FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 30, No 3 (2015), 325–339

ON λ-SPIRAL-LIKE FUNCTIONS INVOLVING A CONVOLUTION STRUCTURE

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Abstract. By using a subordination condition, a new class $\mathcal{S}_p^{\lambda}(b;g;h)$ of p-valent functions involving a convolution structure is defined. Among others, this class includes the λ-spiral-like and λ-Robertson classes of functions. Based on first-order differential subordination and its properties, various results pertaining to the class $\mathcal{S}_p^{\lambda}(b;g;h)$ and its subclass $\mathcal{S}_p^{\lambda}(b;g;A,B)$ are derived. Several consequences of our results yield certain new results. We also point out the relationship with other known results.

Keywords: p-valent functions, subordination, convolution, λ-spirallike functions.

1. Introduction

Let $\mathcal{A}_p$ denotes a class of p-valent functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, p \in \mathbb{N} = \{1, 2, ...\}, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{T}_p$, a subclass of $\mathcal{A}_p$ whose members are of the form

$$f(z) = z^p - \sum_{n=p+1}^{\infty} |a_n| z^n. \tag{1.2}$$

The convolution (Hadamard product) of $f(z)$ of the form (1.1) and $g(z)$ of the form

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n, z \in \mathbb{U} \tag{1.3}$$

Received March 24, 2015; Accepted April 24, 2015
2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C50

is defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n = (g * f)(z), z \in \mathbb{U}. \tag{1.4}$$

The above convolution leads us to consider various linear operators for the class $\mathcal{A}_p$. Indeed, we infer that $f(z) * \frac{z^p}{1-z} = f(z)$ and $f(z) * \frac{(p+(1-p)z)z^p}{p(1-z)^2} = \frac{zf'(z)}{p}$.

Let $p(z)$ and $q(z)$ analytic in $\mathbb{U}$ be such that $p(0) = q(0)$. We say $p(z)$ is subordinate to $q(z)$ for $z \in \mathbb{U}$ and write $p(z) \prec q(z)$, $z \in \mathbb{U}$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$, and $|w(z)| < 1, z \in \mathbb{U}$ such that $p(z) = q(w(z)), z \in \mathbb{U}$. Furthermore, if the function $q(z)$ is univalent in $\mathbb{U}$, then we have following equivalence:

$$p(z) \prec q(z) \Leftrightarrow p(0) = q(0) \text{ and } p(\mathbb{U}) \subset q(\mathbb{U}).$$

Let $q(z), z \in \mathbb{U}$ be convex. We denote by $\mathcal{P}(q)$, a class of analytic functions $p(z)$ such that $p(z) \prec q(z)$ in $\mathbb{U}$. The class $\mathcal{P}\left(\frac{1+Az}{1+Bz}\right) = \mathcal{P}(A, B), -1 \leq B < A \leq 1$ is the Janowski class [7] of analytic functions $p(z)$, and in particular, the class $\mathcal{P}(1, -1) = \mathcal{P}$ is the class of analytic functions $p(z)$ with positive real part in $\mathbb{U}$.

For a non-zero complex number $b, |\lambda| < \pi/2$ and for some given function $g \in \mathcal{A}_p$, we define here a new class $\mathcal{S}_p^\lambda(b; g; h)$ consisting of λ-spiral-like functions $f \in \mathcal{A}_p$ satisfying the subordination condition that

$$1 + \frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{z(f * g)'(z)}{p(f * g)(z)} - 1\right) \prec h(z), z \in \mathbb{U}, \tag{1.5}$$

where $h \in \mathcal{P}$.

Observe that if $f \in \mathcal{S}_p^\lambda(b; g; h)$, then by putting $d = \frac{e^{i\lambda}}{b\cos\lambda}$, we get

$$\frac{z(f * g)'(z)}{p(f * g)(z)} \prec 1 + \left(\frac{h(z) - 1}{d}\right), z \in \mathbb{U}. \tag{1.6}$$

We denote $\mathcal{S}_p^\lambda\left(b; g; \frac{1+Az}{1+Bz}\right)$ by $\mathcal{S}_p^\lambda(b; g; A, B)$, whose members satisfy the condition that

$$\frac{z(f * g)'(z)}{p(f * g)(z)} \prec \frac{1 + \left(B + \frac{A-B}{d}\right)z}{1 + Bz}, z \in \mathbb{U}, \tag{1.7}$$

and $\mathcal{S}_p^\lambda(b; g; 1, -1)$ by $\mathcal{S}_p^\lambda(b; g)$. Further, we denote $\mathcal{S}_p^\lambda\left(b; \frac{z^p}{1-z}; A, B\right)$ by $\mathcal{R}_p^\lambda(b; A, B)$ and $\mathcal{S}_p^\lambda\left(b; \frac{(p+(1-p)z)z^p}{p(1-z)^2}; A, B\right)$ by $Q_p^\lambda(b; A, B)$. It may be observed that

$$f \in Q_p^\lambda(b; A, B) \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{R}_p^\lambda(b; A, B).$$

