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$\alpha-$SERIES FOR QUADRUPLED FIXED POINT

Animesh Gupta

Abstract. This manuscript has two aims: first, we extend the definitions of compatibility and weakly reciprocally continuity, for a quadvariate mapping F and a self-mapping g akin to a compatible mapping as introduced by Choudhary and Kundu (Nonlinear Anal. 73:2524-2531, 2010) for a bivariate mapping F and a self-mapping g. Further, using these definitions, we establish quadrupled coincidence and fixed point results by applying the new concept of an α-series for a sequence of mappings, introduced by Sihag et al. (Quaest. Math. 37:1-6, 2014), in the setting of partially ordered metric spaces.

Keywords: α-series; compatible mappings; quadrupled coincidence point; quadrupled fixed point; partially ordered metric space.

1. Introduction

Since the year 1922, Banachs contraction principle, due to its simplicity and applicability, has been a very popular tool in modern analysis, especially in nonlinear analysis including its applications to differential and integral equations, variational inequality theory, complementarity problems, equilibrium problems, minimization problems and many others. Also, many authors have improved, extended and generalized this contraction principle in several ways. Existence of fixed points in ordered metric spaces has been initiated in 2004 by Ran and Reurings [30] further studied by Nieto and Rodriguez - Lopez [29]. Samet and Vetro [37] introduced the notion of fixed point of N order in case of single-valued mappings. It should be noted that through the coupled fixed point (for N = 2) and tripled fixed point (for N = 3)technique we cannot solve a system with the following form:

$$\begin{aligned} x^4 + 6yzw - 9x + 12 &= 0, \\ y^4 + 6xzw - 9y + 12 &= 0, \\ z^4 + 6yxw - 9z + 12 &= 0 \\ w^4 + 6yxz - 9w + 12 &= 0. \end{aligned}$$

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In particular for $N = 4$ (Quadruple case) i.e., Let $(X, \preceq)$ be partially ordered set and (X, d) be a complete metric space. We consider the following partial order on the product space $X^4 = X \times X \times X \times X$

$$(u, v, r, t) \preceq (x, y, z, w) \ \ iff \ \ x \preceq u, \ \ y \preceq \ \ v, \ \ z \preceq r, \ \ t \preceq w, \tag{1.1}$$

where $(u, v, r, t), (x, y, z, w) \in X^4$.

Regarding this partial order Karapinar [26] give the following definitions,

Definition 1.1. Let $(X, \preceq)$ be a partially ordered set and $F : X^4 \to X$. We say that F has the mixed monotone property if $F(x, y, z, w)$ is monotone non-decreasing in x and z and it is monotone non-increasing in y and w, that is, for any $x, y, z, w \in X$

$$\begin{aligned}
x_1, x_2 \in X, \ \ x_1 &\preceq x_2 \Longrightarrow F(x_1, y, z, w) \preceq F(x_2, y, z, w) \\
y_1, y_2 \in X, \ \ y_1 &\preceq y_2 \Longrightarrow F(x, y_2, z, w) \preceq F(x, y_1, z, w) \\
z_1, z_2 \in X, \ \ z_1 &\preceq z_2 \Longrightarrow F(x, y, z_1, w) \preceq F(x, y, z_2, w) \\
w_1, w_2 \in X, \ \ w_1 &\preceq w_2 \Longrightarrow F(x, y, z, w_2) \preceq F(x, y, z, w_1)
\end{aligned} \tag{1.2}$$

Definition 1.2. An element $(x, y, z, w) \in X^4$ is called a quadruple fixed point of $F : X^4 \to X$ if

$$\begin{aligned}
F(x, y, z, w) &= x, \ \ F(y, z, w, x) = y, \\
F(z, w, x, y) &= z, \ \ F(w, x, y, z) = w.
\end{aligned} \tag{1.3}$$

Definition 1.3. Let (X, d) be a complete metric space. It is called metric on X^4, the mapping $d : X \times X \to X$ with

$$d[(x, y, z, t), (u, v, w, s)] = d(x, u) + d(y, v) + d(z, w) + d(t, s).$$

Akin to the concept of g-mixed monotone property for a quadvariate mapping, $F : X^4 \to X$ and a self-mapping, $g : X \to X$, is as follows.

Definition 1.4. Let $(X, \preceq)$ be a partially ordered set and $F : X^4 \to X$ and $g : X \to X$. We say that F has the $g-$mixed monotone property if $F(x, y, z, t)$ is monotone nondecreasing in x and z, and if it is monotone non-increasing in y and t, that is, for any $x, y, z, t \in X$,

$$x_1, x_2 \in X, \ \ g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y, z, t) \preceq F(x_2, y, z, t),$$

$$y_1, y_2 \in X, \ \ g(y_1) \preceq g(y_2) \Rightarrow F(x, y_1, z, t) \succeq F(x, y_2, z, t),$$

$$z_1, z_2 \in X, \ \ g(z_1) \preceq g(z_2) \Rightarrow F(x, y, z_1, t) \preceq F(x, y, z_2, t)$$

and

$$t_1, t_2 \in X, \ \ g(t_1) \preceq g(t_2) \Rightarrow F(x, y, z, t_1) \succeq F(x, y, z, t_2).$$

Now, we introduce the concept of compatible mapping for a quadvariate mapping F and a self-mapping g akin to compatible mapping as introduced by Choudhary and Kundu [8] for a bivariate mapping F and a self-mapping g.

