FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 1 (2016), 139–159

ON COMPARATIVE GROWTH RELATIONSHIP OF ITERATED ENTIRE FUNCTIONS FROM THE VIEWPOINT OF SLOWLY CHANGING FUNCTIONS

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Abstract. A positive continuous function $L \equiv L(r)$ is called slow if $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. Lakshminarasimhan [15] introduced the idea of functions of L-bounded index. Later Lahiri and Bhattacharjee [17] worked on the entire functions (i.e., functions analytic in the finite complex plane) of L-bounded index and of nonuniform L-bounded index. The growth of an entire function f with respect to another entire function g is defined as the ratio of their maximum moduli for sufficiently large values of r. The same may be defined in terms of maximum terms as well as Nevanlinna's characteristic functions of entire functions. In this paper we would like to investigate some comparative growth analyses of iterated entire functions (as defined by Lahiri and Banerjee [16]) on the basis of their maximum terms, maximum moduli and Nevanlinna's characteristic functions and obtain some powerful results with a scope of further research in the concerned area.
Keywords: Iterated entire function, maximum term, maximum modulus, Nevanlinna's characteristic function, growth, slowly changing function, generalised L^*-order (generalised L^*-lower order).

1. Introduction.

Let f be an entire function defined in the finite complex plane $\mathbb{C}$. Lahiri and Banerjee [16] first initiated the theory of iteration of an entire function f with respect to another entire function g in $\mathbb{C}$ in the following way:

$$\begin{aligned} f(z) &= f_1(z) \\ f(g(z)) &= f(g_1(z)) = f_2(z) \\ f(g(f(z))) &= f(g(f_1(z))) = f(g_2(z)) = f_3(z) \end{aligned}$$

Received April 18, 2015; Accepted June 23, 2015
2010 *Mathematics Subject Classification.* Primary 30D35; Secondary 30D30

$$\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots$$

$$f(g(f\ldots\ldots\ldots(f(z) \text{ or } g(z))\ldots\ldots\ldots\ldots)) = f_n(z), \text{ according as } n \text{ is odd or even,}$$

and so

$$\begin{aligned} g(z) &= g_1(z) \\ g(f(z)) &= g(f_1(z)) = g_2(z) \\ g(f(g(z))) &= g(f(g_1(z))) = g(f_2(z)) = g_3(z) \end{aligned}$$

$$\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots$$

$$g(f(g_{n-2}(z))) = g(f_{n-1}(z)) = g_n(z) \ .$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions in $\mathbb{C}$.

We just recall that the the maximum modulus $M(r, f)$ and maximum term $\mu(r, f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ are respectively defined as $M(r, f) = \max_{|z|=r} |f(z)|$ and $\mu(r, f) = \max_{n \geq 0} (|a_n| r^n)$. For any two entire functions f and g defined in $\mathbb{C}$, the growth of f with respect to g can be estimated from the ratios $\frac{M(r,f)}{M(r,g)}$ as $r \to \infty$ and $\frac{\mu(r,f)}{\mu(r,g)}$ as $r \to \infty$. Several researchers like Clunie [5], Singh [25], Song and Yang [23], Lahiri and Sharma [18], Datta and Biswas {[7],[8]} all studied the growth properties of entire functions in several directions. Later Banerjee and Datta {[2],[3]} used the concepts of iteration in the area of the growth properties of entire functions. Further Banerjee and Datta in [4] investigated the same in the light of maximum terms of entire functions. But the area still remains virgin to study the scope of the comparative analysis of growths of iterated entire functions in terms of slowly changing functions. In fact, the treatment of slowly changing functions in this area is a weaker supposition rather than others and this is the reason for which the study of growth of iterated entire functions in terms of such a type of functions must lead to some powerful results in the concerned field. Datta et al. {[9],[10],[11], [13]} already studied the growths of composite entire functions in the view of slowly changing functions in several directions and this motivation leads us to further proceed in the field of iterated entire functions.

2. Definitions and Notations.

In order to study the theory of growth properties of iterated entire functions in terms of slowly changing functions, it is very much necessary to mention some relevant definitions and notations. The following notation is frequently used in this paper:

$\log^k x = \log\left(\log^{k-1} x\right)$ for $k = 1, 2, 3, \ldots.$ and $\log^0 x = x$.

Taking this into account, the growth indicators $\rho_f^{[p]}$ (respectively $\lambda_f^{[p]}$) [21] of an entire function f is defined as

$$\rho_f^{[p]} = \limsup_{r\to\infty} \frac{\log^p M(r,f)}{\log r} \left(\text{respectively } \lambda_f^{[p]} = \liminf_{r\to\infty} \frac{\log^p M(r,f)}{\log r}\right).$$

For $p = 2$, the above growth indicator reduces to

$$\rho_f = \limsup_{r\to\infty} \frac{\log^2 M(r,f)}{\log r} \left(\text{respectively } \lambda_f = \liminf_{r\to\infty} \frac{\log^2 M(r,f)}{\log r}\right)$$

which is particularly know as *order (*respectively *lower order)* of f. If $\rho_f < \infty$ then f is of finite order. Also $\rho_f = 0$ means that f is of order zero. In this connection, Datta and Biswas [6] gave the following definition :

Definition 2.1. [6] *Let f* be *an entire function of order zero. Then the quantities* ρ_f^{**} and λ_f^{**} of f *are defined by:*

$$\rho_f^{**} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\log r} \textit{ and } \lambda_f^{**} = \liminf_{r\to\infty} \frac{\log M(r,f)}{\log r}.$$

The rate of growth of an entire function generally depends upon *order (lower order)* of it. The entire function with higher *order* is of faster growth than that of lesser *order*. But if orders of two entire functions are same, then it is impossible to detect the function with faster growth. In that case, it is necessary to compute another class of growth indicators of entire functions called their *types* [21]. So the *type* σ_f of an entire function f is defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty .$$

Somasundaram and Thamizharasi [24] introduced the notions of *L-order* and *L-type* for entire functions. The more generalised concept for *L-order* and *L-type* of entire functions are *L^*-order* and *L^*-type*. Their definitions are as follows:

Definition 2.2. [24] The *L^*-order* $\rho_f^{L^*}$ and *the L^*-lower order* $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^2 M(r,f)}{\log\left[re^{L(r)}\right]} \textit{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^2 M(r,f)}{\log\left[re^{L(r)}\right]}.$$

Definition 2.3. The L^*-type $\sigma_f^{L^*}$ of an entire function f is defined as

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \ 0 < \rho_f^{L^*} < \infty .$$

In the line of Sato [21] , Datta and Biswas [6] one can define the generalised L^*-order $\rho_f^{[p]L^*}$ and generalised L^*-lower order $\lambda_f^{[p]L^*}$ of entire f in the following manner :

Definition 2.4. Let p be an integer ≥ 1. The generalised L^*-order $\rho_f^{[p]L^*}$ and generalised L^*-lower order $\lambda_f^{[p]L^*}$ of an entire function f are defined as

$$\rho_f^{[p]L^*} = \limsup_{r\to\infty}\frac{\log^{p} M(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{[p]L^*} = \liminf_{r\to\infty}\frac{\log^{p} M(r,f)}{\log\left[re^{L(r)}\right]}$$

respectively.