One may notice that a function $f \in \mathcal{S}_p^{\lambda}(b; g; A, B)$ must evidently satisfy the condition that

$$\mathrm{Re}\left(1 + \frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{z(f*g)'(z)}{p(f*g)(z)} - 1\right)\right) > \frac{1-A}{1-B}, z \in \mathbb{U}.$$

The class $\mathcal{S}_p^{\lambda}(1; g; 1-2\alpha, -1), 0 \leq \alpha < 1$ is represented by $\mathcal{S}_{p,\alpha}^{\lambda}(g)$ whose members satisfy the condition that

$$\frac{z(f*g)'(z)}{p(f*g)(z)} \prec 1 + \frac{\left(2(1-\alpha)e^{-i\lambda}\cos\lambda\right)z}{1-z}, z \in \mathbb{U}.$$

To make this paper relatively self-contained (and for the reader's convenience), we deem it worthwhile here to give a rather comprehensive description of the important special cases of the classes $\mathcal{R}_p^{\lambda}(b; A, B), Q_p^{\lambda}(b; A, B), \mathcal{S}_p^{\lambda}(b; g)$ and $\mathcal{S}_{p,\alpha}^{\lambda}(g)$, which were studied earlier and some of which are used in the sequel.

Indeed, for $0 \leq \alpha < 1$, the class $\mathcal{R}_p^{\lambda}(b; (1-\alpha)A + \alpha B, B) = S_p^{\lambda}(A, B, b)$ was recently studied by Dileep and Latha [5]. The class $\mathcal{R}_1^{\lambda}(b; A, B) = S^{\lambda}(A, B, b)$ was studied in [18], and the class $\mathcal{R}_1^{0}(1; A, B) = S^{*}(A, B)$ is the familiar Janowski class of starlike functions [7]. Class $Q_p^{0}(b; 1, -1) = C_p(b)$ is a class of p-valently convex functions of complex order, studied by Aouf [2]. The class $\mathcal{S}_1^{\lambda}\left(b; \frac{z}{1-z}\right) = S^{\lambda}(b)$ was studied by Al-Oboudi and Haidan [3] (see also [1]), whereas, the class $\mathcal{S}_1^{\lambda}\left(1; \frac{z}{1-z}\right) = S^{\lambda}$ is the class of λ-spiral-like univalent functions, introduced by Spacek [24]. Also, the Class $\mathcal{S}_1^{0}\left(b; \frac{z}{1-z}\right) = S(b)$ is the class of starlike functions of complex order which was studied by Nasr and Aouf [12]. Further, for $0 \leq \alpha < 1$, the class $\mathcal{S}_1^{\lambda}\left(1-\alpha; \frac{z}{1-z}\right)$ is the class of λ-spiral-like univalent functions of order α studied by Libera [9]. On the other hand, for $0 \leq \alpha < 1$, the class $\mathcal{S}_1^{0}\left(1-\alpha; \frac{z}{1-z}\right) = S^{*}(\alpha)$ is the well known class of starlike functions of order α studied by Robertson [21]. Moreover, the class $\mathcal{S}_1^{\lambda}\left(b; \frac{z}{(1-z)^2}\right) = C^{\lambda}(b)$ is a λ-Robertson class of complex order studied by Aouf *et al.* [1], and the Class $\mathcal{S}_1^{\lambda}\left(1; \frac{z}{(1-z)^2}\right) = C^{\lambda}$ was studied earlier by Robertson [22]. For $0 \leq \alpha < 1$, the class $\mathcal{S}_1^{\lambda}\left(1-\alpha; \frac{z}{(1-z)^2}\right) = C^{\lambda}(1-\alpha)$ is the class of λ-Robertson functions of order α studied by Chichra [4], whereas, for $0 \leq \alpha < 1$, the class $\mathcal{S}_1^{0}\left(1-\alpha; \frac{z}{(1-z)^2}\right) = K(\alpha)$ is the class of convex functions of order α studied earlier by Robertson [21]. The class $\mathcal{S}_1^{0}\left(b; \frac{z}{(1-z)^2}\right) = C(b)$ is the class of convex functions of complex order studied by Nasr and Aouf [13], and the class $\mathcal{S}_{1,\alpha}^{\lambda}\left(\frac{z}{1-z}\right) = S_p^{\alpha}(\lambda)$ was introduced by Kwon and Owa [8] (see also [14]), and finally, the class $\mathcal{S}_{1,\alpha}^{\lambda}\left(\frac{z}{(1-z)^2}\right) = K_p^{\alpha}(\lambda)$ was introduced by Owa *et al.* [14].

Suppose $\psi : \mathbb{C}^2 \to \mathbb{C}$ be analytic in a domain D, and let h be univalent in $\mathbb{U}$. Also, let $p(z)$ be analytic in $\mathbb{U}$ with $(p(z), zp'(z)) \in D$ when $z \in \mathbb{U}$, then $p(z)$ is said to satisfy the first-order differential subordination if

$$\psi\,(p(z), zp'(z)) \prec h(z). \tag{1.8}$$

The univalent function q is said to be a dominant of the differential subordination (1.8) if $p \prec q$ for all p satisfying (1.8). If $\widetilde{q}$ is a dominant of (1.8) and $\widetilde{q} \prec q$ for all dominants q of (1.8), then $\widetilde{q}$ is said to be the best dominant of (1.8). The theory of differential subordination was introduced by Miller and Mocanu in [11].