Definition 1.5. Let mapping F and g where $F : X^4 \to X$ and $g : X \to X$ are said to be compatible if

$$\lim_{n\to+\infty} d(g(F(x_n, y_n, z_n, t_n)), F(g(x_n), g(y_n), g(z_n), g(t_n))) = 0,$$

$$\lim_{n\to+\infty} d(g(F(y_n, z_n, t_n, x_n)), F(g(y_n), g(z_n), g(t_n), g(x_n))) = 0,$$

$$\lim_{n\to+\infty} d(g(F(z_n, t_n, x_n, y_n)), F(g(z_n), g(t_n), g(x_n), g(y_n))) = 0,$$

and

$$\lim_{n\to+\infty} d(g(F(t_n, x_n, y_n, z_n)), F(g(t_n), g(x_n), g(y_n), g(z_n))) = 0,$$

whenever $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{t_n\}$ are sequences in X, such that

$$\lim_{n\to+\infty} F(x_n, y_n, z_n, t_n) = \lim_{n\to+\infty} g(x_n) = x,$$

$$\lim_{n\to+\infty} F(y_n, z_n, t_n, x_n) = \lim_{n\to+\infty} g(y_n) = y,$$

$$\lim_{n\to+\infty} F(z_n, t_n, x_n, y_n) = \lim_{n\to+\infty} g(z_n) = z,$$

and

$$\lim_{n\to+\infty} F(t_n, x_n, y_n, z_n) = \lim_{n\to+\infty} g(t_n) = t,$$

for all $x, y, z, t \in X$.

Definition 1.6. The mappings $F : X^4 \to X$ and $g : X \to X$ are called:
(i) Reciprocally continuous if

$$\lim_{n\to+\infty} g(F(x_n, y_n, z_n, t_n)) = g(x) \text{ and } \lim_{n\to+\infty} F(g(x_n), g(y_n), g(z_n), g(t_n)) = F(x, y, z, t),$$

$$\lim_{n\to+\infty} g(F(y_n, z_n, t_n, x_n)) = g(y) \text{ and } \lim_{n\to+\infty} F(g(y_n), g(z_n), g(t_n), g(x_n)) = F(y, z, t, x)$$

$$\lim_{n\to+\infty} g(F(z_n, t_n, x_n, y_n)) = g(z) \text{ and } \lim_{n\to+\infty} F(g(z_n), g(t_n), g(x_n), g(y_n)) = F(z, t, x, y),$$

and

$$\lim_{n\to+\infty} g(F(t_n, x_n, y_n, z_n)) = g(t) \text{ and } \lim_{n\to+\infty} F(g(t_n), g(x_n), g(y_n), g(z_n)) = F(t, x, y, z),$$

whenever $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{t_n\}$ are sequences in X, such that

$$\lim_{n\to+\infty} F(x_n, y_n, z_n, t_n) = \lim_{n\to+\infty} g(x_n) = x,$$

$$\lim_{n\to+\infty} F(y_n, z_n, t_n, x_n) = \lim_{n\to+\infty} g(y_n) = y,$$

$$\lim_{n\to+\infty} F(z_n, t_n, x_n, y_n) = \lim_{n\to+\infty} g(z_n) = z,$$

and

$$\lim_{n\to+\infty} F(t_n, x_n, y_n, z_n) = \lim_{n\to+\infty} g(t_n) = t,$$

for all $x, y, z, t \in X$.

(ii)Weakly reciprocally continuous if

$$\lim_{n\to+\infty} g(F(x_n, y_n, z_n, t_n)) = g(x) \; or \; \lim_{n\to+\infty} F(g(x_n), g(y_n), g(z_n), g(t_n)) = F(x, y, z, t),$$

$$\lim_{n\to+\infty} g(F(y_n, z_n, t_n, x_n)) = g(y) \; or \; \lim_{n\to+\infty} F(g(y_n), g(z_n), g(t_n), g(x_n)) = F(y, z, t, x)$$

$$\lim_{n\to+\infty} g(F(z_n, t_n, x_n, y_n)) = g(z) \; or \; \lim_{n\to+\infty} F(g(z_n), g(t_n), g(x_n), g(y_n)) = F(z, t, x, y),$$

and

$$\lim_{n\to+\infty} g(F(t_n, x_n, y_n, z_n)) = g(t) \; or \; \lim_{n\to+\infty} F(g(t_n), g(x_n), g(y_n), g(z_n)) = F(t, x, y, z),$$

whenever $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ are sequences in X, such that

$$\lim_{n\to+\infty} F(x_n, y_n, z_n, t_n) = \lim_{n\to+\infty} g(x_n) = x,$$

$$\lim_{n\to+\infty} F(y_n, z_n, t_n, x_n) = \lim_{n\to+\infty} g(y_n) = y,$$

$$\lim_{n\to+\infty} F(z_n, t_n, x_n, y_n) = \lim_{n\to+\infty} g(z_n) = z,$$

and

$$\lim_{n\to+\infty} F(t_n, x_n, y_n, z_n) = \lim_{n\to+\infty} g(t_n) = t,$$

for all $x, y, z, t \in X$.