Now it is needless to mention the concept of Nevanlinna's characteristic function (after the name of Rolf Nevanlinna(1926)) $T(r, f)$ of entire function f defined as $T(r,f) = \frac{1}{2\pi}\int\limits_{0}^{2\pi} \log^{+}\left|f(re^{i\theta})\right| d\theta$ where $\log^{+} x = \max(\log x, 0)$ for all $x \geqslant 0$ which is very much helpful to carry out the results of this paper. Using the inequalities $\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f)$ $\{cf.$ **[26]** $\}$, for $0 \leq r < R$ and $T(r,f) \leq \log^{+} M(r,f) \leq \frac{R+r}{R-r}T(R,f)$ $\{cf.$ [14]$\}$ for $0 \leq r < R < \infty$ respectively one may verify that

$$\begin{aligned}
\rho_f^{[p]L^*} &= \limsup_{r\to\infty}\frac{\log^{p}\mu(r,f)}{\log\left[re^{L(r)}\right]} = \limsup_{r\to\infty}\frac{\log^{p-1} T(r,f)}{\log\left[re^{L(r)}\right]} \\
\text{and } \lambda_f^{[p]L^*} &= \liminf_{r\to\infty}\frac{\log^{p}\mu(r,f)}{\log\left[re^{L(r)}\right]} = \liminf_{r\to\infty}\frac{\log^{p-1} T(r,f)}{\log\left[re^{L(r)}\right]} .
\end{aligned}$$

For further clarification of definitions and notations used in this paper, one may consult [27].

3. Some Theoretical Results.

In this section we present some lemmas which will be needed in the sequel.

Lemma 3.1. [25] *Let f and g be any two entire functions with $g(0) = 0$. Then for all sufficiently large values of r,*

$$\mu(r, f\circ g) \geq \frac{1}{2}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - \left|g(0)\right|, f\right) .$$

Lemma 3.2. [5] *If f and g are any two entire functions then for all sufficiently large values of r,*

$$M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - \left|g(0)\right|, f\right) \leq M(r, f\circ g) \leq M\left(M(r,g), f\right) .$$

Lemma 3.3. [20] *Let f and g be any two entire functions. Then we have*

$$T\left(r, f\circ g\right) \geq \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4}, g\right) + O(1), f\right).$$

Although the following lemmas have already been published in **[12]**, we again give the proofs of the same here because the techniques employed in the lemmas have frequently been used in the theorems proved.

Lemma 3.4. [12] Let f and g be any two entire functions such that $\rho_f^{[p]L^*} < \infty$ and $\rho_g^{[q]L^*} < \infty$ where p and q are any two positive integers. Then for any $\varepsilon > 0$ and for all sufficiently large values of r,

$$\begin{aligned} & \log^{p+\left(\frac{n-2}{2}\right)\{(p-1)+(q-1)\}}\mu\left(r, f_n\right) \\ \leq\ & \left(\rho_f^{[p]L^*} + \varepsilon\right)\left(\log\mu\left(\beta r, g\right) + L\left(\mu\left(r, g\right)\right)\right) + O(1) \text{ when } n \text{ is even} \end{aligned}$$

and

$$\begin{aligned} & \log^{p+\frac{(n-3)}{2}(p-1)+\frac{(n-1)}{2}(q-1)}\mu\left(r, f_n\right) \\ \leq\ & \left(\rho_g^{[q]L^*} + \varepsilon\right)\left[\log\mu\left(\beta r, f\right) + L\left(\mu\left(r, f\right)\right)\right] + O(1) \text{ when } n \text{ is odd with } n \neq 1 \end{aligned}$$

and $\beta > 1$.

Proof. Let us consider n to be an even number.
Then in view of Lemma 3.2 and the inequality $\mu\left(r, f\right) \leq M\left(r, f\right) \leq \frac{R}{R-r}\mu\left(R, f\right)$ $\{cf.\ [26]\}$, for $0 \leq r < R$ we get for all sufficiently large values of r that

$$\begin{aligned} \log\mu\left(r, f_n\right) &\leq \log M\left(r, f_n\right) \\ \textit{i.e., } \log\mu\left(r, f_n\right) &\leq \log M\left(M\left(r, g_{n-1}\right), f\right) \\ \textit{i.e., } \log^p\mu\left(r, f_n\right) &\leq \log^p M\left(M\left(r, g_{n-1}\right), f\right) \\ \textit{i.e., } \log^p\mu\left(r, f_n\right) &\leq \left(\rho_f^{[p]L^*} + \varepsilon\right)\log\left[M\left(r, g_{n-1}\right)e^{L\left(M\left(r, g_{n-1}\right)\right)}\right] \\ \textit{i.e., } \log^p\mu\left(r, f_n\right) &\leq \left(\rho_f^{[p]L^*} + \varepsilon\right)\left[\log M\left(r, g_{n-1}\right) + L\left(M\left(r, g_{n-1}\right)\right)\right] \\ \textit{i.e., } \log^p\mu\left(r, f_n\right) &\leq \left(\rho_f^{[p]L^*} + \varepsilon\right)\left[\log M\left(M\left(r, f_{n-2}\right), g\right) + L\left(M\left(r, g_{n-1}\right)\right)\right] \end{aligned}$$

$$\begin{aligned} & \textit{i.e., } \log^p\mu\left(r, f_n\right) \\ \leq\ & \log M\left(M\left(r, f_{n-2}\right), g\right)\left[\left(\rho_f^{[p]L^*} + \varepsilon\right) + \frac{\left(\rho_f^{[p]L^*} + \varepsilon\right)L\left(M\left(r, g_{n-1}\right)\right)}{\log M\left(M\left(r, f_{n-2}\right), g\right)}\right] \end{aligned}$$

$$i.e., \ \log^{p+(q-1)} \mu(r, f_n) \leq \log^{q} M(M(r, f_{n-2}), g) + O(1)$$

$$\begin{aligned} & i.e., \ \log^{p+(q-1)} \mu(r, f_n) \\ \leq \ & \left(\rho_g^{[q]L^*} + \varepsilon\right)[\log M(M(r, g_{n-3}), f) + L(M(r, f_{n-2}))] + O(1) \end{aligned}$$

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Therefore,

$$\begin{aligned} & \log^{p+\left(\frac{n-2}{2}\right)\{(p-1)+(q-1)\}} \mu(r, f_n) \\ \leq \ & \left(\rho_f^{[p]L^*} + \varepsilon\right)[\log \mu(\beta r, g) + L(\mu(r, g))] + O(1) \text{ when } n \text{ is even.} \end{aligned}$$

Similarly,

$$\begin{aligned} & \log^{p+\frac{(n-3)}{2}(p-1)+\frac{(n-1)}{2}(q-1)} \mu(r, f_n) \\ \leq \ & \left(\rho_g^{[q]L^*} + \varepsilon\right)[\log \mu(\beta r, f) + L(\mu(r, f))] + O(1) \text{ when } n \text{ is odd and } n \neq 1 . \end{aligned}$$