In this paper, we define a class $\mathcal{S}_p^{\lambda}\,(b; g; h)$ of p-valent analytic functions whose convolution with some p-valent analytic function $g(z)$ satisfy a subordination condition. This class includes several classes of λ-spiral-like functions and λ-Robertson class of functions with complex order. Using the first-order differential subordination, we derive a subordination result for the class $\mathcal{S}_p^{\lambda}\,(b; g; h)$. Subordination results for the subclass $\mathcal{S}_p^{\lambda}\,(b; g; A, B)$ of $\mathcal{S}_p^{\lambda}\,(b; g; h)$ are also derived for $B \neq 0$ and for $B = 0$. Moreover, a coefficient inequality and a convolution result for the class $\mathcal{S}_p^{\lambda}\,(b; g; A, B)$ are obtained. Also, mentioned are the results based on certain special cases which include some new and known results (obtained earlier by adopting different methods).

2. Main Results

To prove our first main result, we require the following known result on differential subordination.

Lemma 2.1. [10, Theorem 3, p.190]. *Let q be univalent in $\mathbb{U}$ and let θ and ϕ be analytic in a domain D containing $q(\mathbb{U})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set*

$$Q(z) = zq'(z)\phi\,(q(z))\,, \qquad h(z) = \theta\,(q(z)) + Q(z),$$

and suppose that

(i) Q is starlike (univalent) in $\mathbb{U}$ with $Q(0) = 0$ and $Q'(0) \neq 0$,

(iii) $\mathrm{Re}\left(\frac{z\,h'(z)}{Q(z)}\right) = \mathrm{Re}\left(\frac{\theta'(q(z))}{\phi(q(z))} + \frac{z\,Q'(z)}{Q(z)}\right) > 0, z \in \mathbb{U}.$

If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subset D$ and

$$\theta\,(p(z)) + zp'(z)\phi\,(p(z)) \prec \theta\,(q(z)) + zq'(z)\phi\,(q(z))\,, \tag{2.1}$$

then $p \prec q$ and q is the best dominant of (2.1).

Throughout this paper, we assume that only the principal values of the powers are considered in our investigations.

Theorem 2.1. *Let $f \in \mathcal{A}_p$ and $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$ ($\neq 0$ in $\mathbb{U}$) be univalent in $\mathbb{U}$ such that*

$$\operatorname{Re}\left(1 - \frac{z\,q'(z)}{q(z)} + \frac{z\,q''(z)}{q'(z)}\right) > 0 \text{ in } \mathbb{U}, \tag{2.2}$$

and for $0 \neq \beta \in \mathbb{C}$, let

$$h(z) = 1 + \frac{z\,q'(z)}{p\beta q(z)}. \tag{2.3}$$

If $f \in \mathcal{S}_p^{\lambda}(b; g; h)$ be such that $\frac{(f*g)(z)}{z^p} \neq 0$ in $\mathbb{U}$, then for $d = \frac{e^{i\lambda}}{b\cos\lambda}$:

$$\left(\frac{(f*g)(z)}{z^p}\right)^{\beta d} \prec q(z), z \in \mathbb{U}, \tag{2.4}$$

and $q(z)$ is the best dominant.

Proof. Let us consider

$$s(z) = \left(\frac{(f*g)(z)}{z^p}\right)^{\beta d}, \tag{2.5}$$

and

$$\phi(w) = \frac{1}{p\beta w}, \theta(w) = 1,$$

then $s(z)$ is analytic in $\mathbb{U}$ with $s(0) = q(0)$ and θ and ϕ are analytic in a domain D ($0 \notin D$). In order to apply Lemma 2.1, we observe from (2.3) that $h(z) = \theta(q(z)) + Q(z)$, where $Q(z) = \frac{z\,q'(z)}{p\beta q(z)}$ is such that $Q(0) = 0$, $Q'(0) \neq 0$. Using (2.2), we find that

$$\operatorname{Re}\left(\frac{z\,h'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{z\,Q'(z)}{Q(z)}\right) = \operatorname{Re}\left(1 - \frac{z\,q'(z)}{q(z)} + \frac{z\,q''(z)}{q'(z)}\right) > 0$$

in $\mathbb{U}$. Further, on differentiating (2.5) logarithmically, we obtain in view of the subordination condition (1.5) that

$$\theta(s(z)) + zs'(z)\phi(s(z)) = 1 + \frac{zs'(z)}{p\beta s(z)} = 1 + d\left(\frac{z\,(f*g)'(z)}{p\,(f*g)(z)} - 1\right) \prec h(z), z \in \mathbb{U}, \tag{2.6}$$

where $h(z)$ is given by (2.3). Now applying Lemma 2.1, we conclude that the subordination (2.6) implies the result (2.4), where $q(z)$ is the best dominant of this subordination. This proves Theorem 2.1. □