Definition 1.7. Let $(X, d, \preceq)$ be a partially ordered metric space. We say that X is regular if the following conditions hold:
(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \preceq x$ for all $n \geq 0$,
(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y \preceq y_n$ for all $n \geq 0$.

Definition 1.8. Let $\{a_n\}$ be a sequence of non-negative real numbers. We say that a series $\Sigma_{n=1}^{+\infty} a_n$ is an $\alpha-$series, if there exist $0 < \alpha < 1$ and $n_\alpha \in N$ such that $\Sigma_{i=1}^{k} a_i \leq \alpha k$ for each $k \geq n_\alpha$.

Remark 1.1. Each convergent series of non-negative real terms is an $\alpha-$series.However, there are also divergent series that are $\alpha-$series. For example, $\Sigma_{n=1}^{+\infty} \frac{1}{n}$ is an $\alpha-$series.

2. Main Results

Let $(X, \preceq)$ be a partially ordered set, g be a self-mapping on X and $\{T_i\}_{i\in N}$ be a sequence of mappings from X^4 into X such that $T_i(X^4) \subseteq g(X)$ and

$$\begin{aligned} T_i(x,y,z,t) &\preceq T_{i+1}(u,v,w,s),\\ T_{i+1}(v,w,s,u) &\preceq T_i(y,z,t,x)\\ T_i(z,t,x,y) &\preceq T_{i+1}(w,s,u,v),\\ T_{i+1}(s,u,v,w) &\preceq T_i(t,x,y,z) \end{aligned} \tag{2.1}$$

for $x,y,z,t,u,v,w,s \in X$ with $g(x) \preceq g(u)$, $g(v) \preceq g(y)$, $g(z) \preceq g(w)$ and $g(s) \preceq g(t)$.

In the proof of our main theorem, we consider sequences that are constructed in the following way.

Let $x_0, y_0, z_0, t_0 \in X$ be such that

$$g(x_0) \preceq T_0(x_0,y_0,z_0,t_0), \quad g(y_0) \succeq T_0(y_0,z_0,t_0,x_0),$$
$$g(z_0) \preceq T_0(z_0,t_0,x_0,y_0) \ \ and \ \ g(t_0) \succeq T_0(t_0,x_0,y_0,z_0).$$

Since $T_0(X^4) \subseteq g(X)$, we can choose $x_1, y_1, z_1, t_1 \in X$ such that

$$g(x_1) = T_0(x_0,y_0,z_0,t_0),$$

$$g(y_1) = T_0(y_0,z_0,t_0,x_0),$$

$$g(z_1) = T_0(z_0,t_0,x_0,y_0)$$

and

$$g(t_1) = T_0(t_0,x_0,y_0,z_0).$$

Again we can choose $x_2, y_2, z_2, t_2 \in X$ such that

$$g(x_2) = T_1(x_1,y_1,z_1,t_1),$$

$$g(y_2) = T_1(y_1,z_1,t_1,x_1),$$

$$g(z_2) = T_1(z_1,t_1,x_1,y_1)$$

and

$$g(t_2) = T_1(t_1,x_1,y_1,z_1).$$

Continuing like this, we can construct three sequences $\{x_n\}, \{y_n\}$, and $\{x_n\}$ such that

$$\begin{aligned} g(x_{n+1}) &= T_n(x_n,y_n,z_n,t_n),\\ g(y_{n+1}) &= T_n(y_n,z_n,t_n,x_n),\\ g(z_{n+1}) &= T_n(z_n,t_n,x_n,y_n),\\ g(t_{n+1}) &= T_n(t_n,x_n,y_n,z_n) \end{aligned} \tag{2.2}$$

for all $n \geq 0$.

Now, by using mathematical induction, we prove that

$$\begin{aligned} &g(x_n) \preceq g(x_{n+1}), \quad g(y_n) \succeq g(y_{n+1}), \\ &g(z_n) \preceq g(z_{n+1}), \quad g(t_n) \succeq g(t_{n+1}) \end{aligned} \tag{2.3}$$

for all $n \geq 0$.

Since

$$g(x_0) \preceq T_0(x_0, y_0, z_0, t_0),$$

$$g(y_0) \succeq T_0(y_0, z_0, t_0, x_0),$$

$$g(z_0) \preceq T_0(z_0, t_0, x_0, y_0)$$

and

$$g(t_0) \succeq T_0(t_0, x_0, y_0, z_0).$$

In view of

$$g(x_1) = T_0(x_0, y_0, z_0, t_0),$$

$$g(y_1) = T_0(y_0, z_0, t_0, x_0),$$

$$g(z_1) = T_0(z_0, t_0, x_0, y_0)$$

and

$$g(t_1) = T_0(t_0, x_0, y_0, z_0),$$

we have $g(x_n) \preceq g(x_{n+1})$, $g(y_n) \succeq g(y_{n+1})$, $g(z_n) \preceq g(z_{n+1})$ and $g(t_n) \succeq g(t_{n+1})$, that is, 2.3 holds for $n = 0$. We presume that 2.3 holds for some $n > 0$. Now, by 2.2 and 2.3, one deduces that