This proves the lemma. □ □

Lemma 3.5. [12] Let f and g be any two entire functions such that $\rho_f^{[p]L^*} < \infty$ and $\rho_g^{[q]L^*} < \infty$ where p and q are any two positive integers. Then for any $\varepsilon > 0$ and for all sufficiently large values of r,

$$\begin{aligned} & \log^{p+\left(\frac{n-2}{2}\right)\{(p-1)+(q-1)\}} M(r, f_n) \\ \leq \ & \left(\rho_f^{[p]L^*} + \varepsilon\right)(\log M(r, g) + L(M(r, g))) + O(1) \text{ when } n \text{ is even} \end{aligned}$$

and

$$\begin{aligned} & \log^{p+\frac{(n-3)}{2}(p-1)+\frac{(n-1)}{2}(q-1)} M(r, f_n) \\ \leq \ & \left(\rho_g^{[q]L^*} + \varepsilon\right)[\log M(r, f) + L(M(r, f))] + O(1) \text{ when } n \text{ is odd and } n \neq 1 . \end{aligned}$$

We omit the proof of the lemma because it can be carried out in the line of Lemma 3.4 and with the help of Lemma 3.2.

Similarly, the following lemma can be carried out in the line of Lemma 3.4 and in view of the inequality $T(r, f) \leq \log^{+} M(r, f) \leq \frac{R+r}{R-r} T(R, f)$ $\{cf.$ [14]$\}$ for $0 \leq r < R < \infty$.

Lemma 3.6. [12] Let f and g be any two entire functions such that $\rho_f^{[p]L^*} < \infty$ and $\rho_g^{[q]L^*} < \infty$ where p and q are any two positive integers. Then for any $\varepsilon > 0$ and for all sufficiently large values of r,

$$\begin{aligned} &\log^{\frac{n}{2}(p-1)+\left(\frac{n-2}{2}\right)(q-1)} T(r, f_n) \\ \leq\ &\left(\rho_f^{[p]L^*} + \varepsilon\right)(\log M(r, g) + L(M(r, g))) + O(1) \text{ when } n \text{ is even} \end{aligned}$$

and

$$\begin{aligned} &\log^{\frac{(n-1)}{2}\{(p-1)+(q-1)\}} T(r, f_n) \\ \leq\ &\left(\rho_g^{[q]L^*} + \varepsilon\right)[\log M(r, f) + L(M(r, f))] + O(1) \text{ when } n \text{ is odd and } n \neq 1\,. \end{aligned}$$

We omit the proof of the lemma.

Lemma 3.7. *[12] Let f and g be any two entire functions such that $0 < \lambda_f^{[p]L^*} < \infty$ and $0 < \lambda_g^{[q]L^*} < \infty$ where p and q are any two positive integers. Then for any $\varepsilon\left(0 < \varepsilon < \min\left\{\lambda_f^{[p]L^*}, \lambda_g^{[q]L^*}\right\}\right)$ and for all sufficiently large values of r,*

$$\begin{aligned} &\log^{p+\left(\frac{n-2}{2}\right)\{(p-1)+(q-1)\}} \mu(r, f_n) \\ \geq\ &\left(\lambda_f^{[p]L^*} - \varepsilon\right)\left(\log \mu\left(\frac{r}{2^{n-1}}, g\right) + L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right)\right) + O(1) \\ &\qquad\qquad \textit{when } n \textit{ is even} \end{aligned}$$

and

$$\begin{aligned} &\log^{p+\frac{(n-3)}{2}(p-1)+\frac{(n-1)}{2}(q-1)} \mu(r, f_n) \\ \geq\ &\left(\lambda_g^{[q]L^*} - \varepsilon\right)\left(\log \mu\left(\frac{r}{2^{n-1}}, f\right) + L\left(\mu\left(\frac{r}{2^{n-1}}, f\right)\right)\right) + O(1) \\ &\qquad\qquad \textit{when } n \textit{ is odd and } n \neq 1\,. \end{aligned}$$

Proof. We choose ε in such a way that $\varepsilon\left(0 < \varepsilon < \min\left\{\lambda_f^{[p]L^*}, \lambda_g^{[q]L^*}\right\}\right)$.
Also let us consider n to be an even number .
Now in view of Lemma 3.1 and the inequality $\mu(r, f) \leq M(r, f) \leq \frac{R}{R-r}\mu(R, f)$ $\{cf.$ **[26]** $\}$, for $0 \leq r < R$ we get for all sufficiently large values of r that

$$\begin{aligned} \mu(r, f_n) &= \mu(r, f \circ g_{n-1}) \\ \text{i.e., } \mu(r, f_n) &\geq \frac{1}{2}\mu\left(\frac{1}{16}\mu\left(\frac{r}{2}, g_{n-1}\right), f\right) \\ \text{i.e., } \log^{p} \mu(r, f_n) &\geq \log^{p} \mu\left(\frac{1}{16}\mu\left(\frac{r}{2}, g_{n-1}\right), f\right) + O(1) \end{aligned}$$

$$\begin{aligned} i.e., \ \log^{p}\mu(r,f_n) &\geq \left(\lambda_f^{[p]L^*}-\varepsilon\right)\log\left[\left(\frac{1}{16}\mu\left(\frac{r}{2},g_{n-1}\right)\right)e^{L\left(\frac{1}{16}\mu\left(\frac{r}{2},g_{n-1}\right)\right)}\right] \\ &\quad + O(1) \\ i.e., \ \log^{p}\mu(r,f_n) &\geq \left(\lambda_f^{[p]L^*}-\varepsilon\right)\left[\log\mu\left(\frac{r}{2},g_{n-1}\right)+L\left(\mu\left(\frac{r}{2},g_{n-1}\right)\right)\right] \\ &\quad + O(1) \end{aligned}$$

$$\begin{aligned} & i.e., \ \log^{p}\mu(r,f_n) \\ \geq\ & \left(\lambda_f^{[p]L^*}-\varepsilon\right)\left[\log\mu\left(\frac{1}{16}\mu\left(\frac{r}{2^2},f_{n-2}\right)+O(1),g\right)+L\left(\mu\left(\frac{r}{2},g_{n-1}\right)\right)\right] \\ & + O(1) \end{aligned}$$

$$\begin{aligned} & i.e., \ \log^{p}\mu(r,f_n) \\ & \geq \log\mu\left(\frac{1}{16}\mu\left(\frac{r}{2^2},f_{n-2}\right),g\right)\left[\left(\lambda_f^{[p]L^*}-\varepsilon\right)+\frac{\left(\lambda_f^{[p]L^*}-\varepsilon\right)L\left(\mu\left(\frac{r}{2},g_{n-1}\right)\right)+O(1)}{\log\mu\left(\frac{1}{16}\mu\left(\frac{r}{2^2},f_{n-2}\right),g\right)}\right] \end{aligned}$$

$$\begin{aligned} i.e., \ \log^{p+(q-1)}\mu(r,f_n) &\geq \log^{q}\mu\left(\frac{1}{16}\mu\left(\frac{r}{2^2},f_{n-2}\right),g\right)+O(1) \\ i.e., \ \log^{p+(q-1)}\mu(r,f_n) &\geq \left(\lambda_g^{[q]L^*}-\varepsilon\right)\log\left\{\left(\frac{1}{16}\mu\left(\frac{r}{2^2},f_{n-2}\right)\right)e^{L\left(\mu\left(\frac{r}{2^2},f_{n-2}\right)\right)}\right\} \\ &\quad + O(1) \end{aligned}$$

$$i.e., \ \log^{p+(q-1)}\mu(r,f_n) \geq \left(\lambda_g^{[q]L^*}-\varepsilon\right)\left[\log\mu\left(\frac{r}{2^2},f_{n-2}\right)+L\left(\mu\left(\frac{r}{2^2},f_{n-2}\right)\right)\right]$$