In our next result, we use a lemma which is as follows:

Lemma 2.2. [22] *The function $(1-z)^{\beta} \equiv e^{\beta\log(1-z)}$, $\beta \neq 0$, is univalent in $\mathbb{U}$ if and only if β is either in the closed disk $\left|\beta - 1\right| \leq 1$, or in the closed disk $\left|\beta + 1\right| \leq 1$.*

By choosing $0 \neq \beta \in \mathbb{C}$ and

$$q(z) = (1 + Bz)^{\frac{p(A-B)\beta}{B}}, -1 \leq B < A \leq 1\ (B \neq 0)$$

in Theorem 2.1, we get the following result with the use of Lemma 2.2.

Theorem 2.2. *Let $f \in \mathcal{S}_p^{\lambda}(b; g; A, B)$ with $B \neq 0$ be such that $\frac{(f*g)(z)}{z^p} \neq 0$ in $\mathbb{U}$, and $0 \neq \beta \in \mathbb{C}$ be such that*

$$\text{(2.7)} \qquad \textit{either} \quad \left| \frac{p(A-B)\beta}{B} - 1 \right| \leq 1 \ \textit{or} \ \left| \frac{p(A-B)\beta}{B} + 1 \right| \leq 1,$$

then for $d = \frac{e^{i\lambda}}{b \cos \lambda}$:

$$\text{(2.8)} \qquad \left(\frac{(f * g)(z)}{z^p} \right)^{\beta d} \prec (1 + Bz)^{\frac{p(A-B)\beta}{B}}, z \in \mathbb{U},$$

and $(1 + Bz)^{\frac{p(A-B)\beta}{B}}$ is the best dominant.

Proof. Let $f \in \mathcal{S}_p^{\lambda}(b; g; A, B)$ and $s(z)$ be given by (2.5). For $B \neq 0$, $0 \neq \beta \in \mathbb{C}$, let

$$\text{(2.9)} \qquad q(z) = (1 + Bz)^{\frac{p(A-B)\beta}{B}}.$$

then $s(0) = q(0)$ and

$$\frac{z\,q'(z)}{q(z)} = \frac{p(A-B)\beta z}{1 + Bz}.$$

By Lemma 2.2, under the condition (2.7), $q(z)$ is univalent (see also [16], [22]) and on letting $Q(z) = \frac{zq'(z)}{p\beta q(z)}$, we get $Q(z) = \frac{(A-B)z}{1+Bz}$, which is univalent with $Q(0) = 0$, $Q'(0) = A - B \neq 0$, and $\mathrm{Re}\left(\frac{z\,Q'(z)}{Q(z)}\right) = \mathrm{Re}\left(\frac{1}{1+Bz}\right) > 0$, $z \in \mathbb{U}$.

Following similar lines of the proof of Theorem 2.1, we get the result (2.8) on applying Lemma 2.1, which proves Theorem 2.2. □

In the case, when $B = 0$, $0 \neq \beta \in \mathbb{C}$ and $q(z) = e^{p\beta Az}$, $0 < A \leq 1$ in Theorem 2.1, then similar to Theorem 2.2, we can easily prove the following result.

Theorem 2.3. *Let $f \in \mathcal{S}_p^{\lambda}(b; g; A, 0)$ be such that $\frac{(f*g)(z)}{z^p} \neq 0$ in $\mathbb{U}$, and $0 \neq \beta \in \mathbb{C}$ be such that $|\beta| < \frac{\pi}{pA}$, then for $d = \frac{e^{i\lambda}}{b \cos \lambda}$:*

$$\left(\frac{(f * g)(z)}{z^p} \right)^{\beta d} \prec e^{p\beta Az}, z \in \mathbb{U},$$

and $e^{p\beta Az}$ is the best dominant.

For a non-zero complex number a, with $\beta = \frac{a}{d}$, Theorems 2.2 and 2.3, yield the following corollaries.

Corollary 2.1.

(i) Let $f \in \mathcal{S}_p^{\lambda}(b; g; A, B)$ *with* $B \neq 0$ *be such that* $\frac{(f*g)(z)}{z^p} \neq 0$ *in* $\mathbb{U}$*, and for* $d = \frac{e^{i\lambda}}{b\cos\lambda}$*, a non-zero complex number* a *be such that*

$$\text{either} \quad \left|\frac{p(A-B)a}{Bd} - 1\right| \leq 1 \text{ or } \left|\frac{p(A-B)a}{Bd} + 1\right| \leq 1,$$

then

$$\left(\frac{(f*g)(z)}{z^p}\right)^a \prec (1+Bz)^{\frac{p(A-B)a}{Bd}}, z \in \mathbb{U},$$

and $(1+Bz)^{\frac{p(A-B)a}{Bd}}$ *is the best dominant.*

(ii) Let $f \in \mathcal{S}_p^{\lambda}(b; g; A, 0)$ *be such that* $\frac{(f*g)(z)}{z^p} \neq 0$ *in* $\mathbb{U}$*, and for* $d = \frac{e^{i\lambda}}{b\cos\lambda}$*, a non-zero complex number* a *be such that* $\left|\frac{a}{d}\right| < \frac{\pi}{pA}$*, then*

$$\left(\frac{(f*g)(z)}{z^p}\right)^a \prec e^{\frac{paA}{d}z}, z \in \mathbb{U},$$

and $e^{\frac{paA}{d}z}$ *is the best dominant.*

Remark 2.1.