$$\begin{aligned} g(x_{n+1}) &= T_n(x_n, y_n, z_n, t_n) \preceq T_{n+1}(x_{n+1}, y_{n+1}, z_{n+1}, t_{n+1}) = g(x_{n+2}) \\ g(y_{n+2}) &= T_{n+1}(y_{n+1}, z_{n+1}, t_{n+1}, x_{n+1}) \preceq T_n(y_n, z_n, t_n, x_n) = g(y_{n+1}) \\ g(z_{n+1}) &= T_n(z_n, t_n, x_n, y_n) \preceq T_{n+1}(z_{n+1}, t_{n+1}, x_{n+1}, y_{n+1}) = g(z_{n+2}) \\ g(t_{n+2}) &= T_{n+1}(t_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}) \preceq T_n(t_n, x_n, y_n, z_n) = g(t_{n+1}). \end{aligned}$$

Thus by mathematical induction, we conclude that 2.3 holds for all $n0$. Therefore, we have

$$\begin{aligned} &g(x_0) \preceq g(x_1) \preceq g(x_2) \preceq \cdots \preceq g(x_{n+1}) \preceq \ldots, \\ &g(y_0) \succeq g(y_1) \succeq g(y_2) \succeq \cdots \succeq g(y_{n+1}) \succeq \ldots, \\ &g(z_0) \preceq g(z_1) \preceq g(z_2) \preceq \cdots \preceq g(z_{n+1}) \preceq \ldots, \\ &g(t_0) \succeq g(t_1) \succeq g(t_2) \succeq \cdots \succeq g(t_{n+1}) \succeq \ldots. \end{aligned}$$

In view of the above considerations, we revise Definitions 1.4 and 1.5 as follows.

Definition 2.1. Let (X,d) be a metric space. $\{T_i\}_{i\in N}$ and g are compatible if

$$\lim_{n\to+\infty} d(g(T_n(x_n,y_n,z_n,t_n)),T_n(g(x_n),g(y_n),g(z_n),g(t_n)))=0,$$
$$\lim_{n\to+\infty} d(g(T_n(y_n,z_n,t_n,x_n)),T_n(g(y_n),g(z_n),g(t_n),g(x_n)))=0,$$
$$\lim_{n\to+\infty} d(g(T_n(z_n,t_n,x_n,y_n)),T_n(g(z_n),g(t_n),g(x_n),g(y_n)))=0,$$
$$\lim_{n\to+\infty} d(g(T_n(t_n,x_n,y_n,z_n)),T_n(g(t_n),g(x_n),g(y_n),g(z_n)))=0,$$

whenever $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ are sequences in X, such that

$$\lim_{n\to+\infty} T_n(x_n,y_n,z_n,t_n)=\lim_{n\to+\infty} g(x_{n+1})=x,$$
$$\lim_{n\to+\infty} T_n(y_n,z_n,t_n,x_n)=\lim_{n\to+\infty} g(y_{n+1})=y,$$
$$\lim_{n\to+\infty} T_n(z_n,t_n,x_n,y_n)=\lim_{n\to+\infty} g(z_{n+1})=z,$$
$$\lim_{n\to+\infty} T_n(t_n,x_n,y_n,z_n)=\lim_{n\to+\infty} g(t_{n+1})=t,$$

for all $x,y,z,t\in X$.

Definition 2.2. The mappings $\{T_i\}_{i\in N}$ and $g:X\to X$ are called:
(i) Reciprocally continuous if

$$\lim_{n\to+\infty} g(T_n(x_n,y_n,z_n,t_n))=g(x) \text{ and } \lim_{n\to+\infty} T_n(g(x_n),g(y_n),g(z_n),g(t_n))=T(x,y,z,t),$$
$$\lim_{n\to+\infty} g(T_n(y_n,z_n,t_n,x_n))=g(y) \text{ and } \lim_{n\to+\infty} T_n(g(y_n),g(z_n),g(t_n),g(x_n))=T(y,z,t,x),$$
$$\lim_{n\to+\infty} g(T_n(z_n,t_n,x_n,y_n))=g(z) \text{ and } \lim_{n\to+\infty} T_n(g(z_n),g(t_n),g(x_n),g(y_n))=T(z,t,x,y),$$
$$\lim_{n\to+\infty} g(T_n(t_n,x_n,y_n,z_n))=g(t) \text{ and } \lim_{n\to+\infty} T_n(g(t_n),g(x_n),g(y_n),g(z_n))=T(t,x,y,z),$$

whenever $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{t_n\}$ are sequences in X, such that

$$\lim_{n\to+\infty} T_n(x_n,y_n,z_n,t_n)=\lim_{n\to+\infty} g(x_{n+1})=x,$$
$$\lim_{n\to+\infty} T_n(y_n,z_n,t_n,x_n)=\lim_{n\to+\infty} g(y_{n+1})=y,$$
$$\lim_{n\to+\infty} T_n(z_n,t_n,x_n,y_n)=\lim_{n\to+\infty} g(z_{n+1})=z,$$
$$\lim_{n\to+\infty} T_n(t_n,x_n,y_n,z_n)=\lim_{n\to+\infty} g(t_{n+1})=t,$$

for some $x,y,z,t\in X$.