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Therefore

$$\begin{aligned} & \log^{p+\left(\frac{n-2}{2}\right)\{(p-1)+(q-1)\}}\mu(r,f_n) \\ \geq\ & \left(\lambda_f^{[p]L^*}-\varepsilon\right)\left[\log\mu\left(\frac{r}{2^{n-1}},g\right)+L\left(\mu\left(\frac{r}{2^{n-1}},g\right)\right)\right] \text{ when } n \text{ is even}. \end{aligned}$$

Similarly,

$$\begin{aligned} & \log^{p+\frac{(n-3)}{2}(p-1)+\frac{(n-1)}{2}(q-1)}\mu(r,f_n) \\ \geq\ & \left(\lambda_g^{[q]L^*}-\varepsilon\right)\left[\log\mu\left(\frac{r}{2^{n-1}},f\right)+L\left(\mu\left(\frac{r}{2^{n-1}},f\right)\right)\right] \text{ when } n \text{ is odd and } n \neq 1. \end{aligned}$$

Thus the lemma follows. $\square$ $\square$

Lemma 3.8. *[12] Let f and g be any two entire functions such that $0 < \lambda_f^{[p]L^*} < \infty$ and $0 < \lambda_g^{[q]L^*} < \infty$ where p and q are any two positive integers. Then for any $\varepsilon \left(0 < \varepsilon < \min\left\{\lambda_f^{[p]L^*}, \lambda_g^{[q]L^*}\right\}\right)$ and for all sufficiently large values of r,*

$$\begin{aligned} &\log^{p+\left(\frac{n-2}{2}\right)\{(p-1)+(q-1)\}} M(r, f_n) \\ \geq\ & \left(\lambda_f^{[p]L^*} - \varepsilon\right)\left(\log M\left(\frac{r}{2^{n-1}}, g\right) + L\left(M\left(\frac{r}{2^{n-1}}, g\right)\right)\right) + O(1) \\ & \qquad\qquad\qquad\qquad \textit{when } n \textit{ is even} \end{aligned}$$

and

$$\begin{aligned} &\log^{p+\frac{(n-3)}{2}(p-1)+\frac{(n-1)}{2}(q-1)} M(r, f_n) \\ \geq\ & \left(\lambda_g^{[q]L^*} - \varepsilon\right)\left(\log M\left(\frac{r}{2^{n-1}}, f\right) + L\left(M\left(\frac{r}{2^{n-1}}, f\right)\right)\right) + O(1) \\ & \qquad\qquad\qquad\qquad \textit{when } n \textit{ is odd and } n \neq 1\,. \end{aligned}$$

We omit the proof of the lemma because it can be carried out in the line of Lemma 3.7 and with the help of Lemma 3.2.

Similarly the following lemma can be carried out in the line of Lemma 3.7 and in view of Lemma 3.3.

Lemma 3.9. *[12] Let f and g be any two entire functions such that $0 < \lambda_f^{[p]L^*} < \infty$ and $0 < \lambda_g^{[q]L^*} < \infty$ where p and q are any two positive integers. Then for any $\varepsilon \left(0 < \varepsilon < \min\left\{\lambda_f^{[p]L^*}, \lambda_g^{[q]L^*}\right\}\right)$ and for all sufficiently large values of r,*

$$\begin{aligned} &\log^{\frac{n}{2}(p-1)+\left(\frac{n-2}{2}\right)(q-1)} T(r, f_n) \\ \geq\ & \left(\lambda_f^{[p]L^*} - \varepsilon\right)\left(\log M\left(\frac{r}{4^{n-1}}, g\right) + L\left(M\left(\frac{r}{4^{n-1}}, g\right)\right)\right) + O(1) \\ & \qquad\qquad\qquad\qquad \textit{when } n \textit{ is even} \end{aligned}$$

and

$$\begin{aligned} &\log^{\frac{(n-1)}{2}\{(p-1)+(q-1)\}} T(r, f_n) \\ \geq\ & \left(\lambda_g^{[q]L^*} - \varepsilon\right)\left(\log M\left(\frac{r}{4^{n-1}}, f\right) + L\left(M\left(\frac{r}{4^{n-1}}, f\right)\right)\right) + O(1) \\ & \qquad\qquad\qquad\qquad \textit{when } n \textit{ is odd and } n \neq 1\,. \end{aligned}$$

The proof is omitted.

4. Main Results.

In this section we present the main results of the paper.

Theorem 4.1. *Let f and g be any two entire functions such that $\rho_f^{[p]L^*}$ and $\rho_g^{L^*}$ are both finite and positive where $p \geq 1$.Then for each $\alpha \in (-\infty, \infty)$ and for any even n,*

$$\liminf_{r\to\infty} \frac{\left\{\log^{\frac{np}{2}} \mu(r, f_n)\right\}^{1+\alpha}}{\log^p \mu(\exp(r^\beta), f)} = 0 \text{ and}$$

$$\liminf_{r\to\infty} \frac{\left\{\log^{\frac{np}{2}} \mu(r, f_n)\right\}^{1+\alpha}}{\log^2 \mu(\exp(r^\beta), g)} = 0 \text{ where } \beta > (1+\alpha)\rho_g^{L^*}.$$

Proof. If $1+\alpha < 0$, then the theorem is trivial. So we take $1+\alpha > 0$. Now in view of Lemma 3.4, we have for all sufficiently large values of r that

$$\log^{\frac{np}{2}} \mu(r, f_n) \leq \left(\rho_f^{[p]L^*} + \varepsilon\right)(\log \mu(\beta r, g) + L(\mu(r, g))) + O(1)$$

$$\log^{\frac{np}{2}} \mu(r, f_n) \leq \left(\rho_f^{[p]L^*} + \varepsilon\right)\left[\beta r e^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)} + \left(\rho_f^{[p]L^*} + \varepsilon\right) L(\mu(r, g)) + O(1)$$

i.e.,

$$\begin{aligned} &\left\{\log^{\frac{np}{2}} \mu(r, f_n)\right\}^{1+\alpha} \\ &\leq \left[\left[r e^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)}\left(\rho_f^{[p]L^*} + \varepsilon\right) + \left(\rho_f^{[p]L^*} + \varepsilon\right) L(\mu(r, g)) + O(1)\right]^{1+\alpha}. \end{aligned} \tag{4.1}$$

