta^m\}$, provide a presentation for $\Omega_{Der}(sl_n(\mathbb{C}))$.

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V. Khalili
Department of Mathematics
Faculty of Science, Arak University
Raja Rammohunpur, Darjeeling
Arak 386156-8-8349, Iran
v-khalili@araku.ac.ir