(ii)Weakly reciprocally continuous if

$$\lim_{n\to+\infty} g(T_n(x_n,y_n,z_n,t_n))=g(x),$$
$$\lim_{n\to+\infty} g(T_n(y_n,z_n,t_n,x_n))=g(y),$$
$$\lim_{n\to+\infty} g(T_n(z_n,t_n,x_n,y_n))=g(z),$$
$$\lim_{n\to+\infty} g(T_n(t_n,x_n,y_n,z_n))=g(t),$$

whenever $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{t_n\}$ are sequences in X, such that

$$\lim_{n\to+\infty} T_n(x_n, y_n, z_n, t_n) = \lim_{n\to+\infty} g(x_{n+1}) = x,$$
$$\lim_{n\to+\infty} T_n(y_n, z_n, t_n, x_n) = \lim_{n\to+\infty} g(y_{n+1}) = y,$$
$$\lim_{n\to+\infty} T_n(z_n, t_n, x_n, y_n) = \lim_{n\to+\infty} g(z_{n+1}) = z,$$
$$\lim_{n\to+\infty} T_n(t_n, x_n, y_n, z_n) = \lim_{n\to+\infty} g(t_{n+1}) = t,$$

for some $x, y, z, t \in X$.

Now, we establish the main result of this manuscript as follows.

Theorem 2.1. *Let $(X, d, \leq)$ be a partially ordered metric space. Let g be a self-mapping on X and $\{T_i\}_{i\in N}$ be a sequence of mappings from X^4 into X such that $T_i(X^4) \subseteq g(X)$, $g(X)$ is a complete subset of X, $\{T_i\}_{i\in N}$ and g are compatible, weakly reciprocally continuous, g is monotonic non-decreasing, continuous, satisfying condition 2.1 and the following condition:*

$$\begin{aligned} d(T_i(x,y,z,t), T_j(u,v,w,s)) \quad &\leq \quad \beta_{i,j}[d(g(x), T_i(x,y,z,t)) + d(g(u), T_j(u,v,w,s))] \\ &\quad +\gamma_{i,j} d(g(u), g(x)) \end{aligned} \tag{2.4}$$

for $x, y, z, t, u, v, w, s \in X$ with $g(x) \leq g(u)$, $g(v) \leq g(y)$, $g(z) \leq g(w)$ $g(s) \leq g(t)$ or $g(x) \geq g(u), g(v) \geq g(y)$, $g(z) \geq g(w)$, $g(s) \geq g(t)$. Also $0 \leq \beta_{i,j}, \gamma_{i,j} < 0$ for $i, j \in N$; $\lim_{n\to+\infty} \sup \beta_{i,n} < 1$. Suppose also that there exists $(x_0, y_0, z_0, t_0) \in X^4$ such that

$$g(x_0) \leq T_0(x_0, y_0, z_0, t_0),$$
$$g(y_0) \geq T_0(y_0, z_0, t_0, x_0),$$
$$g(z_0) \leq T_0(z_0, t_0, x_0, y_0)$$

and

$$g(t_0) \geq T_0(t_0, x_0, y_0, z_0).$$

If $\Sigma_{i=1}^{+\infty} \frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}$ is an $\alpha-$series and $g(X)$ is regular, then $\{T_i\}_{i\in N}$ and g have a quadrupled coincidence point, that is, there exists $(x, y, z, t) \in X^4$ such that $g(x) = T_i(x, y, z, t)$, $g(y) = T_i(y, z, t, x)$, $g(z) = T_i(z, t, x, y)$ and $g(t) = T_i(t, x, y, z)$ for $i \in N$.

We consider the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ constructed above and denote

$$\delta_n = d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) + d(g(z_n), g(z_{n+1})) + d(g(t_n), g(t_{n+1})).$$

Then, by 2.4 we get

$$\begin{aligned} d(g(x_1), g(x_2)) &= d(T_0(x_0, y_0, z_0, t_0), T_1(x_1, y_1, z_1, t_1)) \\ &\leq \beta_{0,1}[d(g(x_0), T_0(x_0, y_0, z_0, t_0)) + d(g(x_1), T_1(x_1, y_1, z_1, t_1))] \\ &\quad +\gamma_{0,1} d(g(x_0), g(x_1)) \\ &= \beta_{0,1}[d(g(x_0), d(g(x_1)) + d(g(x_1), d(g(x_2))] \\ &\quad +\gamma_{0,1} d(g(x_0), g(x_1)). \end{aligned}$$

It follows that

$$(1-\beta_{0,1})d(g(x_1),g(x_2)) \leq (\beta_{0,1}+\gamma_{0,1})d(g(x_0),g(x_1))$$

or, equivalently,

$$d(g(x_1),g(x_2)) \leq \left(\frac{\beta_{0,1}+\gamma_{0,1}}{1-\beta_{0,1}}\right)d(g(x_0),g(x_1)).$$