Again we have for a sequence of r tending to infinity and for $\varepsilon(> 0)$ that

$$\log^p \mu\left(\exp\left(r^\beta\right), f\right) \geq \left(\rho_f^{[p]L^*} - \varepsilon\right) \log\left[\exp\left(r^\beta\right) \exp\left\{L\left(\exp\left(r^\beta\right)\right)\right\}\right]$$

$$\text{i.e., } \log^p \mu\left(\exp\left(r^\beta\right)\right), f) \geq \left(\rho_f^{[p]L^*} - \varepsilon\right)\left[r^\beta + L\left(\exp\left(r^\beta\right)\right)\right]. \tag{4.2}$$

So from (4.1) and (4.2), we get for a sequence of r tending to infinity that

$$\frac{\left\{\log^{\frac{np}{2}} \mu(r, f_n)\right\}^{1+\alpha}}{\log^p \mu(\exp(r^\beta), f)}$$

$$\leq \frac{\left[\left[re^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)}\left(\rho_f^{[p]L^*}+\varepsilon\right)+\left(\rho_f^{[p]L^*}+\varepsilon\right)L\left(\mu\left(r,g\right)\right)+O(1)\right]^{1+\alpha}}{\left(\rho_f^{[p]L^*}-\varepsilon\right)\left[r^{\beta}+L\left(\exp\left(r^{\beta}\right)\right)\right]}. \tag{4.3}$$

Let

$$\begin{aligned}\left[e^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)}\left(\rho_f^{[p]L^*}+\varepsilon\right) &= k_1, \left(\rho_f^{[p]L^*}+\varepsilon\right)L\left(\mu\left(r,g\right)\right)=k_2,\\ \left(\rho_f^{[p]L^*}-\varepsilon\right) &= k_3, \left(\rho_f^{[p]L^*}-\varepsilon\right)L\left(\exp\left(r^{\beta}\right)\right)=k_4.\end{aligned}$$

Then from (4.3) we obtain for a sequence of r tending to infinity that

$$\begin{aligned}\frac{\left\{\log^{\frac{np}{2}}\mu\left(r,f_n\right)\right\}^{1+\alpha}}{\log^{p}\mu\left(\exp\left(r^{\beta}\right),f\right)} &\leq \frac{\left[r^{\left(\rho_g^{L^*}+\varepsilon\right)}k_1+k_2+O(1)\right]^{1+\alpha}}{k_3r^{\beta}+k_4}\\ i.e.,\ \frac{\left\{\log^{\frac{np}{2}}\mu\left(r,f_n\right)\right\}^{1+\alpha}}{\log^{p}\mu\left(\exp\left(r^{\beta}\right),f\right)} &\leq \frac{r^{\left(\rho_g^{L^*}+\varepsilon\right)(1+\alpha)}\left[k_1+\frac{k_2+O(1)}{r^{\left(\rho_g^{L^*}+\varepsilon\right)}}\right]^{1+\alpha}}{k_3r^{\beta}+k_4}\end{aligned}$$

where k_1, k_2, k_3 and k_4 are finite.
Since $\left(\rho_g^{L^*}+\varepsilon\right)(1+\alpha)<\beta$, one can verify

$$\liminf_{r\to\infty}\frac{\left\{\log^{\frac{np}{2}}\mu\left(r,f_n\right)\right\}^{1+\alpha}}{\log^{p}\mu\left(\exp\left(r^{\beta}\right),f\right)}=0$$

where we choose $\varepsilon(>0)$ such that

$$0<\varepsilon<\min\left\{\rho_f^{[p]L^*},\frac{\beta}{1+\alpha}-\rho_g^{L^*}\right\},$$

which proves the first part of the theorem.
Similarly, the second part of the theorem follows from the following inequality in place of (4.2)

$$i.e.,\ \log^{2}\mu\left(\exp\left(r^{\beta}\right),g\right)\geq\left(\rho_g^{L^*}-\varepsilon\right)\left[r^{\beta}+L\left(\exp\left(r^{\beta}\right)\right)\right]$$

for a sequence of r tending to infinity.
This proves the theorem. □ □

Remark 4.1. In view of Lemma 3.5 and under the same conditions, Theorem 4.1 is still valid with $M\left(r,f_n\right)$, $M\left(\exp\left(r^{\beta}\right),f\right)$ and $M\left(\exp\left(r^{\beta}\right),g\right)$ as respetively replaced by $\mu\left(r,f_n\right)$, $\mu\left(\exp\left(r^{\beta}\right),f\right)$ and $\mu\left(\exp\left(r^{\beta}\right),g\right)$.

Remark 4.2. Using Lemma 3.6 and the conditions of Theorem 4.1, one may easily deduce the followings

$$\liminf_{r\to\infty}\frac{\left\{\log^{\frac{np-2}{2}}T\left(r,f_n\right)\right\}^{1+\alpha}}{\log^{p-1}T\left(\exp\left(r^{\beta}\right),f\right)}=0$$

$$\text{and }\liminf_{r\to\infty}\frac{\left\{\log^{\frac{np-2}{2}}T\left(r,f_n\right)\right\}^{1+\alpha}}{\log T\left(\exp\left(r^{\beta}\right),g\right)}=0\ \text{ where }\beta>(1+\alpha)\,\rho_g^{L^*}.$$

Remark 4.3. In Theorem 4.1, Remark 4.1 and Remark 4.2 if we take the condition " $0<\lambda_f^{[p]L^*}\leq\rho_f^{[p]L^*}<\infty$ and $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$ " in place of " $\rho_f^{[p]L^*}$ and $\rho_g^{L^*}$ are both finite and positive " the theorem remains true with "lim" replaced by "lim inf".

The following theorem can be carried out in the line of Theorem 4.1 and with the help of Lemma 3.4. Therefore its proof is omitted.

Theorem 4.2. *Let f and g be any two entire functions with $\rho_f^{L^*}$ and $\rho_g^{[q]L^*}$ are both finite and positive where $q\geq 1$.Then for each $\alpha\in(-\infty,\infty)$ and for any odd $n\ (\neq 1)$,*

$$\liminf_{r\to\infty}\frac{\left\{\log^{\frac{nq-q+2}{2}}\mu\left(r,f_n\right)\right\}^{1+\alpha}}{\log^{2}\mu\left(\exp\left(r^{\beta}\right),f\right)}=0\ \textit{and}$$

$$\liminf_{r\to\infty}\frac{\left\{\log^{\frac{nq-q+2}{2}}\mu\left(r,f_n\right)\right\}^{1+\alpha}}{\log^{q}\mu\left(\exp\left(r^{\beta}\right),g\right)}=0\ \textit{ where }\beta>(1+\alpha)\,\rho_f^{L^*}.$$

Remark 4.4. In view of Lemma 3.5 under the same conditions, Theorem 4.2 remains true with $M\left(r,f_n\right)$, $M\left(\exp\left(r^{\beta}\right),f\right)$ and $M\left(\exp\left(r^{\beta}\right),g\right)$ as respetively replaced by $\mu\left(r,f_n\right)$, $\mu\left(\exp\left(r^{\beta}\right),f\right)$ and $\mu\left(\exp\left(r^{\beta}\right),g\right)$.