(1) The results (i) of Corollary 2.1 coincide with the results of Aouf *et al.* [1, Theorem 1, p. 95 and Corollaries 1, 2, p. 96] involving the classes $S^{\lambda}(b)$ and $C^{\lambda}(b)$, respectively, which also include the results of Obradovic *et al.* [15] for the classes $S(b)$, S^{λ} and $S^{\lambda}(1-\alpha)$, $0 \leq \alpha < 1$.

(2) The results (i) and (ii) of Corollary 2.1 coincide with the result of Obradovic and Owa [16, Theorem 2, p. 363] for the class $S^*(A, B)$ and its subclass $S^*(\alpha)$, $0 \leq \alpha < 1$.

For real β, Theorem 2.2 simplifies to the following form:

Corollary 2.2. *Let* $f \in \mathcal{S}_p^{\lambda}(b; g; A, B)$ *with* $B \neq 0$ *be such that* $\frac{(f*g)(z)}{z^p} \neq 0$ *in* $\mathbb{U}$*. Then for* $d = \frac{e^{i\lambda}}{b\cos\lambda}$ *and for positive real*

$$\beta = \frac{|B|}{p(A-B)}, \tag{2.10}$$

$$\operatorname{Re}\left(\frac{(f*g)(z)}{z^p}\right)^{\beta d} > \begin{cases} 1 - p(A-B)\beta, B > 0 \\ \frac{1}{1+p(A-B)\beta}, B < 0 \end{cases}, z \in \mathbb{U}. \tag{2.11}$$

Proof. From (2.8), we obtain that

$$\begin{aligned} \operatorname{Re}\left(\frac{(f*g)(z)}{z^p}\right)^{\beta d} &\geq \inf_{z\in\mathbb{U}} \operatorname{Re}(1+Bz)^{\frac{p(A-B)\beta}{B}} \\ &> \begin{cases} 1-|B|, B>0 \\ \frac{1}{1+|B|}, B<0 \end{cases}, \end{aligned}$$

which proves the result (2.11) upon using (2.10). □

Remark 2.2. By setting $b = 1 = p$, replacing β by $\beta\cos\lambda$, Corollary 2.2 for $B = -1, A = 1-2\alpha$, and for $g(z) = \frac{z}{1-z}$, corresponds to the known results of Obradovic and Owa [17, Theorem 1, p. 440], (for the case when $n = 1$). Also, on setting $b = 1 = p$, replacing β by $\frac{\beta}{2}$, Corollary 2.2 for $B = -1, A = 1$, and for $g(z) = \frac{z}{1-z}$, correspond to the known result [17, Corollary 1, p. 442] (for the when case $n = 1$).

A more compact form of the result occurs when $\beta = \frac{B}{p(A-B)}$ in (2.8) of Theorem 2.2, and this result is given by the following corollary.

Corollary 2.3. *Let $f \in \mathcal{S}_p^\lambda(b; g; A, B)$ with $B \neq 0$ be such that $\frac{(f*g)(z)}{z^p} \neq 0$ in $\mathbb{U}$, then for $d = \frac{e^{i\lambda}}{b\cos\lambda}$:*

$$\left(\frac{(f*g)(z)}{z^p}\right)^{\frac{dB}{p(A-B)}} \prec 1 + Bz, z \in \mathbb{U},$$

and hence, the Marx-Strohhacker type inequality:

$$\left|\left(\frac{(f*g)(z)}{z^p}\right)^{\frac{dB}{p(A-B)}} - 1\right| < |B|, z \in \mathbb{U}. \tag{2.12}$$

Remark 2.3. Corollary 2.3 is the known result of Dileep and Latha ([5], Theorem 3.3, p. 543) for the class $S_p^\lambda(A, B, b)$.