Also, one obtains

$$d(g(x_2),g(x_3)) \leq \left(\frac{\beta_{1,2}+\gamma_{1,2}}{1-\beta_{1,2}}\right)d(g(x_1),g(x_2))$$

$$d(g(x_2),g(x_3)) \leq \left(\frac{\beta_{1,2}+\gamma_{1,2}}{1-\beta_{1,2}}\right)\left(\frac{\beta_{0,1}+\gamma_{0,1}}{1-\beta_{0,1}}\right)d(g(x_0),g(x_1)).$$

Repeating the above procedure, we have

$$d(g(x_n),g(x_{n+1})) \leq \prod_{i=0}^{n-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right)d(g(x_0),g(x_1)). \tag{2.5}$$

Using similar arguments as above, one can also show that

$$d(g(y_n),g(y_{n+1})) \leq \prod_{i=0}^{n-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right)d(g(y_0),g(y_1)) \tag{2.6}$$

$$d(g(z_n),g(z_{n+1})) \leq \prod_{i=0}^{n-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right)d(g(z_0),g(z_1)) \tag{2.7}$$

and

$$d(g(t_n),g(t_{n+1})) \leq \prod_{i=0}^{n-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right)d(g(t_0),g(t_1)). \tag{2.8}$$

Adding 2.5, 2.6,2.7 and 2.8, we have

$$\begin{aligned}\delta_n &= d(g(x_n),g(x_{n+1}))+d(g(y_n),g(y_{n+1}))+d(g(z_n),g(z_{n+1}))+d(g(t_n),g(t_{n+1}))\\ &\leq \prod_{i=0}^{n-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right)[d(g(x_0),g(x_1)+d(g(y_0),g(y_1))+d(g(z_0),g(z_1))+d(g(t_0),g(t_1))]\\ &= \prod_{i=0}^{n-1}\left(\frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}\right)\delta_0.\end{aligned}$$

Moreover, for $p > 0$ and by repeated use of the triangle inequality, one obtains

$$\begin{aligned}
& d(g(x_n), g(x_{n+p})) + d(g(y_n), g(y_{n+p})) + d(g(z_n), g(z_{n+p})) + d(g(t_n), g(t_{n+p})) \\
\leq\ & d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) + d(g(z_n), g(z_{n+1})) + d(g(t_n), g(t_{n+1})) \\
& + d(g(x_{n+1}), g(x_{n+2})) + d(g(y_{n+1}), g(y_{n+2})) + d(g(z_{n+1}), g(z_{n+2})) \\
& + d(g(t_{n+1}), g(t_{n+2})) \\
& \cdots + d(g(x_{n+p-1}), g(x_{n+p})) + d(g(y_{n+p-1}), g(y_{n+p})) + d(g(z_{n+p-1}), g(z_{n+p})) \\
& + d(g(t_{n+p-1}), g(t_{n+p})) \\
\leq\ & \prod_{i=0}^{n-1} \left(\frac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}} \right) \delta_0 + \prod_{i=0}^{n} \left(\frac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}} \right) \delta_0 \\
& + \cdots + \prod_{i=0}^{n+p-2} \left(\frac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}} \right) \delta_0 \\
=\ & \Sigma_{k=0}^{p-1} \prod_{i=0}^{n+k-1} \left(\frac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}} \right) \delta_0 \\
=\ & \Sigma_{k=n}^{n+p-1} \prod_{i=0}^{k-1} \left(\frac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}} \right) \delta_0.
\end{aligned}$$

Let α and n_α be as in Definition 1.8, then, for $n \geq n_\alpha$, and using the fact that the geometric mean of non-negative numbers is less than or equal to the arithmetic mean, it follows that

$$\begin{aligned}
& d(g(x_n), g(x_{n+p})) + d(g(y_n), g(y_{n+p})) + d(g(z_n), g(z_{n+p})) + d(g(t_n), g(t_{n+p})) \\
\leq\ & \Sigma_{k=n}^{n+p-1} \left[\frac{1}{k} \prod_{i=0}^{k-1} \left(\frac{\beta_{i,i+1} + \gamma_{i,i+1}}{1 - \beta_{i,i+1}} \right) \right]^k \delta_0 \\
\leq\ & \left(\Sigma_{k=n}^{n+p-1} \alpha^k \right) \delta_0 \\
\leq\ & \frac{\alpha^n}{1 - \alpha} \delta_0
\end{aligned}$$

Now, taking the limit as $n \to +\infty$, one deduces that

$$\lim_{n \to +\infty} [d(g(x_n), g(x_{n+p})) + d(g(y_n), g(y_{n+p})) + d(g(z_n), g(z_{n+p})) + d(g(t_n), g(t_{n+p}))] = 0,$$

which further implies that

$$\begin{aligned}
\lim_{n \to +\infty} d(g(x_n), g(x_{n+p})) &= 0, \\
\lim_{n \to +\infty} d(g(y_n), g(y_{n+p})) &= 0, \\
\lim_{n \to +\infty} d(g(z_n), g(z_{n+p})) &= 0, \\
\lim_{n \to +\infty} d(g(t_n), g(t_{n+p})) &= 0.
\end{aligned}$$