Remark 4.5. Using Lemma 3.6 and the conditions of Theorem 4.2 one may easily compute the followings

$$\liminf_{r\to\infty}\frac{\left\{\log^{\frac{q(n-1)}{2}}T\left(r,f_n\right)\right\}^{1+\alpha}}{\log T\left(\exp\left(r^{\beta}\right),f\right)}=0\text{ and}$$

$$\liminf_{r\to\infty}\frac{\left\{\log^{\frac{q(n-1)}{2}}T\left(r,f_n\right)\right\}^{1+\alpha}}{\log^{[q-1]}T\left(\exp\left(r^{\beta}\right),g\right)}=0\ \text{ where }\beta>(1+\alpha)\,\rho_f^{L^*}.$$

Remark 4.6. In Theorem 4.1, Remark 4.1 and Remark 4.2 if we take the condition " $0<\lambda_f^{L^*}\leq\rho_f^{L^*}<\infty$ and $0<\lambda_g^{[q]L^*}\leq\rho_g^{[q]L^*}<\infty$ " in place of " $\rho_f^{L^*}$ and $\rho_g^{[q]L^*}$ are both finite and positive " the theorem remains true with "lim" replaced by "lim inf".

Theorem 4.3. *Let f and g be any two entire functions such that $0 < \lambda_f^{[p]L^*} \leq \rho_f^{[p]L^*} < \infty$ where p is any positive integer and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$. Then for any even number n,*

$$\limsup_{r\to\infty} \frac{\log^{\frac{np+2}{2}} \mu(r, f_n)}{\log^p \mu(r, f) + L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right)} \geq \frac{\rho_g^{L^*}}{\rho_f^{[p]L^*}}.$$

Proof. In view of Lemma 3.7, we have for all sufficiently large values of r that

$$\log^{\frac{np}{2}} \mu(r, f_n) \geq \left(\lambda_f^{[p]L^*} - \varepsilon\right)\left(\log \mu\left(\frac{r}{2^{n-1}}, g\right) + L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right)\right) + O(1)$$

which implies

$$\log^{\frac{np}{2}} \mu(r, f_n) \geq \left(\lambda_f^{[p]L^*} - \varepsilon\right) \log \mu\left(\frac{r}{2^{n-1}}, g\right). \left\{\frac{L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right) + O(1)}{\log \mu\left(\frac{r}{2^{n-1}}, g\right)}\right\}$$

and further

$$\begin{aligned}\log^{\frac{np+2}{2}} \mu(r, f_n) \geq{}& \log^2 \mu\left(\frac{r}{2^{n-1}}, g\right) \\ &+ \left(\frac{\lambda_g^{L^*} - \varepsilon}{\rho_f^{[p]L^*} + \varepsilon}\right) L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right) \\ &- \log\left[\exp\left\{\left(\frac{\lambda_g^{L^*} - \varepsilon}{\rho_f^{[p]L^*} + \varepsilon}\right) L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right)\right\}\right] \\ &+ \log\left\{\frac{L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right) + O(1)}{\log \mu\left(\frac{r}{2^{n-1}}, g\right)}\right\}\end{aligned}$$

i.e.,

$$\begin{aligned}\log^{\frac{np+2}{2}} \mu(r, f_n) \geq{}& \log^2 \mu\left(\frac{r}{2^{n-1}}, g\right) \\ &+ \left(\frac{\lambda_g^{L^*} - \varepsilon}{\rho_f^{[p]L^*} + \varepsilon}\right) L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right) \\ &+ \log\left\{\frac{L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right) + O(1)}{\exp\left\{\left(\frac{\lambda_g^{L^*} - \varepsilon}{\rho_f^{[p]L^*} + \varepsilon}\right) L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right)\right\} \log \mu\left(\frac{r}{2^{n-1}}, g\right)}\right\}\end{aligned}$$

Finally

$$\log^{\frac{np+2}{2}} \mu(r, f_n) \geq \log^2 \mu\left(\frac{r}{2^{n-1}}, g\right) + \left(\frac{\lambda_g^{L^*} - \varepsilon}{\rho_f^{[p]L^*} + \varepsilon}\right) L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right). \tag{4.4}$$

Now from (4.4) it follows for a sequence of values of r tending to infinity that

$$\log^{\frac{np+2}{2}} \mu(r, f_n) \geq \left(\rho_g^{L^*} - \varepsilon\right) \log\left\{\frac{r}{2^{n-1}} e^{L(r)}\right\} + \left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_f^{[p]L^*} + \varepsilon}\right) L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right).$$

Now we get for all sufficiently large values of r that

$$\begin{aligned} \log^{p} \mu(r, f) &\leq \left(\rho_f^{[p]L^*} + \varepsilon\right) \log\left\{r e^{L(r)}\right\} \\ \text{i.e., } \log^{p} \mu(r, f) &\leq \left(\rho_f^{[p]L^*} + \varepsilon\right) \log\left\{\frac{r}{2^{n-1}} e^{L(r)}\right\} + \log 2^{n-1}. \end{aligned} \tag{4.5}$$

Hence from (4.) and (4.5), it follows for all sufficiently large values of r that

$$\begin{aligned} \log^{\frac{np+2}{2}} \mu(r, f_n) \geq{}& \left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_f^{[p]L^*} + \varepsilon}\right) \left(\log^{p} \mu(r, f) - \log 2^{n-1}\right) \\ &+ \left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_f^{[p]L^*} + \varepsilon}\right) L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right) \end{aligned}$$

i.e.,

$$\begin{aligned} \log^{\frac{np+2}{2}} \mu(r, f_n) \geq{}& \left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_f^{[p]L^*} + \varepsilon}\right) \left[\log^{p} \mu(r, f) + L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right)\right] \\ &- \left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_f^{[p]L^*} + \varepsilon}\right) \log 2^{n-1} \end{aligned}$$

$$\text{i.e., } \frac{\log^{\frac{np+2}{2}} \mu(r, f_n)}{\log^{p} \mu(r, f) + L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right)}$$

$$\geq \left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_f^{[p]L^*} + \varepsilon}\right) - \frac{\left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_f^{[p]L^*} + \varepsilon}\right) \log 2^{n-1}}{\log^{p} \mu(r, f) + L\left(\mu\left(\frac{r}{2^{n-1}}, g\right)\right)}. \tag{4.6}$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (4.6) that

$$\limsup_{r\to\infty}\frac{\log^{\frac{np+2}{2}}\mu(r,f_n)}{\log^{p}\mu(r,f)+L\left(\mu\left(\frac{r}{2^{n-1}},g\right)\right)}\geq\frac{\rho_g^{L^*}}{\rho_f^{[p]L^*}}.$$

This proves the theorem. □ □

In the line of Theorem 4.3, the following theorem can be proved:

Theorem 4.4. *Let f and g be any two entire functions with $0<\lambda_f^{[p]L^*}\leq\rho_f^{[p]L^*}<\infty$ where $p\geq1$ and $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$. Then for any even number n,*

$$\liminf_{r\to\infty}\frac{\log^{\frac{np+2}{2}}\mu(r,f_n)}{\log^{p}\mu(r,f)+L\left(\mu\left(\frac{r}{2^{n-1}},g\right)\right)}\geq\frac{\lambda_g^{L^*}}{\rho_f^{[p]L^*}}.$$

The proof is omitted.