Further, for real $\beta = \frac{B}{p(A-B)}$, and setting $g(z) = \frac{z^p}{1-z}$ and $g(z) = \frac{(p+(1-p)z)z^p}{p(1-z)^2}$, respectively, Theorem 2.2 yields the following results:

Corollary 2.4. *Let $f \in \mathcal{R}_p^\lambda(b; A, B)$ with $B \neq 0$ be such that $\frac{f(z)}{z^p} \neq 0$ in $\mathbb{U}$, then for $d = \frac{e^{i\lambda}}{b\cos\lambda}$:*

$$\left(\frac{f(z)}{z^p}\right)^{\frac{dB}{p(A-B)}} \prec 1 + Bz, z \in \mathbb{U},$$

and hence, the Marx-Strohhacker type inequality:

$$\left|\left(\frac{f(z)}{z^p}\right)^{\frac{dB}{p(A-B)}} - 1\right| < |B|, z \in \mathbb{U}.$$

Corollary 2.5. *Let $f \in Q_p^{\lambda}(b; A, B)$ with $B \neq 0$ be such that $\frac{f'(z)}{z^{p-1}} \neq 0$ in $\mathbb{U}$, then for $d = \frac{e^{i\lambda}}{b\cos\lambda}$:*

$$\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{dB}{p(A-B)}} \prec 1 + Bz, z \in \mathbb{U},$$

and hence, the Marx-Strohhacker type inequality:

$$\left|\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{dB}{p(A-B)}} - 1\right| < |B|, z \in \mathbb{U}.$$

Remark 2.4. For $p = 1$, Corollary 2.4 gives the known result of Polatoglu and Sen ([18], Theorem 2, p. 93) .

3. Coefficient Inequality

Theorem 3.1. *Let $f \in \mathcal{A}_p$ be of the form* (1.1) *and* $0 \neq b \in \mathbb{C}$, $|\lambda| < \pi/2$, $-1 \leq B < A \leq 1$. *If the coefficients of $f(z)$ satisfy for some $g \in \mathcal{A}_p$ of the form (1.3) and for $d = \frac{e^{i\lambda}}{b\cos\lambda}$, the inequality*

$$\sum_{n=p+1}^{\infty} \left\{\left(\frac{n}{p} - 1\right)\frac{(1 + |B|)\,|d|}{A - B} + 1\right\}|a_n b_n| \leq 1, \tag{3.1}$$

then $f \in \mathcal{S}_p^{\lambda}(b; g; A, B)$. Furthermore, the inequality (3.1) *is necessary if $f * g \in \mathcal{T}_p$ satisfies for $d = \frac{e^{i\lambda}}{b\cos\lambda}(|b| \leq 1), B < 0$, the subordination:*

$$\frac{z\,(f * g)'\,(z)}{p\,(f * g)\,(z)} \prec \frac{1 + \left(B + \frac{A-B}{|d|}\right)z}{1 + Bz}, z \in \mathbb{U}. \tag{3.2}$$

The inequality is sharp for the functions given by

$$f(z) = z^p - \frac{A - B}{\left\{\left(\frac{n}{p} - 1\right)(1 + |B|)\,|d| + A - B\right\}|b_n|}z^n, n \in \{p + 1, p + 2, ...\}. \tag{3.3}$$

Proof. Let $f, g \in \mathcal{A}_p$, respectively, be of the form (1.1) and (1.3). To show that $f \in \mathcal{S}_p^{\lambda}(b; g; A, B)$, we need to show in view of (1.7) that for some Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$:

$$|w(z)| = \left|\frac{z\,(f * g)'\,(z) - p\,(f * g)\,(z)}{Bz\,(f * g)'\,(z) - p\left(B + \frac{A-B}{d}\right)(f * g)\,(z)}\right| < 1, z \in \mathbb{U},$$

which yields that

$$|w(z)| < \frac{\sum_{n=p+1}^{\infty}\left(\frac{n}{p}-1\right)|a_n b_n|}{\frac{A-B}{|d|}-\sum_{n=p+1}^{\infty}\left\{\left(\frac{n}{p}-1\right)|B|+\frac{A-B}{|d|}\right\}|a_n b_n|} \leq 1,$$

provided that (3.1) holds. Conversely, if $f * g \in \mathcal{T}_p$ satisfies for $d = \frac{e^{i\lambda}}{b\cos\lambda}$ $(|b| \leq 1), B < 0$, the subordination (3.2), then we have

$$\operatorname{Re}\left(\frac{z\,(f*g)'\,(z)}{p\,(f*g)\,(z)}\right) > \frac{1-\left(B+\frac{A-B}{|d|}\right)}{1-B}, z \in \mathbb{U}$$

or,

$$\operatorname{Re}\left(\frac{z\,(f*g)'\,(z)}{p\,(f*g)\,(z)}\right) > \frac{1+|B|-\frac{A-B}{|d|}}{1+|B|}, z \in \mathbb{U}. \tag{3.4}$$

Assuming $\frac{z\left(f*g\right)'(z)}{p\left(f*g\right)}$ to be real for real values of z, and using the inequality (3.4), we get

$$(1+|B|)\,z\,(f*g)'\,(z) - p\left(1+|B|-\frac{A-B}{|d|}\right)(f*g)\,(z) > 0,$$

and this inequality upon using the corresponding series expansions, and then letting $z \to 1^-$ along the real line gives the desired inequality (3.1). Sharpness of the result can be verified for the function given by (3.3). □

If we put $A = 1-2\alpha, 0 \leq \alpha < 1, B = -1, b = 1$ in Theorem 3.1, we get the following coefficient inequality for the class $\mathcal{S}^{\lambda}_{p,\alpha}(g)$.