Thus $\{g(x_n)\}, \{g(y_n)\}, \{g(z_n)\}$ and $\{g(t_n)\}$ are Cauchy sequences in X. Since $g(X)$ is complete, then there exists $(x, y, z, t) \in X^4$, with $g(x) = x, g(y) = y, g(z) = z$ and $g(t) = t$, such that

$$\begin{aligned}
\lim_{n\to+\infty} T_n(x_n, y_n, z_n, t_n) &= \lim_{n\to+\infty} g(x_{n+1}) = x,\\
\lim_{n\to+\infty} T_n(y_n, z_n, t_n, x_n) &= \lim_{n\to+\infty} g(y_{n+1}) = y,\\
\lim_{n\to+\infty} T_n(z_n, t_n, x_n, y_n) &= \lim_{n\to+\infty} g(z_{n+1}) = z,\\
\lim_{n\to+\infty} T_n(t_n, x_n, y_n, z_n) &= \lim_{n\to+\infty} g(t_{n+1}) = t.
\end{aligned}$$

Now, as $\{T_i\}_{i\in N}$ and g are weakly reciprocally continuous, we have

$$\lim_{n\to+\infty} g(T_n(x_n, y_n, z_n, t_n)) = g(x),$$

$$\lim_{n\to+\infty} g(T_n(y_n, z_n, t_n, x_n)) = g(y)$$

$$\lim_{n\to+\infty} g(T_n(z_n, t_n, x_n, y_n) = g(z)$$

and

$$\lim_{n\to+\infty} g(T_n(t_n, x_n, y_n, z_n)) = g(z).$$

On the other hand, the compatibility of $\{T_i\}_{i\in N}$ and g yields

$$\begin{aligned}
&\lim_{n\to+\infty} d(g(T_n(x_n, y_n, z_n, t_n)), T_n(g(x_n), g(y_n), g(z_n), g(t_n))) = 0,\\
&\lim_{n\to+\infty} d(g(T_n(y_n, z_n, t_n, x_n)), T_n(g(y_n), g(z_n), g(t_n), g(x_n))) = 0,\\
&\lim_{n\to+\infty} d(g(T_n(z_n, t_n, x_n, y_n)), T_n(g(z_n), g(t_n), g(x_n), g(y_n))) = 0,\\
&\lim_{n\to+\infty} d(g(T_n(t_n, x_n, y_n, z_n)), T_n(g(t_n), g(x_n), g(y_n), g(z_n))) = 0.
\end{aligned}$$

Then we have

$$\lim_{n\to+\infty} T_n(g(x_n), g(y_n), g(z_n), g(t_n)) = g(x), \tag{2.9}$$

$$\lim_{n\to+\infty} T_n(g(y_n), g(z_n), g(t_n), g(x_n)) = g(y), \tag{2.10}$$

$$\lim_{n\to+\infty} T_n(g(z_n), g(t_n), g(x_n), g(y_n)) = g(z), \tag{2.11}$$

and

$$\lim_{n\to+\infty} T_n(g(t_n), g(x_n), g(y_n), g(z_n)) = g(t), \tag{2.12}$$

Since $\{g(x_n)\}$ and $\{g(z_n)\}$ are non-decreasing also $\{g(y_n)\}$ and $\{g(t_n)\}$ are non-increasing, using the regularity of X, we have $g(x_n) \preceq x, \quad y \preceq g(y_n)$, $g(z_n) \preceq z$ and $t \preceq g(t_n)$ for all $n \geq 0$. Then by 2.4, one obtains

$$\begin{aligned} & d(T_i(x,y,z,t), T_n(g(x_n), g(y_n), g(z_n), g(t_n))) \\ \leq \quad & \beta_{i,n}[d(g(x), T_i(x,y,z,t)) \\ & +d(g(g(x_n), T_n(g(x_n), g(y_n), g(z_n), g(t_n)))] \\ & +\gamma_{i,n} d(g(g(x_n), g(x)). \end{aligned}$$

Taking the limit as $n \to +\infty$, we obtain $T_i(x,y,z,t) = g(x)$ as $\beta_{i,n} < 1$. Similarly, it can be proved that $g(y) = T_i(y,z,t,x)$, $g(z) = T_i(z,t,x,y)$ and $g(t) = T_i(t,x,y,z)$. Thus, (x,y,z,t) is a quadrupled coincidence point of $\{T_i\}_{i\in N}$ and g.

Now, we give useful conditions for the existence and uniqueness of a quadrupled common fixed point.

Theorem 2.2. *In addition to the hypotheses of Theorem 2.1, suppose that the set of coincidence points is comparable with respect to g, then $\{T_i\}_{i\in N}$ and g have a unique quadrupled common fixed point, that is, there exists $(x,y,z,t) \in X^4$ such that $x = g(x) = T_i(x,y,z,t)$, $y = g(y) = T_i(y,z,t,x)$, $z = g(z) = T_i(z,t,x,y)$ and $t = g(t) = T_i(t,x,y,z)$ for $i \in N$.*