Remark 4.7. In view of Lemma 3.8 and under the same conditions , Theorem 4.3 and Theorem 4.4 are still valid with maximum moduli as replaced by maximum terms.

Remark 4.8. Following Theorem 4.3 and Theorem 4.4 and also using Lemma 3.9 and in view of the inequality $T(r,f)\leq\log^{+}M(r,f)$ $\{cf.$ [14]$\}$ one may respectively obtain the followings conclusions :

$$\limsup_{r\to\infty}\frac{\log^{\frac{np}{2}}T(r,f_n)}{\log^{p-1}T(r,f)+L\left(\exp\left\{T\left(\frac{r}{4^{n-1}},g\right)\right\}\right)}\geq\frac{\rho_g^{L^*}}{\rho_f^{[p]L^*}}$$

and

$$\liminf_{r\to\infty}\frac{\log^{\frac{np}{2}}T(r,f_n)}{\log^{p-1}T(r,f)+L\left(\exp\left\{T\left(\frac{r}{4^{n-1}},g\right)\right\}\right)}\geq\frac{\lambda_g^{L^*}}{\rho_f^{[p]L^*}}.$$

In the line of Theorem 4.3 and Theorem 4.4, we may state the following two theorems without their proofs :

Theorem 4.5. *Let f and g be any two entire functions such that $0<\lambda_f^{L^*}\leq\rho_f^{L^*}<\infty$ and $0<\lambda_g^{[q]L^*}\leq\rho_g^{[q]L^*}<\infty$ where q is any positive integer . Then for any odd number $n\,(\neq1)$,*

$$\limsup_{r\to\infty}\frac{\log^{\frac{q(n-1)+4}{2}}\mu(r,f_n)}{\log^{q}\mu(r,g)+L\left(\mu\left(\frac{r}{2^{n-1}},f\right)\right)}\geq\frac{\rho_f^{L^*}}{\rho_g^{[q]L^*}}.$$

Theorem 4.6. *Let f and g be any two entire functions with $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and $0 < \lambda_g^{[q]L^*} \leq \rho_g^{[q]L^*} < \infty$ where $q \geq 1$. Then for any odd number $n\ (\neq 1)$,*

$$\liminf_{r\to\infty} \frac{\log^{\frac{q(n-1)+4}{2}} \mu(r, f_n)}{\log^q \mu(r,g) + L\left(\mu\left(\frac{r}{2^{n-1}}, f\right)\right)} \geq \frac{\lambda_f^{L^*}}{\rho_g^{[q]L^*}}.$$

Replacing maximum terms by maximum modulus in Theorem 4.5 and Theorem 4.6, we may obtain the following two remarks:

Remark 4.9. For any two entire functions f and g with $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and $0 < \lambda_g^{[q]L^*} \leq \rho_g^{[q]L^*} < \infty$ where q is any positive integer. Then for any odd number $n\ (\neq 1)$,

$$\limsup_{r\to\infty} \frac{\log^{\frac{q(n-1)+4}{2}} M(r, f_n)}{\log^q M(r,g) + L\left(M\left(\frac{r}{2^{n-1}}, f\right)\right)} \geq \frac{\rho_f^{L^*}}{\rho_g^{[q]L^*}}.$$

Remark 4.10. If f and g be any two entire functions with $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and $0 < \lambda_g^{[q]L^*} \leq \rho_g^{[q]L^*} < \infty$ where $q \geq 1$. Then for any odd number $n\ (\neq 1)$,

$$\liminf_{r\to\infty} \frac{\log^{\frac{q(n-1)+4}{2}} M(r, f_n)}{\log^q M(r,g) + L\left(M\left(\frac{r}{2^{n-1}}, f\right)\right)} \geq \frac{\lambda_f^{L^*}}{\rho_g^{[q]L^*}}.$$

Remark 4.11. Under the same conditions, Remark 4.9 and Remark 4.10 must be valid with Nevanlinna's characteristic function (taking into account the necessary changes of successive logarithms both in numerators and denominators of the ratios) as replaced by maximum moduli.

Theorem 4.7. *Let f and g be any two entire functions such that $0 < \rho_f^{[p]L^*} < \infty$ where $p \geq 1$ and $\sigma_g^{L^*} < \infty$. Then for any $\beta > 1$ and for any even number n,*
(a) if $L(\mu(\beta r, g)) = o\{\log \mu(r,g)\}$ then

$$\liminf_{r\to\infty} \frac{\log^{\frac{np}{2}} \mu(r, f_n)}{\log \mu(r,g) + L(\mu(\beta r, g))} \leq \beta^{2\rho_g^{L^*}}.\rho_f^{[p]L^*}$$

and (b) if $\log \mu(r,g) = o\{L(\mu(\beta r, g))\}$ then

$$\frac{\log^{\frac{np}{2}} \mu(r, f_n)}{\log \mu(r,g) + L(\mu(\beta r, g))} \leq \rho_f^{[p]L^*}.$$

Proof. In view of Lemma 3.4 and the inequality $\mu(r,f) \leq M(r,f)$ $\{cf.$ **[26]** $\}$ we get for all sufficiently large values of r that

$$\text{Log}^{\frac{np}{2}} \mu(r, f_n) \leq \left(\rho_f^{[p]L^*} + \varepsilon\right)\{\text{Log} M(\beta r, g) + L(\mu(\beta r, g))\} + O(1). \tag{4.7}$$

Using the definition of L^*-type, we obtain from (4.7) for all sufficiently large values of r that

$$\log^{\frac{np}{2}} \mu(r, f_n) \leq \left(\rho_f^{[p]L^*} + \varepsilon\right)\left(\sigma_g^{L^*} + \varepsilon\right)\left\{\beta r e^{L(r)}\right\}^{\rho_g^{L^*}} + \left(\rho_f^{[p]L^*} + \varepsilon\right) L(\mu(\beta r, g)) + O(1) . \tag{4.8}$$

Again from the definition of L^*-type and taking $R = \beta r$ the inequality $M(r, f) \leq \frac{R}{R-r}\mu(R, f)$ $\{cf.$ **[26]** $\}$, we get for a sequence of values of r tending to infinity that

$$\begin{aligned} \log \mu(r, g) &\geq \log M\left(\frac{r}{\beta}, g\right) + O(1) \\ &\geq \left(\sigma_g^{L^*} - \varepsilon\right)\left\{\left(\frac{r}{\beta}\right) e^{L\left(\frac{r}{\beta}\right)}\right\}^{\rho_g^{L^*}} + O(1) \end{aligned} \tag{4.9}$$

$$i.e., \ \left\{\beta r e^{L(r)}\right\}^{\rho_g^{L^*}} \leq \beta^{2\rho_g^{L^*}} \frac{\log \mu(r, g)}{\left(\sigma_g^{L^*} - \varepsilon\right)} .$$