Corollary 3.1. *Let $f \in \mathcal{A}_p$ be of the form* (1.1) *and* $|\lambda| < \pi/2, 0 \leq \alpha < 1$. *If the coefficients of $f(z)$ satisfy for some $g \in \mathcal{A}_p$ of the form* (1.3) *the inequality*

$$\sum_{n=p+1}^{\infty}\left[\left(\frac{n-p}{1-\alpha}\right)\frac{\sec\lambda}{p}+1\right]|a_n b_n| \leq 1, \tag{3.5}$$

then $f \in \mathcal{S}^{\lambda}_{p,\alpha}(g)$. Inequality (3.5) is necessary for $0 \leq \alpha < 1$, if $f * g \in \mathcal{T}_p$ satisfies the subordination:

$$\frac{z\,(f*g)'\,(z)}{p\,(f*g)\,(z)} \prec 1+\frac{2\,(1-\alpha)\cos\lambda z}{1-z}, z \in \mathbb{U}. \tag{3.6}$$

Remark 3.1. For $p = 1, g(z) = \frac{z}{1-z}$, the above inequality (3.5) of Corollary 3.1 coincides with the result of Kwon and Owa [8, Theorem 2.3, p. 21] which was obtained by a different method.

4. Subordination Result

In this section we obtain a subordination theorem involving convolution by using the definition and a lemma due to Wilf [25], which is presented here in the following form.

Definition 4.1. A sequence $\left\{A_{n+p-1}\right\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if whenever $h(z)=\sum_{n=1}^{\infty} c_{n+p-1}z^n, c_p=1$ is analytic, univalent and convex in $\mathbb{U}$, we have the subordination:

$$\sum_{n=1}^{\infty} A_{n+p-1}c_{n+p-1}z^n \prec h(z), z\in\mathbb{U}.$$

Lemma 4.1. *The sequence* $\left\{A_{n+p-1}\right\}_{n=1}^{\infty}$ *is subordinating factor sequence if and only if*

$$\mathrm{Re}\left(1+2\sum_{n=1}^{\infty} A_{n+p-1}z^n\right)>0, z\in\mathbb{U}.$$

Let $\mathcal{P}_p^{\lambda}(b;g;A,B)$ denote a subclass of the class $\mathcal{S}_p^{\lambda}(b;g;A,B)$, if the functions, therein, are such that $f*g\in\mathcal{T}_p$ satisfies for $d=\frac{e^{i\lambda}}{b\cos\lambda}$ $(|b|\le 1, B<0,)$ the subordination condition (3.2).

Theorem 4.1. *Let* $f\in\mathcal{P}_p^{\lambda}(b;g;A,B)$ *and* $\frac{t(z)}{z^{p-1}}$ *be a convex function. Then*

$$\frac{p\left(A-B\right)+\left(1+|B|\right)|d|}{2\left[2p\left(A-B\right)+\left(1+|B|\right)|d|\right]}\frac{\left(f*g*t\right)(z)}{z^{p-1}}\prec\frac{t(z)}{z^{p-1}}, z\in\mathbb{U}. \tag{4.1}$$

In particular,

$$\mathrm{Re}\left(\frac{\left(f*g\right)(z)}{z^{p-1}}\right)>-\frac{2p\left(A-B\right)+\left(1+|B|\right)|d|}{p\left(A-B\right)+\left(1+|B|\right)|d|}, z\in\mathbb{U}.$$

The quantity

$$\frac{p\left(A-B\right)+\left(1+|B|\right)|d|}{2p\left(A-B\right)+\left(1+|B|\right)|d|}$$

cannot be replaced by any larger value.

Proof. Let $f\in\mathcal{P}_p^{\lambda}(b;g;A,B)$ be of the form (1.1) with $g(z)$ given by (1.3) and $\frac{t(z)}{z^{p-1}}\in K$ (a class of convex functions) be of the form

$$\frac{t(z)}{z^{p-1}}=z+\sum_{n=2}^{\infty} c_{n+p-1}z^n. \tag{4.2}$$

Then, it easily follows that

$$\begin{aligned}
&\frac{p(A-B)+(1+|B|)|d|}{2\left[2p(A-B)+(1+|B|)|d|\right]}\frac{(f*g*t)(z)}{z^{p-1}}\\
=\;&\frac{p(A-B)+(1+|B|)|d|}{2\left[2p(A-B)+(1+|B|)|d|\right]}\left(z+\sum_{n=2}^{\infty}a_{n+p-1}b_{n+p-1}c_{n+p-1}z^{n}\right).
\end{aligned}$$

Thus, by Definition 4.1, the assertion of the theorem holds if the sequence

$$\left\{\frac{p(A-B)+(1+|B|)|d|}{2\left[2p(A-B)+(1+|B|)|d|\right]}a_{n+p-1}b_{n+p-1}\right\}_{n=1}^{\infty}$$

with $a_p = 1 = b_p$ is a subordinating factor sequence. In view of Lemma 4.1, this will be the case if and only if

$$\text{Re}\left[1+\sum_{n=1}^{\infty}\frac{p(A-B)+(1+|B|)|d|}{2p(A-B)+(1+|B|)|d|}a_{n+p-1}b_{n+p-1}z^{n}\right]>0,\, z\in\mathbb{U}. \tag{4.3}$$