Proof. From Theorem 2.1, the set of quadrupled coincidence points is non-empty. Now, we show that if (x,y,z,t) and (u,v,w,s) are quadrupled coincidence points, that is, if $g(x) = T_i(x,y,z,t)$, $g(y) = T_i(y,z,t,x)$, $g(z) = T_i(z,t,x,y)$, $g(t) = T_i(t,x,y,z)$, $g(u) = T_i(u,v,w,s)$, $g(v) = T_i(v,w,s,u)$, $g(w) = T_i(w,s,u,v)$ and $g(s) = T_i(s,u,v,w)$ then $g(x) = g(u)$, $g(y) = g(v)$, $g(z) = g(w)$ and $g(t) = g(s)$. Since the set of coincidence points is comparable, applying condition 2.4 to these points, we get

$$\begin{aligned} d(g(x), g(u)) &= d(T_i(x,y,z,t), T_j(u,v,w,s)) \\ &\leq \beta_{i,j}[d(g(x), T_i(x,y,z,t)) + d(g(u), T_j(u,v,w,s))] \\ &\quad +\gamma_{i,j} d(g(x), g(u)), \end{aligned}$$

and so as $\gamma i, j < 1$, it follows that $d(g(x), g(u)) = 0$, that is, $g(x) = g(u)$. Similarly, it can be proved that $g(y) = g(v)$, $g(z) = g(w)$ and $g(t) = g(s)$. Hence, $\{T_i\}_{i\in N}$ and g have a unique quadrupled point of coincidence. It is well known that two compatible mappings are also weakly compatible, that is, they commute at their coincidence points. Thus, it is clear that $\{T_i\}_{i\in N}$ and g have a unique quadrupled common fixed point whenever $\{T_i\}_{i\in N}$ and g are weakly compatible. This finishes the proof. □

If g is the identity mapping, as a consequence of Theorem 2.1, we state the following corollary.

Corollary 2.1. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $\{T_i\}_{i\in N}$ be a sequence of mappings from $X\times X\times X\times X$ into X such that $\{T_i\}_{i\in N}$ satisfies, for $x, y, z, t, u, v, w, s \in X$ with $x \preceq u, \quad v \preceq y, \quad z \preceq w \quad s \preceq t$ or $x \succeq u, v \succeq y, \quad z \succeq w, \quad s \succeq t.$, the following conditions:*

(i) $T_n(x,y,z,t) \preceq T_{n+1}(u,v,w,s)$,

(ii) *for* $0 \leq \beta_{i,j}, \gamma_{i,j} < 1$ *and* $i, j \in N$

$$d(T_i(x,y,z,t), T_j(u,v,w,s)) \leq \beta_{i,j}[d(x, T_i(x,y,z,t)) + d(u, T_j(u,v,w,s))] + \gamma_{i,j} d(u,x).$$

$$x_0 \preceq T_0(x_0,y_0,z_0,t_0), \quad y_0 \succeq T_0(y_0,z_0,t_0,x_0),$$
$$z_0 \preceq T_0(z_0,t_0,x_0,y_0) \;\; and \;\; t_0 \succeq T_0(t_0,x_0,y_0,z_0).$$

If $\sum_{i=1}^{+\infty} \frac{\beta_{i,i+1}+\gamma_{i,i+1}}{1-\beta_{i,i+1}}$ *is an* $\alpha-$*series and* X *is regular, then* $\{T_i\}_{i\in N}$ *has a quadrupled fixed point, that is, there exists* $(x,y,z,t) \in X^4$ *such that* $x = T_i(x,y,z,t), y = T_i(y,z,t,x)$, $z = T_i(z,t,x,y)$ *and* $t = T_i(t,x,y,z)$ *for* $i \in N$.

Example 2.1. Take $X = [0,1]$ endowed with usual metric $d = |x - y|$ for all $x, y \in X$ and $\preceq$ be defined as greater than /equal to the $(X, d, \preceq)$ be partial order metric space. Let $T_i : X^4 \to X$ be mapping defined as $T_i(x,y,z,t) = \frac{x+y+z+t}{4i}$; $i \in N$ and g is self -mapping defined as $g(x) = x$. Clearly, $T_i(x,y,z,t) \subseteq g(X)$, $g(X)$ is a complete subset of X.
By choosing the sequences $\{x_n\} = \frac{1}{n}$, $\{y_n\} = \frac{1}{n+1}$, $\{z_n\} = \frac{1}{n+2}$ and $\{t_n\} = \frac{1}{n+3}$ one can easily observe that $\{T_i\}_{i\in N}$ and g are compatible, weakly reciprocally continuous; g is monotonic nondecreasing, continuous, as well as satisfying condition 2.1.

Again by taking $0 \leq \beta_{i,j}, \gamma_{i,j} < 1$, it is easy to check inequality 2.4 holds, thus all the hypotheses of Theorem 2.1 are satisfied and $(0,0,0,0), (1,1,1,1)$ are the quadrupled coincident points of g and T_i. Moreover,using the same T_i and g in Theorem 2.2, $(0,0,0,0)$ is the unique fixed point of g and T_i.

Remark 2.1. Open problem: In this paper, we prove quadrupled fixed point results. The idea can be extended to multidimensional cases. But the technicalities in the proofs therein will be different. We consider this as an open problem.

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Animesh Gupta
Ward No. 2
Gandhi Chowk
Pachmarhi - 461881, Dist. Hoshangabad
Madhya Pradesh - India
dranimeshgupta10@gmail.com