Now from (4.8) and (4.9), it follows for a sequence of values of r tending to infinity that

$$\log^{\frac{np}{2}} \mu(r, f_n) \leq \beta^{2\rho_g^{L^*}}.\left(\rho_f^{[p]L^*} + \varepsilon\right)\left(\sigma_g^{L^*} + \varepsilon\right)\frac{\log \mu(r, g)}{\left(\sigma_g^{L^*} - \varepsilon\right)} + \left(\rho_f^{[p]L^*} + \varepsilon\right) L(\mu(\beta r, g)) + O(1)$$

i.e.,

$$\frac{\log^{\frac{np}{2}} \mu(r, f_n)}{\log \mu(r, g) + L(\mu(\beta r, g))} \leq \frac{\frac{\beta^{2\rho_g^{L^*}}.\left(\rho_f^{[p]L^*}+\varepsilon\right)\left(\sigma_g^{L^*}+\varepsilon\right)}{\left(\sigma_g^{L^*}-\varepsilon\right)}}{1 + \frac{L(\mu(\beta r, g))}{\log \mu(r, g)}} + \frac{\left(\rho_f^{[p]L^*} + \varepsilon\right)}{1 + \frac{\log \mu(r, g)}{L(\mu(\beta r, g))}} . \tag{4.10}$$

If $L(\mu(\beta r, g)) = o\{\log \mu(r, g)\}$ then from (4.10) we get that

$$\liminf_{r\to\infty} \frac{\log^{\frac{np}{2}} \mu(r, f_n)}{\log \mu(r, g) + L(\mu(\beta r, g))} \leq \frac{\beta^{2\rho_g^{L^*}}.\left(\rho_f^{[p]L^*} + \varepsilon\right)\left(\sigma_g^{L^*} + \varepsilon\right)}{\left(\sigma_g^{L^*} - \varepsilon\right)} .$$

Since ε (> 0) is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{\frac{np}{2}} \mu(r, f_n)}{\log \mu(r, g) + L(\mu(\beta r, g))} \leq \beta^{2\rho_g^{L^*}}.\rho_f^{[p]L^*} .$$

Thus the first part of the theorem follows.
Again if $\log \mu(r, g) = o\{L(\mu(\beta r, g))\}$ then from (4.10) it follows that

$$\liminf_{r\to\infty} \frac{\log^{\frac{np}{2}} \mu(r, f_n)}{\log \mu(r, g) + L(\mu(\beta r, g))} \leq \left(\rho_f^{[p]L^*} + \varepsilon\right) .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to\infty}\frac{\log^{\frac{np}{2}}\mu(r,f_n)}{\log\mu(r,g)+L(\mu(\beta r,g))}\leq\rho_f^{[p]L^*},$$

which is the second part of the theorem. Thus the theorem is established. □

The following theorem can be carried out in the line of Theorem 4.7 and with the help of Lemma 3.4. Therefore its proof is omitted.

Theorem 4.8. *Let f and g be any two entire functions with $0<\rho_g^{[q]L^*}<\infty$ where $q\geq 1$ and $\sigma_f^{L^*}<\infty$. Then for any $\beta>1$ and any odd number n except 1,*
(a) if $L(\mu(\beta r,f))=o\{\log\mu(r,f)\}$ then

$$\liminf_{r\to\infty}\frac{\log^{\frac{q(n-1)+2}{2}}\mu(r,f_n)}{\log\mu(r,f)+L(\mu(\beta r,f))}\leq\beta^{2\rho_f^{L^*}}\cdot\rho_g^{[q]L^*}$$

and (b) if $\log\mu(r,f)=o\{L(\mu(\beta r,f))\}$ then

$$\frac{\log^{\frac{q(n-1)+2}{2}}\mu(r,f_n)}{\log\mu(r,f)+L(\mu(\beta r,f))}\leq\rho_g^{[q]L^*}.$$

Remark 4.12. In view of Lemma 3.5 and under the same conditions, Theorem 4.7 and Theorem 4.8 still stand with $M(r,f_n)$, $M(r,g)$ and $M(r,f)$ respectively as changed by $\mu(r,f_n)$, $\mu(r,g)$ and $\mu(r,f)$.

Remark 4.13. Using Lemma 3.6 and the conditions of Theorem 4.7, one may easily deduce the followings:
(a) if $L\left(\exp\left\{\frac{\beta+1}{\beta-1}T(\beta r,g)\right\}\right)=o\{\log M(r,g)\}$ then

$$\liminf_{r\to\infty}\frac{\log^{\frac{np-2}{2}}T(r,f_n)}{T(r,g)+L\left(\exp\left\{\frac{\beta+1}{\beta-1}T(\beta r,g)\right\}\right)}\leq\rho_f^{[p]L^*}$$

and (b) if $\log M(r,g)=o\left\{L\exp\left\{\frac{\beta+1}{\beta-1}T(\beta r,g)\right\}\right\}$ then

$$\frac{\log^{\frac{np-2}{2}}T(r,f_n)}{T(r,g)+L\left(\exp\left\{\frac{\beta+1}{\beta-1}T(\beta r,g)\right\}\right)}\leq\rho_f^{[p]L^*}.$$

Remark 4.14. In view of Lemma 3.6 and using the conditions of Theorem 4.8, the following conclusions may be obtained:
(a) if $L\left(\exp\left\{\frac{\beta+1}{\beta-1}T(\beta r,f)\right\}\right)=o\{\log M(r,f)\}$ then

$$\liminf_{r\to\infty}\frac{\log^{\frac{q(n-1)}{2}}T(r,f_n)}{T(r,f)+L\left(\exp\left\{\frac{\beta+1}{\beta-1}T(\beta r,f)\right\}\right)}\leq\rho_g^{[q]L^*}$$

and (b) if $\log M(r,f) = o\left\{L\exp\left\{\frac{\beta+1}{\beta-1}T(\beta r,f)\right\}\right\}$ then

$$\frac{\log^{\frac{q(n-1)}{2}} T(r,f_n)}{T(r,f) + L\left(\exp\left\{\frac{\beta+1}{\beta-1}T(\beta r,f)\right\}\right)} \leq \rho_g^{[q]L^*}.$$

5. Concluding Remarks and Scope for Further Research.

Different growth indicators of entire f in terms of slowly changing functions have frequently been used in this paper just taking into consideration a comparison with the exp function. But cases may be risen out if one is interested in finding out what are the applications of growth indicators of an entire function f with respect to an arbitrary entire function g. Keeping this in mind, the notion of *relative order (* respectively *relative lower order)* as initiated by L. Bernal [1] may be a further scope of penetration in the field of the growth of iterated entire functions in view of slowly changing functions. Still it remains open to the researchers of this branch to investigate such a type of results in the light of *relative (p,q) th order (*respectively *relative (p,q)-th lower order) with* any two positive integers p, q as introduced by Sánchez Ruiz et al. [19] and also r*elative proximate order (*respectively *relative proximate lower order)* in order to obtain sharper estimations of the same.

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