Now, for $|z| = r$, we see that

$$\begin{aligned}
&\text{Re}\left[1+\sum_{n=1}^{\infty}\frac{p(A-B)+(1+|B|)|d|}{2p(A-B)+(1+|B|)|d|}a_{n+p-1}b_{n+p-1}z^{n}\right]\\
=\;&\text{Re}\left[1+\frac{p(A-B)+(1+|B|)|d|}{2p(A-B)+(1+|B|)|d|}z+\right.\\
&\left.\frac{p(A-B)}{2p(A-B)+(1+|B|)|d|}\sum_{n=2}^{\infty}\left(1+\frac{(1+|B|)|d|}{p(A-B)}\right)a_{n+p-1}b_{n+p-1}z^{n}\right]\\
\geq\;&1-\frac{p(A-B)+(1+|B|)|d|}{2p(A-B)+(1+|B|)|d|}r-\\
&\frac{p(A-B)}{2p(A-B)+(1+|B|)|d|}\sum_{n=p+1}^{\infty}\left(1+\frac{(n-p)(1+|B|)|d|}{p(A-B)}\right)|a_nb_n|\,r^{n-p+1}\\
\geq\;&1-\frac{p(A-B)+(1+|B|)|d|}{2p(A-B)+(1+|B|)|d|}r-\frac{p(A-B)}{2p(A-B)+(1+|B|)|d|}r^{2},\ \text{by (3.1)}\\
>\;&0.
\end{aligned}$$

Hence, (4.3) is true, which proves the desired assertion (4.1).

In particular, if $f\in\mathcal{P}_p^{\lambda}(b;g;A,B)$ and $\frac{t(z)}{z^{p-1}}=\frac{z}{1-z}$, we obtain from (4.1):

$$\text{Re}\left(\frac{(f*g)(z)}{z^{p-1}}\right)>-\frac{2p(A-B)+(1+|B|)|d|}{p(A-B)+(1+|B|)|d|},\, z\in\mathbb{U}.$$

Sharpness can be seen for the function $f_p\in\mathcal{P}_p^{\lambda}(b;g;A,B)$ given by

$$\left(f_p*g\right)(z)=z^{p}-\frac{p(A-B)}{p(A-B)+(1+|B|)|d|}z^{p+1}.$$

Since, for this function f_p, and for $\frac{t(z)}{z^{p-1}} = \frac{z}{1-z}$, from the relation (4.1), we get

$$F(z) := \frac{p(A-B)+(1+|B|)|d|}{2[2p(A-B)+(1+|B|)|d|]} \frac{\left(f_p * g\right)(z)}{z^{p-1}} \prec \frac{z}{1-z}, z \in \mathbb{U},$$

and it can be verified that

$$\begin{aligned} &\min_{|z|\le 1} \operatorname{Re}(F(z)) \\ &= \min_{|z|\le 1} \operatorname{Re}\left(\frac{p(A-B)+(1+|B|)|d|}{2[2p(A-B)+(1+|B|)|d|]} z - \frac{p(A-B)}{2[2p(A-B)+(1+|B|)|d|]} z^2\right) \\ &= -\frac{1}{2}. \end{aligned}$$

This shows that the quantity $\frac{p(A-B)+(1+|B|)|d|}{2[2p(A-B)+(1+|B|)|d|]}$ is best possible. □

By putting $g(z) = \frac{z^p}{1-z}$, $A = 1-2\alpha, B = -1, b = 1$ in Theorem 4.1, we get the following corollary.

Corollary 4.1. *Let $f \in \mathcal{S}^{\lambda}_{p,\alpha}(g)$ and $\frac{t(z)}{z^{p-1}}$ be a convex function $\forall z \in \mathbb{U}$. Then*

$$\frac{p(1-\alpha)+\sec\lambda}{2(p(1-\alpha)+\sec\lambda)} \frac{(f*t)(z)}{z^{p-1}} \prec \frac{t(z)}{z^{p-1}}, z \in \mathbb{U}. \tag{4.4}$$

In particular

$$\operatorname{Re}\left(\frac{f(z)}{z^{p-1}}\right) > -\frac{2p(1-\alpha)+\sec\lambda}{p(1-\alpha)+\sec\lambda}, z \in \mathbb{U}.$$

The quantity

$$\frac{p(1-\alpha)+\sec\lambda}{2(p(1-\alpha)+\sec\lambda)}$$

cannot be replaced by any larger value.

Remark 4.1. For $p = 1$, Corollary 4.1 coincides with the result of Kwon and Owa ([8], Theorem 2.4, p.22) which also includes the result of Singh [23] (for the case when $\alpha = 0$); see also [6] and [19].

ACKNOWLEDGEMENT: *The authors express their sincerest thanks to the referee for giving various useful suggestions to improve this paper.*

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